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EXPLICIT SOLUTIONS OF AN EPIDEMIOLOGICAL MODEL OF THE SIR TYPE

FRANCESCO CALOGERO

Department of Physics, University of Rome "La Sapienza", Rome, Italy
and
INFN, Sezione di Roma 1

ANDREA GIANSANTI

Department of Physics, University of Rome "La Sapienza", Rome, Italy
and
INFN, Sezione di Roma 1

FARRIN PAYANDEH

Department of Physics, Payame Noor University (PNU), PO BOX 19395-3697
Tehran, Iran

This paper is dedicated to the memory of Emma Previato. We all admire her scientific achievements. Some of us did have the privilege to personally know her.

Abstract. A system of 4 nonlinearly-coupled Ordinary Differential Equations has been recently introduced to investigate the evolution of human respiratory virus epidemics. In this paper we prove that some *explicit* solutions of that system can be obtained by *algebraic* operations, provided the parameters of the model satisfy certain *constraints*.

E-mail addresses: francesco.calogero@uniroma1.it, andrea.giansanti@uniroma1.infn.it, payandeh@pnu.ac.ir.

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1. INTRODUCTION

Since the classic paper by Kermack and McKendric [1] the mathemtical theory of epidemics rests on compartmental SIR models in which the population is divided into three compartments: the number of susceptible $S(t)$, infected $I(t)$ and recovered $R(t)$ individuals at time t. The evolution of the dynamical variables above is ruled by nonlinear systems of ordinary differential equations (ODEs). The original structure of the SIR models has been variously extended to include more compartments (e.g. SEIR models, in which also the number of exposed individuals $E(t)$ is considered) and has been largely used in modelling and forecasting the still ongoing COVID-19 pandemic (see [2][3][4] for an assessment review). Analytic solutions for this nowadays large family of models are known only in a limited number of cases and, in general, the quantitative epidemiological data analysis is mainly based either on the numerical integration of systems of ODEs of the SIR type or on fitting the data to analytic approximate *ansatze*, suggested by empirical observations. Searching for analytical solutions of compartmental models of epidemics has an intrinsic mathematical interests and may allow control on the use of approximate methods and in the analysis of stability. In this paper we provide families of special solutions of a nonlinear system of ODEs that has been recently proposed to model the COVID-19 epidemics. The model has ben proposed to forecast the evolution of the virulence of the infection by a very active group lead by Eugene Koonin at the NCBI center of the National Institutes of Health, in the US. The model adds, to the usual S and I compartments, $A(t)$ the number of asymptomatic infectious hosts and the number of symptomatic infectious hosts $C(t)$ [5]. This model was designed to afford the problem of incorporating the latent period between when an individual is exposed to a pathogen and when that individual becomes infected and contagious, a time scale that is left out in the basic SIR models [2]. From the mathematical point of view, our choice of the model in [5] was motivated by the observation that the proposed system of first order ODEs has a right-hand side that is homogeneous in the 4 dependent dynamical variables, a property that has been previously associated to families of analytic solutions [6].

Here are the equations:

$$\dot{\tilde{x}}_1 = -k_D\tilde{x}_1 + \alpha k_R\left(\tilde{x}_3 + \tilde{x}_4\right), \tag{1a}$$

$$\dot{\tilde{x}}_2 = k_B\tilde{x}_1 + \left[k_B - k_D - f\left(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\right)\right]\tilde{x}_2 + \left[k_B + (1-\alpha)\,k_R\right]\left(\tilde{x}_3 + \tilde{x}_4\right), \tag{1b}$$

$$\dot{\tilde{x}}_3 = f\left(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\right)\tilde{x}_2 - \left(k_R + k_D + k_P\right)\tilde{x}_3, \tag{1c}$$

$$\dot{\tilde{x}}_4 = k_P\tilde{x}_3 - \left(k_R + k_D + k_{DV}\right)\tilde{x}_4, \tag{1d}$$

where

$$f\left(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\right) = \frac{k_I\left(\tilde{x}_3 + \beta\tilde{x}_4\right)}{\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 + \beta\tilde{x}_4}\;. \tag{1e}$$

Notation. We maintained the original notation of [5], except for the following replacement of the 4 dependent variables $I\left(t\right)$ (number of *Immune* hosts), $S\left(t\right)$ (number of *Susceptible* hosts), $A\left(t\right)$ (number of *Asymptomatic* and infectious hosts), $C\left(t\right)$

(number of *Symptomatic* and infectious hosts) used there, and the use of a superimposed dot (instead of an appended prime) to denote differentiation with respect to the dependent variable t ("time"):

(2)

$$I(t) = \tilde{x}_1(t) \ , \ S(t) = \tilde{x}_2(t) \ , \ A(t) = \tilde{x}_3(t) \ , \ C(t) = \tilde{x}_4(t) \ ; \ \dot{\tilde{x}}(t) \equiv d\tilde{x}(t)/dt \ .$$

Note that in the following we occasionally *omit* to indicate *explicitly* the t-dependence of the dependent variables: and see below Remark 1 for our (notational) motivation of the tilde superimposed on these coordinates $\tilde{x}_n(t)$. □

For the epidemiological significance of this model see [5], as well as for references to analogous models, considered in references [3] and [4].

