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ON POWERFUL VALUES OF POLYNOMIALS OVER NUMBER FIELDS

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ABSTRACT. Let $\mathcal{B} = \{b_i\}_{i=1}^{\infty}$ be a fixed sequence of pairwise distinct elements of a number field k. Given the integers $2 \leq s \leq r$, assuming a quantitative version of Vojta's conjecture on the bounded degree algebraic numbers over a number field k, we provide lower and upper bounds for the cardinal number of the set $\mathbf{G}_{r,s}^{\mathcal{B}_M}$ of polynomials $f \in k[x]$ of degree $r \geq 2$ whose irreducible factors have multiplicity strictly less than s, and $f(b_1), \cdots, f(b_M)$ are nonzero s-powerful elements in k, where $M = 2r^2 + 6r + 1$ if $r = s$, and $2sr^2 + sr + 1$ otherwise. Moreover, considering certain conditions on $\mathcal{B}$, we show the existence of an integer $M_0 > M$ such that no polynomial in $\mathbf{G}_{r,s}^{\mathcal{B}_M}$ takes s-powerful values at all $b_1, \cdots, b_n$ for $n \geq M_0$.

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1. Introduction and main results

Let k be a number field and denote by $\mathcal{O}_k$ the ring of integers in k. Given integer $s \geq 2$, an element α of $\mathcal{O}_k$ is called *s-powerful* if for each prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$ dividing the principle ideal (α) we have $\text{ord}_{\mathfrak{p}}(\alpha) \geq s$. This definition immediately extends to elements of k. Clearly, any s-power in k is an s-powerful element. Similarly, given a polynomial $f \in k[x]$ of degree $r \geq s$, we say that f is an *s-powerful polynomial* if each irreducible factor of f has multiplicity at least s. It is clear that any s-power polynomial in $k[x]$ is s-powerful.

The powerful values of polynomials have been studied by several authors in the literature, for example see [3], [11], [15], [19] and [10]. In the recent work [10], H. Pasten considered the problem for number fields as well as function fields. In the case of number fields, he showed that a certain conjecture by Vojta on Diophantine approximation for number field extensions of bounded degree, see Conjecture 3.3(i) in Sec. 3, implies the finiteness of the set of all monic polynomial $f \in k[x]$ of degree $r \geq 2$ whose irreducible factors have multiplicity strictly less than $s \leq r$ and all $f(b_1), \cdots f(b_M)$ are s-powerful for mutually distinct fixed elements $b_i \in k$, where k

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is a fixed number field and $\bar{M} = 2r^2 + 9r + 1$, if $r = s$; and $\bar{M} = 2sr^2 + (2s+1)r + 1$ otherwise, see [10, Thm. 2.1]. As a corollary, he concluded the existence of positive constants M_0 such that if $f(1), \cdots, f(M_0)$ are s-powerful for some polynomial $f \in k[x]$, then it has a factor with multiplicity at least s, see [10, Cor. 2.2]. He also achieved some consequences in logic related to Hilbert's tenth problem, such as the Büchi's n-square problem.

In this paper, we assume the quantitative version of the Vojta's conjecture on algebraic points of bounded degree over number field, see Conjecture 3.3(ii) in Section 3. We fix an arbitrary sequence $\mathcal{B} = \{b_1, b_2, \cdots\}$ of pairwise distinct elements in k. Given integers $2 \le s \le r < n$, we let $\mathbf{F}_{r,s}^{\mathcal{B}_n}$ to be the set of all monic polynomials $f \in k[x]$ of degree r such that $f(b_i)$ is a nonzero s-powerful element in k for each $b_i \in \mathcal{B}_n$, where $\mathcal{B}_n = \{b_1, \cdots, b_n\} \subset \mathcal{B}$. Denote by $\mathbf{G}_{r,s}^{\mathcal{B}_n}$ the subset of $\mathbf{F}_{r,s}^{\mathcal{B}_n}$ containing polynomials for which all irreducible factors have multiplicity strictly less than s. It is clear that $\mathbf{F}_{r,s}^{\mathcal{B}_{n+1}} \subseteq \mathbf{F}_{r,s}^{\mathcal{B}_n}$ and hence $\mathbf{G}_{r,s}^{\mathcal{B}_{n+1}} \subseteq \mathbf{G}_{r,s}^{\mathcal{B}_n}$ for all $n \ge 1$. One can think about $\mathbf{G}_{r,s}^{\mathcal{B}_n}$ as the set of exceptions to the rule: *if f takes s-powerful values too many times then f has factors with exponents $\ge s$.*

The main results of this paper are as follows.

Theorem 1.1. *Assume the Vojta's Conjecture 3.3(ii). Given integers $2 \le s \le r$, let $M := 2r^2 + 6r + 1$ if $r = s$, and $2sr^2 + sr + 1$ otherwise. Then $\mathbf{G}_{r,s}^{\mathcal{B}_M}$ is a finite set. Moreover, there exist positive constants C_0 and C_1, depending on r, s, fundamental quantities of k and elements of $\mathcal{B}_M$, but it is independent of the polynomial $f \in \mathbf{G}_{r,s}^{\mathcal{B}_M}$, such that*

$$C_0 \le \#\mathbf{G}_{r,s}^{\mathcal{B}_M} \le C_1.$$

We remark that the number M given by this theorem depends only on the integers r an s, but it is independent of the sequence $\mathcal{B}$ and its subset $\mathcal{B}_M$. In contrast, the proof of Theorem 1.1, which is similar to [10, Thm. 2.1], shows that C_0 and C_1 depend on $b_i \in \mathcal{B}_M$, the integers r, s, and basic quantities of the number field k. We remark that the number M in Theorem 1.1 is smaller than the number $\bar{M}$ given in [10, Thm. 2.1]. For, we used a different estimation of the height of logarithmic discriminate of k. Our result also provides a lower and an upper bound for the set under study, which is not provided in [10].