Remark 1. *The system of* 4 *nonlinearly-coupled ODEs Eq.* (1) *features the* 8 *a priori arbitrary (of course time-independent) parameters* k_D, k_R, k_B, k_P, k_{DV}, k_I, α, β, *for whose epidemiological significance we refer to* [5]*; in this paper we focus mainly on some mathematical properties of this system, so we generally assume that these are* 8 *a priori arbitrary (possibly even complex) numbers, although we shall comment occasionally on the relevance of such mathematical treatment on the epidemic model (when these numbers are positive real numbers).*

One observation which is relevant for the mathematical discussion of this model Eq. (1)—which we think is reasonable to state at the very beginning of this paper—is to note that the parameter k_D plays a relatively trivial role in this system, because it can be altogether eliminated from it via the following very simple change of dependent variables:

(3a) $$\tilde{x}_n(t) = x_n(t) \ \exp(-k_D t) \ , \ n = 1, 2, 3, 4 \ ,$$

implying of course

(3b) $$\tilde{x}_n(0) = x_n(0) \ , \ n = 1, 2, 3, 4 \ ;$$

indeed the system of ODEs satisfied by the 4 variables $x_n(t)$ is then *identical* with the original system Eq. (1), except for the elimination of the parameter k_D:

(4a) $$\dot{x}_1 = \alpha k_R (x_3 + x_4) \ ,$$

(4b) $$\dot{x}_2 = k_B x_1 + [k_B - f(x_1, x_2, x_3, x_4)] \, x_2 + [k_B + (1 - \alpha) \, k_R] \, (x_3 + x_4) \ ,$$

(4c) $$\dot{x}_3 = f(x_1, x_2, x_3, x_4) \, x_2 - (k_R + k_P) \, x_3 \ ,$$

(4d) $$\dot{x}_4 = k_P x_3 - (k_R + k_{DV}) \, x_4 \ ,$$

where of course now

(4e) $$f(x_1, x_2, x_3, x_4) = \frac{k_I (x_3 + \beta x_4)}{x_1 + x_2 + x_3 + \beta x_4} .$$

Hence hereafter we shall mainly deal with this, marginally simpler, system Eq. (4).

In the following Section 2 we investigate a very simple solution of this model Eq. (4), characterized by the fact that the 4 components $x_n(t)$ of this solution all evolve proportionally to the same exponential function of time, $\exp(\mu t)$, with μ an appropriate parameter determined in terms of the parameters of the model; implying that the quantity $f(x_1, x_2, x_3, x_4)$ is *time-independent* (see Eq. (4e)), hence that the system Eq. (4), for this class of solutions, reduces to a *linear* system of 4 ODEs.

In the subsequent Section 3 we discuss the solutions characterized by the requirement that each of the 4 components $x_n(t)$ of the solution be *linear* combinations—with *time-independent* coefficients—of 2 exponential functions of time, $\exp(\mu_1 t)$ and $\exp(\mu_2 t)$, and moreover that the quantity $f(x_1, x_2, x_3, x_4)$, see Eq. (4e), be again *time-independent*. The related restrictions on the parameters of the model and the initial-data of this solution are also *explicitly* determined, up to *algebraic* operations.

The subsequent Section 4 outlines the *analogous* treatments when the solution is the sum of 3, or 4, exponentials.

A final Section 5 concludes the paper, by mentioning its applicative relevance and possible further developments of the approach used in this paper.

2. A very simple solution

The right-hand sides of the 4 ODEs Eq. (4) are all *homogeneous* of degree 1 in the 4 dependent variables $x_n(t)$. This implies a well-known (see for instance [6]) consequence, which can be stated as the following

Proposition 1. *The system of* 4 *ODEs Eq.* (4) *features the simple explicit solution*

$$x_n(t) = x_n(0)\ \exp(\mu t)\ ,\ n = 1, 2, 3, 4\ , \tag{5}$$

where $x_n(0)$ *are clearly the* 4 *initial values of the* 4 *dependent variables* $x_n(t)$ *and* μ *is an a priori arbitrary time-independent parameter, provided these* 5 *quantities—i. e.,* $x_n(0)$ *and* μ*, together with the* 7 *parameters of the model Eq.* (4)*—satisfy (as it were, a posteriori) the following* 4 *algebraic equations:*

$$\mu x_1(0) = \alpha k_R [x_3(0) + x_4(0)]\ , \tag{6a}$$

$$\mu x_2(0) = k_B x_1(0) + [k_B - f(0)] x_2(0) + [k_B + (1-\alpha) k_R] [x_3(0) + x_4(0)]\ , \tag{6b}$$

$$\mu x_3(0) = f(0) x_2(0) - (k_R + k_P) x_3(0)\ , \tag{6c}$$

$$\mu x_4(0) = k_P x_3(0) - (k_R + k_{DV}) x_4(0)\ , \tag{6d}$$

where of course (see Eq. (4e) *and Eq.* (5)*)*

$$f(x_1, x_2, x_3, x_4) \equiv f(0) = \frac{k_I [x_3(0) + \beta x_4(0)]}{x_1(0) + x_2(0) + x_3(0) + \beta x_4(0)}. \tag{6e}$$

The validity of Prop. 1 can be easily verified by inserting the solution Eq. (5) in the system Eq. (4) and by then taking advantage of the conditions Eq. (6).

Somewhat less trivial is to ascertain which are the *constraints* on the 4 initial data $x_n(0)$ and on the parameter μ—by solving the system of algebraic equations Eq. (6)—when we consider the model Eq. (4) for an *arbitrary* assignment of its 7 parameters $k_R, k_B, k_P, k_{DV}, k_I, \alpha, \beta$. Remarkably, as we show below, this turns out to be *explicitly* doable by purely *algebraic* operations.

Since all the five equations Eq. (6) are invariant under a *common* rescaling of the 4 initial data $x_n(0)$, it is convenient to assume that one of them, say $x_4(0)$, can be *arbitrarily assigned*, and to focus on the *ratios* of the other 3 to that one, hence on the 3 quantities

(7) $$r_m = x_m(0)/x_4(0) \ , \ x_m(0) = r_m x_4(0) \ , \ m = 1,2,3 \ ;$$

thereby replacing the 5 equations Eq. (6) with the following 5 equations:

(8a) $$\mu \; r_1 = \alpha k_R (r_3 + 1) \ ,$$

(8b) $$\mu \; r_2 = k_B r_1 + [k_B - F(r_1, r_2, r_3)] \, r_2 + [k_B + (1-\alpha) k_R] (r_3 + 1) \ ,$$

(8c) $$\mu \; r_3 = F(r_1, r_2, r_3) \, r_2 - (k_R + k_P) \, r_3 \ ,$$

(8d) $$\mu = k_P r_3 - (k_R + k_{DV}) \ ,$$

where of course (above and hereafter)

(8e) $$F(r_1, r_2, r_3) = k_I (r_3 + \beta) / (r_1 + r_2 + r_3 + \beta) \ .$$

It is now convenient, in order to get rid of the *nonlinear* function $F(r_1, r_2, r_3)$, to sum the 2 equations Eq. (8b) and Eq. (8c), getting thereby

(9) $$\mu \; (r_2 + r_3) = k_B + (1-\alpha) k_R + k_B r_1 + k_B r_2 + (k_B - k_P - \alpha k_R) \, r_3 \ .$$

The 3 equations Eq. (8a), Eq. (8d) and Eq. (9) constitute now a system of *3 linear algebraic* equations for the 3 unknowns r_1, r_2, r_3, which can be easily solved. Indeed from Eq. (8d) we get

(10a) $$r_3 = (\mu + k_{DV} + k_R)/k_P \ ;$$

then from Eq. (8a) and Eq. (10a) we get

(10b) $$r_1 = \alpha k_R (\mu + k_{DV} + k_P + k_R)/(\mu k_P) \ ;$$

and then from Eq. (9), Eq. (10a) and Eq. (10b) we get

(10c) $$r_2 = -\frac{\mu + k_{DV}}{k_P} - \frac{\mu - k_B + k_{DV}}{\mu - k_B} - \frac{[(1+\alpha)\,\mu + \alpha\,(k_{DV} + k_P)]\,k_R + \alpha\,(k_R)^2}{\mu k_P} \ .$$

Note that these are *explicit* expressions of the 3 parameters r_m in terms of the 6 parameters k_R, k_B, k_P, k_{DV}, k_I, α of the system Eq. (4), and moreover of the parameter μ featured by the solution Eq. (5) (where of course now $x_m(0) = r_m x_4(0)$ for $m = 1,2,3$, with $x_4(0)$ remaining as a *free* parameter).

Our remaining task in order to get the special solution Eq. (5) of the system Eq. (4) is to ascertain the permitted values of the parameter μ, as implied by inserting the following expression of $F(r_1, r_2, r_3)$ (obtained by inserting the 3 expressions Eq. (10) of r_1, r_2, r_3 in Eq. (8e)),

(11) $$F(r_1, r_2 r_3) = \frac{-k_I (\mu - k_B)(\mu + k_{DV} + \beta k_P + k_R)}{k_P [(1-\beta)(\mu - k_B) + k_{DV}]} \ ,$$

into any one of the 2 equations Eq. (8b) or Eq. (8c). This yields the following *algebraic* equation of degree 4 (hence *explicitly solvable*) for the quantity μ:

$$\sum_{k=0}^{4} \left(c_k \ \mu^k \right) = 0 \ , \tag{12a}$$

with the following definitions of the 5 parameters c_k:

$$c_4 = k_I - k_P + \beta k_P, \tag{12b}$$

$$\begin{aligned} c_3 = \ & 2k_{DV} k_I - 2k_{DV} k_P + k_I k_P - (k_P)^2 + 2k_I k_R - 2k_P k_R \\ & + \alpha k_I k_R - k_B \left[k_I - (1-\beta) k_P \right] + \beta k_P (k_{DV} + k_I + k_P + 2k_R) \ , \end{aligned} \tag{12c}$$

$$\begin{aligned} c_2 = \ & (k_I - k_P) \left[(k_{DV})^2 + k_R (k_P + k_R) \right] + \alpha k_I k_R (k_P + 2k_R) \\ & + \beta k_P \left[(k_I + k_R)(k_P + k_R) + \alpha k_I k_R \right] \\ & + k_{DV} \left\{ (2+\beta) \, k_I k_P + 2k_I \left(k_R + \alpha k_R \right) - k_P \left[(2-\beta) k_P + (3-\beta) k_R \right] \right\} \\ & + k_B \left\{ \ (1-\beta) k_P (k_P + 2k_R) + k_{DV} (-2k_I + k_P - \beta k_P) \right. \\ & \left. - k_I (k_P + 2k_R + \alpha k_R + \beta k_P) \ \right\} \ , \end{aligned} \tag{12d}$$

$$\begin{aligned} c_1 = \ & (k_{DV} + k_R) \left[k_{DV} k_P (k_I - k_P - k_R) + \alpha k_I k_R (k_{DV} + k_P + k_R) \right] \\ & + \beta k_I k_P \left[k_{DV} k_P + \alpha k_R (k_{DV} + k_P + k_R) \right] \\ & + k_B \left\{ \ - (k_{DV} + k_R) \left[k_{DV} k_I + (k_I - k_P)(k_P + k_R) \right] \right. \\ & - \alpha k_I k_R (2k_{DV} + k_P + 2k_R) \\ & \left. - \beta k_P \left[(k_I + k_R)(k_P + k_R) + k_{DV} (k_I + k_P + k_R) + \alpha k_I k_R \right] \ \right\} \ , \end{aligned} \tag{12e}$$

$$c_0 = -\alpha k_B k_I k_R \left(k_{DV} + k_P + k_R \right) \left(k_{DV} + k_R + \beta k_P \right) \ . \tag{12f}$$