Now, let us consider sequences $\mathcal{C} = \{c_i\}_{i=1}^{\infty}$, $\mathcal{D} = \{d_i\}_{i=1}^{\infty}$, and $\mathcal{E} = \{e_i\}_{i=1}^{\infty}$, associated to the given fixed sequence $\mathcal{B} = \{b_i\}_{i=1}^{\infty}$, which are defined by

$$c_i := b_{i+1} - b_i,\ d_i := \frac{b_i}{b_{i+1}},\ e_i := \frac{1}{b_{i+1}} - \frac{1}{b_i},\ (i = 1, 2, \cdots).$$

As a consequence of the Theorem 1.1, following result shows that if we restrict our attention to the sequences $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ in Theorem 1.1, then we can get rid of the set $\mathbf{G}_{r,s}^{\mathcal{B}_n}$ for a large enough n.

Theorem 1.2. *Assume the Vojta's Conjecture 3.3(ii). Let M denote the integer given by Theorem 1.1. If any of the sequences $\mathcal{C}, \mathcal{D}$, and $\mathcal{E}$ is periodic with period $m \ge 1$, then there exists an integer $M_0 > 0$, depending on r, s, fundamental quantities of k and elements of $\mathcal{B}_M$, such that $\mathbf{G}_{r,s}^{\mathcal{B}_n} = \emptyset$ for $n > M_0$.*

Since all the sequences $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are periodic with period $m = 1$ for the sequence $\mathcal{B} = \{1, 2, \cdots\}$, so Theorem 1.2 implies [10, Cor. 2.2].

The structure of this paper is as follows. In Sec. 2, we give the preliminaries on height functions over number fields and also the main result of [5] that will use in proof of 1.1. In Sec. 3, we provide some definitions and terminologies of Diophantine approximation to ba able for stating the equivalent versions of Vojta's conjecture on bounded degree algebraic points over number fields. The section contains proofs of Theorems 1.1 and 1.2.

2. Preliminaries on heights functions

Given a number field k with algebraic closure $\bar{k}$, we let $\mathcal{P}_k$ denote the set of places of k that splits into two disjoint subsets. One, $\mathcal{P}_k^0$ the set of the finite places, *i.e.*, those corresponding to prime ideals $\mathfrak{p}$ of $\mathcal{O}_k$, and another one of the infinite places denoted by $\mathcal{P}_k^\infty$, *i.e.*, those corresponding to real embedding $\sigma : k \hookrightarrow \mathbb{R}$, called the real infinite places, union with those corresponding to pair of conjugate embedding $\sigma, \bar{\sigma} : k \hookrightarrow \mathbb{C}$ that are called the complex infinite places. For any $v \in \mathcal{P}_k$, denote by $\|\cdot\|_v$ its associated almost absolute value defined by

$$\|\alpha\|_v := \begin{cases} 0 & \text{if } \alpha = 0\ , \\ \#(\frac{\mathcal{O}_k}{\mathfrak{p}_v})^{\operatorname{ord}_{\mathfrak{p}_v}(\alpha)} & \text{if } v \in \mathcal{P}_k^0 \text{ corresponds to a prime } \mathfrak{p}_v \in Spec(\mathcal{O}_k), \\ |\sigma(\alpha)| & \text{if } v \in \mathcal{P}_k^\infty \text{ is a real infinite place,} \\ |\sigma(\alpha)|^2 & \text{if } v \in \mathcal{P}_k^\infty \text{ is a complex infinite place.} \end{cases}$$

Given $P = [\alpha_0 : \cdots : \alpha_n] \in \mathbb{P}_k^n$, the *multiplicative* and *logarithmic* heights are defined by

$$H_k(P) := \prod_{v \in \mathcal{P}_k} \max\{\|\alpha_0\|_v, \cdots, \|\alpha_n\|_v\},$$

$$h_k(P) := \log H_k(P) = \sum_{v \in \mathcal{P}_k} \log \max\{\|\alpha_0\|_v, \cdots, \|\alpha_n\|_v\}.$$

Then, the *multiplicative* and *logarithmic heights* of any $\alpha \in k$ are defined by $H_k(\alpha) = H_k([1 : \alpha])$ and $h_k(\alpha) h_k([1 : \alpha])$, where we identified k with affine space in $\mathbb{P}_k^1$. For any finite extension $K|k$, $\alpha \in k$, and $P \in \mathbb{P}_k^n$, one has

$$H_k(\alpha) = H_K(\alpha)^{1/[K:k]},\ h_k(\alpha) = \frac{1}{[K:k]} h_K(\alpha),$$

$$H_k(P) = H_K(P)^{1/[K:k]},\ h_k(P) = \frac{1}{[K:k]} h_K(P).$$

Considering these facts, one may extend the definition of height function to $\mathbb{P}_{\bar{k}}^n$. In this case, they are called the *absolute multiplicative and additive Weil heights* and denoted by $H(P)$ and $h(P)$, respectively. We note that the action of Galois group of $\mathbb{P}_{\bar{k}}^n$ leaves the absolute heights invariant. Moreover, for each $\alpha, \beta \in \bar{k}^*$ and $n \in \mathbb{Z}$, one has the followings:

$$h(\alpha^n) = |n| h(\alpha),\ h(\alpha\beta) \leq h(\alpha) + h(\beta),\ h(\alpha + \beta) \leq h(\alpha) + h(\beta) + \log 2. \tag{2.1}$$

For any polynomial $f(x) = a_0 + a_1 x + \cdots + a_d x^d \in k[x]$, the *absolute multiplicative* and *additive heights* are defined by

$$H(f) := H([a_0 : a_1 : \cdots : a_d]),\quad h(f) := h([a_0 : a_1 : \cdots : a_d].$$

Giving lower and upper bounds for the cardinal number of the set of bounded degree algebraic points on a projective line is started by [13] and continued by [14], [7], [5], and so on.

Let us recall the main result in [5] that we will use in the proof of Theorem 1.1. Denote by $N(\mathbb{P}_{\bar{k}}^1; r; T)$ the number of points $\alpha \in \mathbb{P}_{\bar{k}}^1$ of degree at most r and

$h(\alpha) \leq T$ for every constant $T > 0$ and integer $r \geq 2$. Let Cl_k be the class number of k, Reg_k the regulator of $\mathcal{O}_k^*$, w_k the number of roots of unity in k, $\zeta_k(s)$ the Dedekind zeta-function of k, $\mathfrak{d}_k$ the absolute value of discriminant of k, m_1 the number of real embedding of k, m_2 the number of pairs of complex embedding of k, and $m = m_1 + 2m_2$ is the degree of k over $\mathbb{Q}$. For more details on these quantities, we refer the reader to [8]. Define

$$a_{k,r} := \frac{Cl_k \cdot \text{Reg}_k}{w_k \zeta_k(r+1)} \cdot \left(\frac{2^{m_1}(2\pi)^{m_2}}{\sqrt{\mathfrak{d}_k}}\right)^{r+1} \cdot (r+1)^{m_1+m_2-1}, \tag{2.2}$$

and denote $b_{k,r} := r \cdot a_{k,r} \cdot T^{mr(r+1)}$ and $T_1 = T^{mr(r+1)-r}$. The proof of following theorem can be found in [5, Thm. 1.0.1].

Theorem 2.1. *Notation being as above, for each $\varepsilon > 0$ one has*

$$b_{k,r} \cdot 2^{-mr(r+1)} - O_\varepsilon(T_1 \cdot T^\varepsilon) \leq N(\mathbb{P}^1_{\bar{k}}; r; T) \leq b_{k,r} \cdot 2^{mr(r+1)} + O(T_1).$$

In particular,

$$2^{-mr(r+1)} + o(1) \leq \frac{N(\mathbb{P}^1_{\bar{k}}; r; T)}{b_{k,r}} \leq 2^{mr(r+1)} + o(1) \quad \text{as} \quad T \to \infty.$$

Without loos of generality, we may suppose that $k \subset \mathbb{C}$ and

$$f(x) = a_0 + a_1 x + \cdots + a_d x^d = a_d \prod_{j=1}^{d} (x - \alpha_j) \in \mathbb{C}[x]. \tag{2.3}$$

In this case, the *Mahler measure* of any $f \in \mathbb{C}[x]$ is defined by

$$M(f) := |a_d| \cdot \prod_{j=1}^{d} \max\{1, |\alpha_j|\},$$

where $|\cdot|$ is the usual absolute value on $\mathbb{C}$. For $\alpha \in \bar{k} = \mathbb{C}$, its Mahler measure is given by $M(\alpha) = M(f_\alpha)$ where $f_\alpha \in k[x]$ denotes its minimal polynomial.

The *logarithmic discriminant* of k is defined by $d_k := \log \mathfrak{d}_k / [k : \mathbb{Q}]$. For a tower of number fields $\mathbb{Q} \subseteq k \subseteq K \subset \bar{k}$ with absolute discriminant $\mathfrak{d}_k$ and $\mathfrak{d}_K$, respectively, the *relative logarithmic discriminant* of $K|k$ is

$$d_k(K) := \frac{1}{[K : k]} \log \mathfrak{d}_{K/k} - \log \mathfrak{d}_k,$$

where $\mathfrak{d}_{K/k}$ is the *relative discriminant* of the extension $K|k$. The *relative logarithmic discriminant* of each $\alpha \in \bar{k}$ is defined by $d_k(\alpha) := d_k(k(\alpha))$. The following proposition gives an upper bound for the logarithmic discriminant $d_k(\alpha)$ that we will use in the proof of 1.1. For a proof, we cite to [1,6].

Proposition 2.2. *Let $k \subset \mathbb{C}$ be a number field and $f \in k[x]$ be a polynomial of degree $d \geq 2$ of the form*

$$f(x) = a_0 + a_1 x + \cdots + a_d x^d = a_d \prod_{j=1}^{d} (x - \alpha_j) \in \mathbb{C}[x].$$

Define $A(d) = d \log d$ if $k = \mathbb{Q}$, and $A(d) = (2d-1)\log d$ otherwise. Then,

(i) *$D(f) = a_d^{2d-2} \prod_{i>j} (\alpha_i - \alpha_j)^2$, and $|D(f)| \leq d^d \cdot M(f)^{2d-2}$;*
(ii) *If $D(f) \neq 0$, then $h(D(f)) \leq 2(d-1)h(f) + A(d)$;*
(iii) *If $\alpha \in \bar{k}$ is of degree $d \geq 2$, then $d_k(\alpha) \leq 2(d-1)h(\alpha) + A(d)$.*

Proof. See [6, Thm. 1] for part (i). The part (ii) is consequence of part (i) for the case $k = \mathbb{Q}$, and it is given by [1, Lem. 3.7] when $k \neq \mathbb{Q}$. The part (iii) is given by [2, Prop. 1.6.9] in the case $k = \mathbb{Q}$; and generally it comes from part (ii). □

3. Vojta's conjecture on bounded degree algebraic points

In this section, we briefly review the basic definitions and results on Diophantine approximation over number fields. For more details, one can refer to [16, 17]. Then, we state the equivalent versions of well-known Vojta's conjecture on bounded degree algebraic points over number fields.