Remark 2. *For completeness let us mention that the results reported just above require the validity of the following inequalities:*

$$\mu \neq 0 \ , \ k_P \neq 0 \ , \ (1-\beta) \left(k_B - \mu \right) - k_{DV} \neq 0. \tag{13}$$

Some of these expressions of the 5 parameters c_k, see Eq. (12), are rather cumbersome (albeit quite explicit), featuring the 7 *a priori arbitrary* parameters k_R, k_B, k_P, k_{DV}, k_I, α, β characterizing the system Eq. (4); and of course much more complicated are the—in principle easily available—*explicit* expressions of the 4 roots μ_n $(n = 1, 2, 3, 4)$ of the fourth-degree equation Eq. (12a). We do not consider useful to report these formulas in this paper; since analogous—much more practical—formulas can be easily obtained from eq. Eq. (12a) in *applicative* contexts, whenever the 7 *a priori arbitrary* parameters k_R, k_B, k_P, k_{DV}, k_I, α, β have been assigned specific numerical values, entailing, via the explicit expressions of the parameters c_k written above (see Eq. (12)), the corresponding numerical values of these parameters c_k, to be then inserted in Eq. (12a) before the standard task of solving this *quartic* equation is performed.

Remark 3. *Let us finally mention that clearly, by setting*

(14) $$\mu = k_D \ ,$$

one is looking (see Eq. (3a) and Eq. (5)) at the equilibrium solution

(15) $$\tilde{x}_n\left(t\right) = \bar{x}_n \ , \ \dot{\bar{x}}_n\left(t\right) = 0 \ , \ n = 1, 2, 3, 4 \ ,$$

of the original pandemic system Eq. (1), as given by the formulas (implied via Eq. (7) by Eq. (10))

(16a) $$\bar{x}_1 = \bar{x}_4 \left[\alpha k_R (k_D + k_{DV} + k_P + k_R)/(k_D k_P)\right] \ ,$$

(16b) $$\bar{x}_2 = \bar{x}_4 \left\{ -\frac{k_D + k_{DV}}{k_P} - \frac{k_D - k_B + k_{DV}}{k_D - k_B} \right.$$

(16c) $$\left. - \frac{\left[\left(1+\alpha\right) k_D + \alpha\left(k_{DV} + k_P\right)\right] k_R + \alpha\left(k_R\right)^2}{k_P k_D} \right\} \ ,$$

(16d) $$\bar{x}_3 = \bar{x}_4 (k_D + k_{DV} + k_R)/k_P \ ,$$

where $\bar{x}_4$ is of course an arbitrary parameter. □

Remark 4. *Note that throughout this paper we assume that the* 4 *roots μ_n of the quartic algebraic equation Eq. (12a) are all different among themselves.* □

To conclude Section 2, let us mention that the *special* solutions Eq. (5) are not very interesting in *applicative* contexts, since they imply that the 4 dependent variables $x_n\left(t\right)$ *all* evolve in the *same*, very *simple*, manner. But fortunately, as shown below, it is also possible to identify other *explicit* solutions of the system of nonlinear ODEs Eq. (4).

3. SOLUTIONS THAT ARE THE LINEAR COMBINATION OF 2 EXPONENTIALS

In this section we investigate the following class of solutions of the system of ODEs Eq. (4):

(17a) $$x_n\left(t\right) = a_{n1} \exp\left(\mu_1 t\right) + a_{n2} \exp\left(\mu_2 t\right) \ , \ n = 1, 2, 3, 4 \ ,$$

where μ_1 and μ_2 are 2 *different* roots of eq. (12); while corresponding values for the 8 time-independent parameters a_{n1} and a_{n2} are obtained below.

Remark 5. *Since there are* 4 *(assumedly different) solutions μ of the fourth-order algebraic eq. Eq. (12a), there are $(4 \cdot 3)/2 = 6$ different assignments of the pair of values μ_1, μ_2. Note moreover that, even if the* 7 *parameters k_R, k_B, k_P, k_{DV}, k_I, α, β of the system of ODEs (4) are all real numbers (as is of course the case in the pandemics case), the* 4 *solutions μ_n of the fourth-order algebraic eq. Eq. (12a) need not be real numbers; but if the* 7 *parameters of the system of ODEs Eq. (4) are all real numbers, then non-real solutions of the algebraic eq. Eq. (12a) must be present in complex conjugate pairs.* □

Remark 6. *Note that we are again assuming, throughout this Section 3, that the quantity $f\left(x_1, x_2, x_3, x_4\right)$ in the system Eq. (4) is time-independent, hence equal to its value at the initial time $t = 0$ (see (Eq. (6e))); although this property, which was obvious in the treatment of Section 2 (see Eq. (4e) and Eq. (5))—and which* *is essential to justify the existence of the subclass of solutions Eq. (17a)—is now instead far from obvious: indeed conditions for it to hold—involving the initial data of these solutions, and also* 1 *constraint on the parameters of the system Eq. (4)—shall have to be ascertained, see below.* □