Given a finite set $S \subset \mathcal{P}_k$ containing $\mathcal{P}_k^\infty$, and the distinct elements $b, \alpha \in k$, the *proximity functions* with respect to S are defined by

$$m_S(\alpha) := \sum_{v \in S} \log^+ \|\alpha\|_v, \text{ and } m_S(b, \alpha) := m_S\left(\frac{1}{\alpha - b}\right).$$

Similarly, the *counting functions* with respect to the set S are defined by

$$N_S(\alpha) := \sum_{v \notin S} \log^+ \|\alpha\|_v, \text{ and } N_S(b, \alpha) := N_S\left(\frac{1}{\alpha - b}\right).$$

By the properties of logarithm function, for any $\alpha \in k$ one has

$$m_S(\alpha) + N_S(\alpha) = \sum_{v \in \mathcal{P}_k} \log^+ \|\alpha\|_v = h(\alpha), \tag{3.1}$$

which is an analogue of first main theorem in classic Value Distribution Theory (or equivalently Nevanlinna Theory). The proximity and counting function of any $\alpha \in \bar{k} \backslash k$ are defined as

$$m_S(\alpha) := \frac{1}{[K : k]} \cdot m_T(\alpha), \text{ and } N_S(\alpha) := \frac{1}{[K : k]} \cdot N_T(\alpha),$$

where K is any finite extension of k containing $k(\alpha)$. These definitions are independent of the choice of the extension K. For an element $b \in k(\alpha)$ distinct from α, one can also define

$$m_S(b, \alpha) := \frac{1}{[k(\alpha) : k]} \cdot m_T(b, \alpha), \text{ and } N_S(b, \alpha) := \frac{1}{[k(\alpha) : k]} \cdot N_T(b, \alpha).$$

It is easy to see that $h(\alpha) = m_S(\alpha) + N_S(\alpha)$ for all $\alpha \in \bar{k}$.

Here is the Vojta's conjecture on algebraic points of bounded degree over number fields, see [16, 17] for more general version.

Conjecture 3.1. *Let k be a number field, $\bar{k}$ its algebraic closure and $S \subset \mathcal{P}_k$ a finite set containing $\mathcal{P}_k^\infty$. Let $b_1, \cdots, b_n$ be pairwise distinct elements of k and $d \geq 2$ an integer. Then, one has the following equivalent statements:*

(i) *For any $\epsilon > 0$, there exists a constant c_ϵ depending on ϵ and previous data, such that the inequality*

$$\sum_{i=1}^{n} m_S(b_i, \alpha) \leq (2 + \epsilon) h(\alpha) + d_k(\alpha) + c_\epsilon,$$

holds for all $\alpha \in \bar{k}$ with $[k(\alpha) : k] \leq d$ and different from all b_i's.

(ii) *For any $\epsilon > 0$, the inequality*

$$\sum_{i=1}^{n} m_S(b_i, \alpha) < (2+\epsilon)h(\alpha) + d_k(\alpha),$$

holds for all but finitely many $\alpha \in \bar{k}$ with $[k(\alpha):k] \leq d$ and different from all b_i's.

Using the qualities 3 and following proof 3.1 as in [16], one can see that the inequality

$$h(\alpha) \leq m_S(b,\alpha) + N_S(b,\alpha) + h(b) + [k(\alpha):\mathbb{Q}] \cdot \log 2, \tag{3.2}$$

holds for any $\alpha \in \bar{k}$ and $b \in k$ distinct from α. Applying this inequality, the Conjecture 3.1 can be restated as follows.

Conjecture 3.2. *Let k be a number field, $\bar{k}$ its algebraic closure and $S \subset \mathcal{P}_k$ a finite set containing $\mathcal{P}_k^{\infty}$. Let $b_1, \cdots, b_n$ be pairwise distinct elements of k and $d \geq 2$ an integer. Then, one has the following equivalent statements:*

(i) *For any $\epsilon > 0$, there exists a constant c_ϵ depending on ϵ and previous data, such that the inequality*

$$(n-2-\epsilon)h(\alpha) \leq d_k(\alpha) + \sum_{i=1}^{n} N_S(b_i,\alpha) + c_\epsilon,$$

holds for all $\alpha \in \bar{k}$ with $[k(\alpha):k] \leq d$ and different from all b_i's.

(ii) *For any $\epsilon > 0$, the inequality*

$$(n-2-\epsilon)h(\alpha) < d_k(\alpha) + \sum_{i=1}^{n} N_S(b_i,\alpha),$$

holds for all but finitely many $\alpha \in \bar{k}$ with $[k(\alpha):k] \leq d$ and different from b_i's.

The *truncated counting function* on $\bar{k}$ is defined by

$$N_S^{(1)}(b,\alpha) := \sum_{w \in M_K^0} \min\{1, \max\{0, \mathrm{ord}_{\mathfrak{p}_w}(\alpha - b)\}\} \cdot \log\left(\# \frac{\mathcal{O}_K}{\mathfrak{p}_w}\right),$$

where $b \in k$ is distinct from $\alpha \in \bar{k}$, $K \supseteq k(\alpha)$ and $\mathfrak{p}_w \subset \mathcal{O}_K$ is a prime ideal corresponding to $w \in M_K^0$ that lies over some $v \in M_k \backslash S$. Here is the truncated version of Vojta's conjecture.

Conjecture 3.3. *Let k be a number field, $\bar{k}$ its algebraic closure and $S \subset \mathcal{P}_k$ a finite set containing $\mathcal{P}_k^{\infty}$. Let $b_1, \cdots, b_n$ be pairwise distinct elements of k and $d \geq 2$ an integer. Then, one has the following equivalent statements:*

(i) *For any $\epsilon > 0$, there exists a constant c_ϵ depending on ϵ and previous data, such that the inequality*

$$(n-2-\epsilon)h(\alpha) \leq d_k(\alpha) + \sum_{i=1}^{n} N_S^{(1)}(b_i,\alpha) + c_\epsilon,$$

holds for all $\alpha \in \bar{k}$ with $[k(\alpha):k] \leq d$ and different from all b_i's.

(ii) *For any $\epsilon > 0$, the inequality*

$$(n - 2 - \epsilon)h(\alpha) < d_k(\alpha) + \sum_{i=1}^{n} N_S^{(1)}(b_i, \alpha),$$

holds for all but finitely many $\alpha \in \bar{k}$ with $[k(\alpha) : k] \leq d$ and different from b_i's.

We notice that the Conjecture 3.3 is a special case of a Vojta's general conjecture on the bounded degree algebraic points on algebraic varieties, see [18, Conj. 25.1]. It is equivalent to the non-truncated version 3.2. Indeed, since $N_S^{(1)}(b, \alpha) \leq N_S(b, \alpha)$ holds by definitions, so the truncated version 3.3 implies the non-truncated one 3.2. The converse is the special case of [18, Thm 3.1], where a more general ABC conjecture is stated.

It is remarkable that the finite sets of elements in $\bar{k}$ of degree at most d for which the inequalities of part (ii) in Conjectures 3.1 3.2 and 3.3 do not hold, depends on b_i's, ϵ, c, k, and d. In practice, determining effectively this finite set of elements is very hard task.

4. Proof of the main results

In this section, we give the proof of Theorems 1.1 and 1.2.

4.1. **Proof of Theorem 1.1.** Suppose that $2 \leq s \leq r$ are integers and $b_1, \cdots, b_M \in k$ are distinct elements, where $M = 2r^2 + 6r + 1$ and $M = 2sr^2 + sr + 1$, otherwise. Then, consider the subset $\mathcal{B}_M := \{b_1, b_2, \cdots, b_M\}$ of the sequence $\mathcal{B} = \{b_1, b_2, \cdots\} \subset k$. Let $f \in \mathbf{G}_{r,s}^{\mathcal{B}_M}$ with factorization $f = f_1^{s_1} \cdots f_t^{s_t}$, where $f_j \in k[x]$ are monic irreducible polynomial of degree $d_j := \deg(f_j)$ and define $s^+ = \max\{s_1, \cdots, s_t\}$. For each $j = 1, \cdots, t$, let $\alpha_j \in \bar{k}$ be an arbitrary root of f_j, $k_j := k(\alpha_j)$ and $g := f_1 \cdots f_t$, which is of degree $d := d_1 + \cdots + d_t$. Let $S \subset \mathcal{P}_k$ be a finite subset of $\mathcal{P}_k$ that is the union of the sets $\mathcal{P}_k^{\infty}$, poles of each $b_i \in \mathcal{B}_M$, and the places above which two or more of b_j's meet. Note that $\alpha_j \neq b_i$ for all i and j, since $f(b_j) \neq 0$.

By Vojta's conjecture 3.3(ii) with $b_i \in \mathcal{B}_M$, the set S, and integer $r \geq 2$, we conclude that for any given $\epsilon > 0$ the following inequality

$$(M - 2 - \epsilon)h(\alpha) < d_k(\alpha) + \sum_{i=1}^{M} N_S^{(1)}(b_i, \alpha) \tag{4.1}$$

holds for all but finitely many $\alpha \in \bar{k}$ with $[k(\alpha) : k] \leq r$ and $\alpha \neq b_i$'s. Let us denote by $N_{k,r}^{\mathcal{B}_M}$ the set of such elements $\alpha \in \bar{k}$ for which (4.1) does not hold, and denote its cardinal number by $n_{k,r}^{\mathcal{B}_M}$. Since we are going to estimate $\#\mathbf{G}_{r,s}^{\mathcal{B}_M}$, so for a while we ignore the polynomials $f \in \mathbf{G}_{r,s}^{\mathcal{B}_M}$ that have some roots in the set $N_{k,r}^{\mathcal{B}_M}$. We recall them in the moment of estimating $\#\mathbf{G}_{r,s}^{\mathcal{B}_M}$.

Thus, assuming Vojta's conjecture 3.3(ii), for any $\epsilon > 0$

$$(M - 2 - \epsilon)h(\alpha_j) < d_k(\alpha_j) + \sum_{i=1}^{M} N_S^{(1)}(b_i, \alpha_j), \tag{4.2}$$

where α_j is a root of f_j for each $j = 1, \cdots, t$. Applying Proposition 2.2(iii) to each of α_j's and using $d_j \leq d$, leads to

$$d_k(\alpha_j) \leq 2(d_j - 1)h(\alpha_j) + A(d_j) \leq 2(d-1)h(\alpha_j) + A(d), \tag{4.3}$$

where $A(d) = d \log d$ if $k = \mathbb{Q}$ and $(2d-1)\log d$ otherwise, for any integer $d \leq r$. Substituting (4.3) in (4.2), and using the fact that $A(d) \leq A(r) \leq 2r \log r$ leads to

$$(M - 2d - \epsilon)h(\alpha_j) < \sum_{i=1}^{M} N_S^{(1)}(b_i, \alpha_j) + c_1, \tag{4.4}$$

where $c_1 := M(B + r \cdot \log 2) + 2r \log r$. Then, multiplying the both side with d_j and summing-up, one can obtain that

$$\sum_{j=1}^{t}(M - 2d - \epsilon)h_{k_j}(\alpha_j) < \sum_{j=1}^{t}\sum_{i=1}^{M} d_j N_S^{(1)}(b_i, \alpha_j) + rc_1. \tag{4.5}$$

We are going to give an upper bound for the term involving truncated function in (4.5). But to do this, we need the following lemma.

Lemma 4.1. *Let $D(g)$ be the discriminant of polynomial $g = f_1 \cdots f_t$ of degree $d \geq 2$ and let $A(d)$ be as above. Then*

$$h(D(g)) \leq 2(d-1)\sum_{j=1}^{t} h_{k_j}(\alpha_j) + 4d(d-1) + A(d).$$

Proof. We assume that α_{ji} are the roots of f_j for $1 \leq i \leq d_j$. Since the absolute heights are invariant by action of Galois group of $\mathbb{P}^1_{\bar{k}}$, so

$$\begin{aligned}\sum_{j=1}^{t} h(f_j) &\leq \sum_{j=1}^{t}\left(\sum_{i=1}^{d_j} h(\alpha_{ji}) + (d_j - 1)\log 2\right) \\ &\leq \sum_{j=1}^{t} d_j h(\alpha_j) + \sum_{j=1}^{t}(d_j - 1)\log 2 \\ &\leq \sum_{j=1}^{t} h_{k_j}(\alpha_j) + (d - t)\log 2.\end{aligned}$$

Hence, using the properties of heights functions [4, Prop. B.7.2], we have

$$\begin{aligned}h(g) = h(f_1 \cdots f_t) &\leq \sum_{j=1}^{t}\left[h(f_j) + (d_j + 1)\log(2)\right] \\ &= \sum_{i=1}^{t} h(f_j) + (d + t)\log 2 \leq \sum_{j=1}^{t} h_{k_j}(\alpha_j) + 2d \log 2.\end{aligned}$$

By Proposition 2.2(ii), we obtain the desired inequality,

$$h(D(g)) \leq 2(d-1)h(g) + A(d) \leq 2(d-1)\sum_{j=1}^{t} h_{k_j}(\alpha_j) + 4d(d-1)\log 2 + A(d).$$

□

Let $\mathcal{D}$ be the reduced divisor on $Spec(\mathcal{O}_k)$ whose support consists of the union of S, zeros of $D(g)$, and poles of α_j's.