The 4 equations Eq. (17a) involve of course the following 4 relations among the 8 parameters a_{n1} and a_{n2} and the 4 initial data $x_n(0)$:

(17b) $$x_n(0) = a_{n1} + a_{n2} \ , \ n = 1, 2, 3, 4 \ .$$

The assumption (see Remark 6) that the function $f(x_1, x_2, x_3, x_4)$ be *time-independent* implies that the system Eq. (4) is again a *linear* system of 4 ODEs with *time-independent* parameters; hence each of the 2 exponential functions in the right-hand side of the *ansatz* Eq. (17a) must satisfy (as it were, *separately*) the system Eq. (4). Therefore each of the 2 sets of 4 parameters a_{n1} and a_{n2} must satisfy the same requirements (see the 4 equations Eq. (6)) satisfied by the initial data $x_n(0)$ in the treatment of the previous Section 2; namely there must now hold the 8 relations

(18a) $$\mu_\ell a_{1\ell} = \alpha k_R (a_{3\ell} + a_{4\ell}) \ , \ \ell = 1, 2 \ ,$$

(18b) $$\mu_\ell a_{2\ell} = k_B a_{1\ell} + [k_B - f(0)] a_{2\ell} + [k_B + (1 - \alpha) k_R] [a_{3\ell} + a_{4\ell}] \ , \ \ell = 1, 2 \ ,$$

(18c) $$\mu_\ell a_{3\ell} = f(0) a_{2\ell} - (k_R + k_P) a_{3\ell} \ , \ \ell = 1, 2 \ ,$$

(18d) $$\mu_\ell a_{4\ell} = k_P a_{3\ell} - (k_R + k_{DV}) a_{4\ell} \ , \ \ell = 1, 2 \ ;$$

where of course we again set

(18e) $$f(x_1, x_2, x_3, x_4) \equiv f(0) = \frac{k_I [x_3(0) + \beta x_4(0)]}{x_1(0) + x_2(0) + x_3(0) + \beta x_4(0)} \ ;$$

but now with the 4 initial data $x_n(0)$ related to the 2 parameters a_{n1} and a_{n2} by the Eq. (17b).

We can therefore now proceed in close analogy to the treatment of the previous Section 2, introducing 6 parameters $b_{m\ell}$ via the following position:

(19) $$a_{m\ell} = b_{m\ell} a_{4\ell} \ , \ b_{m\ell} = a_{m\ell}/a_{4\ell} \ , \ m = 1, 2, 3, \ \ell = 1, 2 \ .$$

These 6 parameters $b_{m\ell}$ $(m = 1, 2, 3, \ \ell = 1, 2)$ are then *explicitly* expressed in terms of the parameters of the system Eq. (4) as follows (see Eq. (10)):

(20a) $$b_{1\ell} = \alpha k_R (\mu_\ell + k_{DV} + k_P + k_R) / (k_P \mu_\ell) \ , \ \ell = 1, 2 \ ,$$

(20b) $$b_{2\ell} = -\frac{\mu_\ell + k_{DV}}{k_p} - \frac{\mu_\ell - k_B + k_{DV}}{\mu_\ell - k_B} - \frac{[(1 + \alpha) \mu_\ell + \alpha (k_{DV} + k_P)] k_R + \alpha (k_R)^2}{k_P \mu_\ell} \ , \ \ell = 1, 2 \ ,$$

(20c) $$b_{3\ell} = (\mu_\ell + k_{DV} + k_R)/k_P \ , \ \ell = 1, 2 \ ;$$

with a_{41} and a_{42} remaining as 2 *free* parameters.

To complete the treatment of this case, it is necessary to identify the *constraints* on the parameters of the system Eq. (4) and on the parameters of the solution under present consideration, see Eq. (17a). These constraints are necessary in order to comply with the requirement—essential for our treatment—that the quantity $h(t)$, related to the quantity $f(x_1, x_2, x_3, x_4)$, see Eq. (4e), by the simple relation

$$f(x_1, x_2, x_3, x_4) = k_I h(t) \tag{21a}$$

—hence reading as follows,

$$h(t) = \frac{x_3(t) + \beta x_4(t)}{x_1(t) + x_2(t) + x_3(t) + \beta x_4(t)} \tag{21b}$$

—be *time-independent*: therefore given in terms of the initial data as follows:

$$h(t) = h(0) = \frac{x_3(0) + \beta x_4(0)}{x_1(0) + x_2(0) + x_3(0) + \beta x_4(0)} \; . \tag{21c}$$

To fulfill this task, we now note that the solutions $x_n(t)$ under consideration in this Section 3 are defined by the relations (Eq. (17a)), hence their insertion in the definition Eq. (21b) implies the following expression of $h(t)$:

$$h(t) = \frac{a_{31} + \beta a_{41} + (a_{32} + \beta a_{42}) \exp[(\mu_2 - \mu_1) t]}{a_{11} + a_{21} + a_{31} + \beta a_{41} + (a_{12} + a_{22} + a_{32} + \beta a_{42}) \exp[(\mu_2 - \mu_1) t]} \; . \tag{21d}$$