Lemma 4.2. *With notation as above, we have:*

$$\sum_{j=1}^{t}\sum_{i=1}^{M} d_j N_S^{(1)}(b_i,\alpha_j) \leq \left(\frac{Ms^+}{s} + d(2d-1)\right)\sum_{j=1}^{t} h_{k_j}(\alpha_j) + rc_2, \tag{4.6}$$

where

$$c_2 := M(B+\log 2) + \sum_{\mathfrak{p}\in S} \log \#(\mathcal{O}_k/\mathfrak{p}) + 2r\log r + A(r).$$

Proof. By changing the order of sums in left-hand side of (4.5) and following the last part of the proof of [10, Lemma 4.9], we have

$$\begin{aligned}\sum_{i=1}^{M}\sum_{j=1}^{t} d_j N_S^{(1)}(b_i,\alpha_j) &\leq \frac{1}{s}\sum_{i=1}^{M}\sum_{j=1}^{t} s_j d_j h(b_i-\alpha_j) + d\deg(\mathcal{D}) \\ &\leq \frac{1}{s}\sum_{i=1}^{M}\left(\sum_{j=1}^{t} s_j d_j [h(b_i)+h(\alpha_j)+\log 2]\right) + d\deg(\mathcal{D}).\end{aligned}$$

Since $t \leq r = \sum_{j=1}^{t} s_j d_j$, and $s_j \leq s^+$, so we have

$$\begin{aligned}\sum_{i=1}^{M}\sum_{j=1}^{t} d_j N_S^{(1)}(b_i,\alpha_j) &\leq \frac{1}{s}\sum_{i=1}^{M}\left(\sum_{j=1}^{t} s_j d_j h(\alpha_j) + r(h(b_i)+\log 2)\right) + d\deg(\mathcal{D}) \\ &\leq \frac{M}{s}\sum_{j=1}^{t} s_j h_{k_j}(\alpha_j) + \frac{Mr(B+\log 2)}{s} + d\deg(\mathcal{D}) \\ &\leq \frac{Ms^+}{s}\sum_{j=1}^{t} h_{k_j}(\alpha_j) + Mr(B+\log 2) + d\deg(\mathcal{D}).\end{aligned}$$

To give an upper bound on the $\deg(\mathcal{D})$ in terms of $h(\alpha_j)$'s, we assume that S' and S_j are the subsets of $\mathcal{P}_k^0$ such that $D(g)$ vanished at $\mathfrak{p}$, α_j has a pole above $\mathfrak{p}$, respectively. We let S'' to be the union of S_j for $j = 1, \cdots, t$. Then, letting $a(S) := \sum_{\mathfrak{p}\in S} \log \#(\mathcal{O}_k/\mathfrak{p})$, we have

$$\begin{aligned}\deg(\mathcal{D}) &= \sum_{\mathfrak{p}\in S''} \log \#(\mathcal{O}_k/\mathfrak{p}) + \sum_{\mathfrak{p}\in S'} \log \#(\mathcal{O}_k/\mathfrak{p}) + a(S) \\ &= \sum_{j=1}^{t}\sum_{\mathfrak{p}\in S_j} \log \#(\mathcal{O}_k/\mathfrak{p}) + \#S' + a(S) \\ &= \sum_{j=1}^{t} h_{k_j}(\alpha_j) + h(D(g)) + a(S).\end{aligned}$$

Using (4.1), and $A(d) \leq A(r)$ for $d \leq r$, we get that

$$\deg(\mathcal{D}) \leq \sum_{j=1}^{t} h_{k_j}(\alpha_j) + 2(d-1)\sum_{j=1}^{t} h_{k_j}(\alpha_j) + a(S) + A(r) + 4r(r-1)$$
$$\leq (2d-1)\sum_{j=1}^{t} h_{k_j}(\alpha_j) + a(S) + A(r) + 4r(r-1).$$

Multiplying the last inequality by d, gives that

$$d\deg(\mathcal{D}) \leq d(2d-1)\sum_{j=1}^{t} h_{k_j}(\alpha_j) + r[a(S) + A(r) + 4r(r-1)].$$

Putting all of the above inequalities together leads to desired one (4.6). □

By Lemma 4.2, one can rewrite (4.5) as follows,

$$\sum_{j=1}^{t}\left(M(1-\frac{s^+}{s}) - 2d^2 - d - \epsilon\right) h_{k_j}(\alpha_j) < r(c_1+c_2). \tag{4.7}$$

Lemma 4.3. *For integers* $2 \leq s \leq r$, *let* $M = 2r^2+6r+1$ *if* $r = s$ *and* $2sr^2+sr+1$ *otherwise. Then, for each* $1 \leq d \leq r$, *we have*

$$M\left(1-\frac{s^+}{s}\right) - 2d^2 - d \geq \frac{1}{r}. \tag{4.8}$$

Proof. First, we have $r - s^+ \geq d-1$. Indeed, if j_0 is an index such that $s_{j_0} = s^+$, then

$$r = \sum_{j=1}^{t} s_j d_j \geq s^+ d_{j_0} + \sum_{j\neq j_0}^{t} d_j \geq s^+ + d_{j_0} - 1 + \sum_{j\neq j_0}^{t} d_j = s^+ + d - 1.$$

Thus,

$$1 - \frac{s^+}{s} \geq \begin{cases} \dfrac{d-1}{r} & \text{if } s = r \\ \dfrac{1}{s} & \text{otherwise.} \end{cases} \tag{4.9}$$

In the case $s = r$, since $M = 2r^2+6r+1$ and $r - s^+ \geq d-1 \geq 1$, so

$$M\left(1-\frac{s^+}{s}\right) - 2d^2 - d \geq M\left(\frac{d-1}{r}\right) - 2d^2 - d$$
$$\geq \frac{d-1}{r}\left(M - \frac{2rd^2+rd}{d-1}\right)$$
$$\geq \frac{d-1}{r}\left(M - 2rd - 3r - \frac{3r}{d-1}\right).$$

The facts $-3r/(d-1) \geq -3r$ and $(d-1)/r \geq 1/r$ for $d-1 \geq 1$, implies that

$$M\left(1-\frac{s^+}{s}\right) - 2d^2 - d \geq \frac{d-1}{r}(M - 2rd - 6r)$$
$$\geq \frac{1}{r}(M - 2r^2 - 6r) \geq \frac{1}{r}.$$

In the case $s < r$, by $M = 2sr^2 + sr + 1$ and $1 - s^+/s \geq 1/s$ we have:

$$\begin{aligned} M\left(1 - \frac{s^+}{s}\right) - 2d^2 - d &\geq M/s - 2d^2 - d \\ &\geq \frac{1}{s}(M - 2sd^2 - sd) \\ &\geq \frac{1}{r}(M - 2sr^2 - sr) \geq \frac{1}{r}. \end{aligned}$$

□

Now, using (4.8) in either cases, the inequality (4.7) can be rewritten as follows,

$$\left(\frac{1}{r} - \epsilon\right) \sum_{j=1}^{t} h_{k_j}(\alpha_j) < r(c_1 + c_2). \tag{4.10}$$

Then, taking $\epsilon := \frac{1}{r+1}$, and $c_3 := r^2(r+1)(c_1 + c_2)$, the inequality (4.10) implies that

$$h(\alpha_j) < d_j h(\alpha_j) = h_{k_j}(\alpha_j) \leq \sum_{j=1}^{t} h_{k_j}(\alpha_j) < c_3.$$

We note that the consonant c_3 depends only on r, s, k and $b_1, \cdots, b_M$, but it is independent of the polynomial $f \in \mathbf{G}_{r,s}^{\mathcal{B}_M}$. Let $N(\mathbb{P}^1_{\bar{k}}; r; c_3)$ be the set of algebraic numbers $\alpha \in \bar{k}$ of degree at most r and height at most c_3, and denote by $n(\mathbb{P}^1_{\bar{k}}; r; c_3)$ its cardinal number. By the famous Northcott's theorem [9], $N(\mathbb{P}^1_{\bar{k}}; r; c_3)$ is a positive number. Letting $c_4 := c_3^{mr(r+1)-r}$ with $m = [k : \mathbb{Q}]$, and applying Theorem 2.1 assuming $\varepsilon := 1$, $T := c_3$ and $T_1 := c_4$, gives us two constants $c_5, c_6 > 0$, depending on r, s and k but not on $b_i \in \mathcal{B}_M$, such that

$$b_{k,r} \cdot 2^{-mr(r+1)} + c_5 \cdot c_4 \cdot c_3 \leq N(\mathbb{P}^1_{\bar{k}}; r; c_3) \leq b_{k,r} \cdot 2^{mr(r+1)} + c_6 \cdot c_4,$$

where $b_{k,r} = r \cdot a_{k,r} \cdot c_3^{mr(r+1)}$ and $a_{k,r}$ is given by (2.2). Let $\mathcal{A}_{k,r}^{\mathcal{B}_M}$ be the union of $N(\mathbb{P}^1_{\bar{k}}; r; c_3)$ and $N_{k,r}^{\mathcal{B}_M}$ defined in the beginning of the proof. Then

$$b_{k,r} \cdot 2^{-mr(r+1)} + c_5 \cdot c_4 \cdot c_3 \leq \#\mathcal{A}_{k,r}^{\mathcal{B}_M} \leq b_{k,r} \cdot 2^{mr(r+1)} + c_6 \cdot c_4 + n_{k,r}^{\mathcal{B}_M}.$$

Since for each $f \in \mathbf{G}_{r,s}^{\mathcal{B}_M}$ has at most r distinct roots in $\mathcal{A}_{k,r}^{\mathcal{B}_M}$, so we conclude that

$$b_{k,r} \cdot 2^{-mr(r+1)} + c_5 \cdot c_4 \cdot c_3 \leq \#\mathbf{G}_{r,s}^{\mathcal{B}_M} \leq r \cdot (b_{k,r} \cdot 2^{mr(r+1)} + c_6 \cdot c_4 + n_{k,r}^{\mathcal{B}_M}).$$

Therefore, we obtain the desired lower and upper bounds for $\#\mathbf{G}_{r,s}^{\mathcal{B}_M}$, i.e.,

$$C_0 \leq \#\mathbf{G}_{r,s}^{\mathcal{B}_M} \leq C_1,$$

where

$$C_0 := b_{k,r} \cdot 2^{-mr(r+1)} + c_5 \cdot c_4 \cdot c_3 + n_{k,r}^{\mathcal{B}_M},$$

and

$$C_1 := r \cdot (b_{k,r} \cdot 2^{mr(r+1)} + c_6 \cdot c_4 + n_{k,r}^{\mathcal{B}_M}).$$

This completes the proof of Theorem 1.1.

4.2. **Proof of Theorem 1.2.** To prove Theorem 1.2, we start with the following lemma.