It is therefore easily seen that the requirement that this expression of $h(t)$ be *time-independent* implies that the 8 parameters $a_{n\ell}$ ($n = 1, 2, 3, 4$; $\ell = 1, 2$) satisfy the following *single constraint* on the 8 parameters $a_{n\ell}$:

$$(a_{12} + a_{22})(a_{31} + \beta a_{41}) - (a_{11} + a_{21})(a_{32} + \beta a_{42}) = 0 \; ; \tag{22}$$

entailing then that

$$h(t) = h(0) = \frac{a_{31} + \beta a_{41}}{a_{11} + a_{21} + a_{31} + \beta a_{41}} = \frac{a_{32} + \beta a_{42}}{a_{12} + a_{22} + a_{32} + \beta a_{42}} \; . \tag{23}$$

By inserting the formulas Eq. (19) in Eq. (22) we then get, for the 6 parameters $b_{m\ell}$ ($m = 1, 2, 3$; $\ell = 1, 2$), the following *single constraint*:

$$(b_{12} + b_{22})(b_{31} + \beta) - (b_{11} + b_{21})(b_{32} + \beta) = 0 \; ; \tag{24}$$

and, via Eq. (20), we finally get the following *single constraint* on the 7 parameters of the system Eq. (4) for the existence of the solution Eq. (17a):

$$\begin{aligned} &(1 - \beta)(k_B)^2 - k_B k_{DV} + k_{DV}(k_{DV} + \beta k_P + k_R + \mu_1) \\ &\quad + [k_{DV} + (1 - \beta)\mu_1]\mu_2 - (1 - \beta) k_B (\mu_1 + \mu_2) = 0 \; , \end{aligned} \tag{25a}$$

provided there hold the following inequalities:

$$k_P \neq 0 \; , \; k_B \neq \mu_1 \; , \;\; k_B \neq \mu_2 \; , \;\; \mu_1 \neq \mu_2 \; . \tag{25b}$$

Remark 7. *Let us recall that there are in general* 6 *different versions of the constraint Eq.* (25) *due to the* 6 *different possible selections of the* 2 *roots* μ_1 *and* μ_2 *(see Remark 5); and that the simplicity of this formula Eq.* (25) *as providing a constraint on the* 7 *parameters of the system Eq.* (4) *is somewhat misleading, due to the explicit but quite complicated dependence on these parameters of the solutions* μ_1 *and* μ_2 *of the fourth-degree algebraic equation Eq.* (12a)*. However, as already mentioned above, all these complicated relations (including those yielding the initial data of the class of solutions considered in this Section 3) become much more easily managed whenever any* 6 *of the* 7 *a priori arbitrary parameters featured by* *the system Eq.* (4) *are assigned specific numerical values, so that the remaining task left is to ascertain the values of the* 7*-th parameter implied by the constraint Eq.* (25) *(as well as those characterizing the initial data* $x_n(0)$ *of the class of solutions considered in this Section 3), thereby identifying the corresponding class of systems Eq.* (4) *featuring the simple explicit solutions of type Eq.* (17a)*.* □

4. Solutions which are the linear superposition of 3 or 4 exponentials

In this Section 4 we treat the subclass of solutions of the system (Eq. (4)) whose time-evolution is a *linear superposition* of 3 or 4 exponentials.

Remark 8. *Throughout this Section 4 we assume that the quantity* $f(x_1, x_2, x_3, x_4)$ *in Eq.* (4) *is time-independent. This property is far from obvious (as in Section 3: see for instance Remark 6). Indeed, conditions for it to hold—involving the initial data of these subclass of solutions, as well as constraints on the parameters of the system Eq.* (4)*—shall have to be ascertained, see below.* □

4.1. **Solutions which are the linear superposition of 3 exponentials.** Here we investigate the following class of solutions of the system of ODEs Eq. (4):

$$x_n(t) = a_{n1}\exp(\mu_1 t) + a_{n2}\exp(\mu_2 t) + a_{n3}\exp(\mu_3 t)\ ,\ n = 1, 2, 3, 4\ , \tag{26a}$$

where μ_1, μ_2, μ_3 are 3 different roots of the 4th-degree algebraic eq. Eq. (12a); while corresponding values for the 12 time-independent parameters $a_{n1,}$ a_{n2}, a_{n3} are obtained below.

Of course these formulas Eq. (26a) imply the following relations among the 4 initial data $x_n(0)$ and the 12 parameters a_{nj} ($n = 1, 2, 3, 4$; $j = 1, 2, 3$):

$$x_n(0) = a_{n1} + a_{n2} + a_{n3}\ ,\ n = 1, 2, 3, 4\ . \tag{26b}$$

Remark 9. *Clearly symbols such as* x_n, $\mu_{n\ell}$, $a_{n\ell}$ *need not have the same significance nor the same values when appearing on this paper. But of course the statements made in Remark 5 concerning the possibility that not all the* 4 *roots of the fourth-order algebraic eq. Eq.* (12a) *be real numbers are generally valid.* □

Remark 10. *Since there are* 4 *(assumedly different) solutions* μ *of the fourth-order algebraic eq. Eq.* (12a)*, there are* 4 *different selections—from the quartet of solutions* μ_n *of eq. Eq.* (12a)*—of the trio of values* μ_1, μ_2, μ_3 *in the ansatz Eq.* (26a)*.* □

Let us now proceed in close analogy to the treatment provided in Section 3. Again we assume that the quantity f in the right-hand sides of the ODEs Eq. (4b) and Eq. (4c) is a *time-independent* parameter, up to identifying below conditions on the parameters of the system Eq. (4) and on the initial data of the solution Eq. (26) under consideration which are *sufficient* to guarantee—as it were, *a posteriori*—that this be the case; hence that the system of ODEs Eq. (4) be equivalent to a system of 4 *linear* ODEs, featuring independent solutions $a\exp(\mu t)$ each depending exponentially on the independent variable t (which can therefore be added without loosing the property to satisfy the system of ODEs Eq. (4)).

We thus obtain—in analogy to the 8 relations Eq. (18)—the following 12 relations:

(27a) $$\mu_j a_{1j} = \alpha k_R \left(a_{3j} + a_{4j}\right) \ , \ j = 1,2,3 \ ,$$

(27b) $$\mu_j a_{2j} = k_B a_{1j} + \left[k_B - f\left(0\right)\right] a_{2j} + \left[k_B + \left(1-\alpha\right) k_R\right] \left(a_{3j} + a_{4j}\right) \ , \ j = 1,2,3 \ ,$$

(27c) $$\mu_j a_{3j} = f\left(0\right) a_{2j} - \left(k_R + k_P\right) a_{3j} \ , \ j = 1,2,3 \ ,$$

(27d) $$\mu_j a_{4j} = k_P a_{3j} - \left(k_R + k_{DV}\right) a_{4j} \ , \ j = 1,2,3 \ .$$

Next we set (in analogy to Eq. (19))

(28) $$a_{mj} = b_{mj} a_{4j} \ , \ b_{mj} = a_{mj}/a_{4j} \ , \ m = 1,2,3, \ j = 1,2,3 \ ,$$

getting thereby the following 9 relations:

(29a) $$b_{1j} = \alpha k_R (\mu_j + k_{DV} + k_P + k_R) / \left(k_P \mu_j\right) \ , \ j = 1,2,3 \ ,$$

(29b) $$b_{2j} = -\frac{\mu_j + k_{DV}}{k_p} - \frac{\mu_j - k_B + k_{DV}}{\mu_j - k_B} - \frac{\left[\left(1+\alpha\right)\mu_j + \alpha\left(k_{DV} + k_P\right)\right] k_R + \alpha \left(k_R\right)^2}{k_P \mu_j} \ , \ j = 1,2,3 \ ,$$

(29c) $$b_{3j} = (\mu_j + k_{DV} + k_R)/k_P \ , \ j = 1,2,3 \ ;$$

with a_{41}, a_{42}, a_{43} remaining as 3 *free* parameters.

We must now investigate the restrictions on the parameters a_{mj} implied by the requirement that the quantity $h\left(t\right)$ be *time-independent*. The analogous formula to Eq. (21d) now reads as follows:

(30a) $$h\left(t\right) = numh\left(t\right)/denh\left(t\right) \ ,$$

(30b) $$numh\left(t\right) = a_{31} + \beta a_{41} + \left(a_{32} + \beta a_{42}\right) \exp\left[\left(\mu_2 - \mu_1\right) t\right] + \left(a_{33} + \beta a_{43}\right) \exp\left[\left(\mu_3 - \mu_1\right) t\right] \ ,$$

(30c) $$denh\left(t\right) = a_{11} + a_{21} + a_{31} + \beta a_{41} + \left(a_{12} + a_{22} + a_{32} + \beta a_{42}\right) \exp\left[\left(\mu_2 - \mu_1\right) t\right] + \left(a_{13} + a_{23} + a_{33} + \beta a_{43}\right) \exp\left[\left(\mu_3 - \mu_1\right) t\right] \ .$$

It is then easily seen that the condition Eq. (22) is now replaced by the following 2 restrictions:

(31) $$\left(a_{11} + a_{21}\right)\left(a_{3k} + \beta a_{4k}\right) = \left(a_{1k} + a_{2k}\right)\left(a_{31} + \beta a_{41}\right) \ , \ k = 2,3 \ .$$

Hence, after the change of parameters Eq. (28), we get the following 2 *constraints,*

(32) $$\left(b_{11} + b_{21}\right)\left(b_{3k} + \beta\right) = \left(b_{1k} + b_{2k}\right)\left(b_{31} + \beta\right) \ , \ k = 2,3 \ ,$$

on the 9 parameters b_{mj} ($m = 1,2,3;\ j = 1,2,3$). Which, via the expressions Eq. (29) of these 9 parameters, entail the following 2 *constraints* on the original 7 parameters of the system Eq. (4):

(33a) $$\left(1-\beta\right)\left(k_B\right)^2 + \left(k_{DV}\right)^2 + \left(1-\beta\right)\mu_1\mu_2 + k_{DV}\left(\beta k_P + k_R + \mu_1 + \mu_2\right) - k_B\left[k_{DV} + \left(1-\beta\right)\left(\mu_1 + \mu_2\right)\right] = 0 \ ,$$

$$k_{DV}\left(k_{DV}+\beta k_P+k_R+\mu_1\right)+\left[k_{DV}+\left(1-\beta\right)\mu_1\right]\mu_3$$
$$-\left(1-\beta\right)k_B\left(\mu_1+\mu_3\right)+k_B\left[\left(1-\beta\right)k_B-k_{DV}\right]=0. \tag{33b}$$

While of course the initial data $x_n(0)$ of the solution under consideration in this Section 4.1 are explicitly given by the formulas Eq. (26b) with Eq. (28) and Eq. (29).

4.2. **Solutions which are the linear superposition of 4 exponentials.** The treatment in Section 4.2 is quite terse, since it is quite analogous to that provided above in Section 4.1; hence we only report the key formulas which play an analogous role to the key formulas in Section 4.1.