Lemma 4.4. *If any of $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ is periodic with period $t \geq 1$, then for each $\ell = qt + p$ with $q \geq 0$ and $1 \leq p \leq t$, we have:*

(i) $c_\ell = c_p$ *implies that* $b_\ell = b_p + q(b_{t+1} - b_1)$; *in particular,*

$$b_{qt} = b_t + (q-1)(b_{t+1} - b_1);$$

(ii) $d_\ell = d_p$ *implies that*

$$b_{\ell+1} = b_{p+1} \left(\frac{b_{t+1}}{b_1}\right)^q; \text{ in particular, } b_{qt} = b_t \left(\frac{b_{t+1}}{b_1}\right)^{q-1};$$

(iii) $e_\ell = e_p$ *implies that*

$$\frac{1}{b_\ell} = \frac{1}{b_p} + q\left(\frac{1}{b_{t+1}} - \frac{1}{b_1}\right);$$

in particular,

$$\frac{1}{b_{qt}} = \frac{1}{b_t} + (q-1)\left(\frac{1}{b_{t+1}} - \frac{1}{b_1}\right).$$

Proof. We prove just the part (i) of this lemma by induction on q, and leave the other cases to the reader. We assume that $\ell = t + p$ and $1 \leq p \leq t$. Then, we have

$$\begin{aligned} b_{t+p} - b_{t+1} &= c_{t+1} + c_{t+2} + \cdots + c_{t+p-1} \\ &= c_1 + c_2 + \cdots + c_{p-1} = b_p - b_1. \end{aligned}$$

Hence, the assertion is true in the case $q = 1$, i.e., $b_{t+p} - b_p = b_{t+1} - b_1$. We assume that it is true for $\ell = qt + p$ with $1 \leq p \leq t$. Since

$$\begin{aligned} b_{(q+1)t+p} - b_{qt+p} &= c_{qt+p} + \cdots + c_{(q+1)t} + c_{(q+1)t+1} + \cdots + c_{(q+1)t+p-1} \\ &= c_p + c_{p+1} + \cdots + c_t + c_1 + \cdots + c_{p-1} = b_{t+1} - b_1, \end{aligned}$$

so $b_{(q+1)t+p} = b_{qt+p} + b_{t+1} - b_1$. By the hypothesis of the induction, we have $b_{qt+p} = b_p + q(b_{t+1} - b_1)$ and hence $b_{(q+1)t+p} - b_p = (q+1)(b_{t+1} - b_1)$ as desired. Clearly, the general case implies the particular one. □

Now, we prove Theorem 1.2 for the sequence $\mathbb{C}$, and leave the details of proof in other cases for readers. Given integers $2 \leq s \leq r$, let M be the integer given by Theorem 1.1. Let $N_0 \# \mathbf{G}_{r,s}^{\mathcal{B}_M}$, and assume that $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are periodic. To prove Theorem 1.2, we show that $\mathbf{G}_{r,s}^{\mathcal{B}_n}$ is empty set for each $n \geq M_0$, where

$$M_0 := \begin{cases} t \cdot (N_0 + M + 2) & \text{if } t \leq M; \\ t \cdot (N_0 + 2) & \text{if } t > M. \end{cases}$$

By contrary, we assume that $\mathbf{G}_{r,s}^{\mathcal{B}_n} \neq \emptyset$ and $f \in \mathbf{G}_{r,s}^{\mathcal{B}_n}$ for some $n \geq M_0$. We prove that there exists $N_0 + 1$ pairwise distinct polynomials $f_1, \cdots, f_{N_0}, f_{N_0+1}$ in $\mathbf{G}_{r,s}^{\mathcal{B}_M}$. This gives a contradiction and complete the proof of the corollary. Let us assume that $\mathcal{C}$ is periodic with period $t \geq 1$. For each $1 \leq j \leq N_0 + 1$, the following polynomials

$$f_j(x) := f(x + j(b_{t+1} - b_1)),$$

are distinct elements of $\mathbf{G}_{r,s}^{\mathcal{B}_M}$. To see this, it is enough to check that $f_j(b_i)$ is an s-powerful element in k, for $1 \leq i \leq M$ and $1 \leq j \leq N_0+1$. If $t > M$, then by definition of f_j's and part (i) of above lemma, we have

$$f_j(b_i) = f(b_i + j(b_{t+1} - b_1)) = f(b_{jt+i}).$$

Since $jt + i \leq (N_0+1)t + M \leq (N_0+2)t \leq n$ and $f \in \mathbf{G}_{r,s}^{\mathcal{B}_n}$, so $f(b_{jt+i})$ and hence $f_j(b_i)$ is an s-powerful element in k. In the case $t \leq M$, for each $1 \leq i \leq M$ we write $i = i_1 t + i_2$ for some $0 \leq i_1 \leq M$ and $0 \leq i_2 \leq t$. Then, by part (i) of above lemma, we have

$$\begin{aligned} f_j(b_i) &= f(b_i + j(b_{t+1} - b_1)) \\ &= f(b_{i_2} + i_1(b_{t+1} - b_1) + j(b_{t+1} - b_1)) \\ &= f(b_{i_2} + (i_1 + j)(b_{t+1} - b_1)) = f(b_{(i_1+j)t+i_2}). \end{aligned}$$

Since $(i_1 + j)t + i_2 \leq (N_0 + M + 1)t + t = (N_0 + M + 2)t \leq n$ and $f \in \mathbf{G}_{r,s}^{\mathcal{B}_n}$, so $f(b_{(i_1+j)t+i_2})$ and hence $f_j(b_i)$ is an s-powerful element in k^*.

In the cases of $\mathcal{D}$ or $\mathcal{E}$, one can get result by similar arguments. Indeed, for each $f \in \mathbf{G}_{r,s}^{\mathcal{B}_n}$ it is enough to consider respectively the following polynomials

$$f_j(x) := f(ux), \text{ with } u = \left(\frac{b_{t+1}}{b_1}\right)^{(j-1)}, \text{ for } 1 \leq j \leq N_0 + 1,$$

$$f_j(x) := v^r f\left(\frac{1}{v}\right), v = \frac{1}{x} + j\left(\frac{1}{b_{t+1}} - \frac{1}{b_1}\right), \quad \text{for } 1 \leq j \leq N_0 + 1.$$

Note that given any polynomial $f \in k[x]$ of degree $r \geq 2$ and any $c \in k$, the function

$$g(x) = w^r f\left(\frac{1}{w}\right), \text{ with } w = \frac{1}{x} + c,$$

is a polynomial of degree r, too.

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