Instead of Eq. (26) we now have

$$x_n(t)=\sum_{q=1}^{4}\left[a_{nq}\exp\left(\mu_q t\right)\right]\ ,\ n=1,2,3,4\ , \tag{34a}$$

$$x_n(0)=\sum_{q=1}^{4}\left(a_{nq}\right)\ ,\ n=1,2,3,4\ . \tag{34b}$$

In place of the 12 equations Eq. (27) we now have the following 16 equations:

$$\mu_q a_{1q}=\alpha k_R\left(a_{3q}+a_{4q}\right)\ ,\ q=1,2,3,4\ , \tag{35a}$$

$$\mu_q a_{2q} = k_B a_{1q}+\left[k_B-f\left(0\right)\right]a_{2q}+\left[k_B+\left(1-\alpha\right)k_R\right]\left(a_{3q}+a_{4q}\right)\ ,$$
$$q = 1,2,3,4\ , \tag{35b}$$

$$\mu_q a_{3q}=f\left(0\right)a_{2q}-\left(k_R+k_P\right)a_{3q}\ ,\ q=1,2,3,4\ , \tag{35c}$$

$$\mu_q a_{4q}=k_P a_{3q}-\left(k_R+k_{DV}\right)a_{4q}\ ,\ q=1,2,3,4\ . \tag{35d}$$

Likewise, in place of the 9 equations Eq. (28), we now write the 12 relations

$$a_{jq}=b_{jq}a_{4j}\ ,\ b_{jq}=a_{jq}/a_{4j}\ ,\ j=1,2,3\ ,\ q=1,2,3,4\ , \tag{36}$$

getting thereby, from Eq. (35), the following 12 relations (analogous to Eq. (29)):

$$b_{1q}=\alpha k_R(\mu_q+k_{DV}+k_P+k_R)/\left(k_P\mu_q\right)\ ,\ q=1,2,3,4\ , \tag{37a}$$

$$b_{2q}=-\frac{\mu_q+k_{DV}}{k_p}-\frac{\mu_q-k_B+k_{DV}}{\mu_q-k_B}$$
$$-\frac{\left[\left(1+\alpha\right)\mu_q+\alpha\left(k_{DV}+k_P\right)\right]k_R+\alpha\left(k_R\right)^2}{k_P\mu_q}\ ,\ q=1,2,3,4\ , \tag{37b}$$

$$b_{3q}=(\mu_q+k_{DV}+k_R)/k_P\ ,\ q=1,2,3,4\ ; \tag{37c}$$

with a_{41}, a_{42}, a_{43}, a_{44} remaining as 4 *free* parameters.

Next come the *constraints* on the parameters of the system Eq. (4) needed in order that the more general solution Eq. (34a), when inserted in the definition Eq. (4e) of the function $f(t)$, hence now reading

$$f\left(t\right)=\frac{k_I\sum_{q=1}^{4}\left[\left(a_{3q}+\beta a_{4q}\right)\exp\left(\mu_q t\right)\right]}{\sum_{q=1}^{4}\left[\left(a_{1q}+a_{2q}+a_{3q}+\beta a_{4q}\right)\exp\left(\mu_q t\right)\right]}\ , \tag{38a}$$

—or, equivalently, see Eq. (36)—

$$f(t) = \frac{k_I \sum_{q=1}^{4} \left[(b_{3q} + 1) \exp(\mu_q t) \right]}{\sum_{q=1}^{4} \left[(b_{1q} + b_{2q} + b_{3q} + \beta) \exp(\mu_q t) \right]} \,, \tag{38b}$$

be *time-independent*.

And since it is easily seen that

$$f(t) = \frac{k_I (b_{31} + 1) \varphi(t)}{(b_{11} + b_{21} + b_{31} + \beta)} \,, \tag{39a}$$

with

$$\varphi(t) = \frac{1 + \sum_{q=2}^{4} \left\{ \left(\frac{b_{3q}+1}{b_{31}+1} \right) \exp\left[(\mu_q - \mu_1) t \right] \right\}}{1 + \sum_{q=2}^{4} \left\{ \left(\frac{b_{1q}+b_{2q}+b_{3q}+\beta}{b_{11}+b_{21}+b_{31}+\beta} \right) \exp\left[(\mu_q - \mu_1) t \right] \right\}} \,, \tag{39b}$$

the requirement that $f(t)$ be *time-independent* amounts to the following 3 constraints:

$$\frac{b_{3q} + 1}{b_{31} + 1} = \frac{b_{1q} + b_{2q} + b_{3q} + \beta}{b_{11} + b_{21} + b_{31} + \beta} \,, \; q = 2, 3, 4 \,, \tag{40}$$

which clearly entail—via the expressions Eq. (29)—3 corresponding *constraints* on the parameters of the original model Eq. (4).

5. CONCLUDING REMARKS

In this paper we have identified certain solutions of the *pandemic* model introduced in the paper [5]; these solutions and the constraints on the parameters of the model required for their validity, are all identified by *algebraic* equations which can be *explicitly* solved; we did not report the corresponding explicit formulas because they are so complicated to be hardly useful when written for *a priori arbitrary* assignments of the parameters of the pandemic model, while they can instead be easily managed for any *specific numerical* assignment of these parameters. We therefore leave the utilization of these findings to the interested pandemics experts.

Additional solutions, more special but perhaps displaying more interesting evolutions, correspond to the special cases in which the algebraic quartic-equation Eq. (12) features 4 roots μ_n which are *not* all different among themselves. This case shall be eventually treated in a separate paper by ourselves or others.

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