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TRIANGULATIONS OF UNORIENTABLE SURFACES

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Abstract. We consider the question of when an unorientable surface without boundary can be tiled by an (l, m, n) triangle. We desire that the automorphism group of the surface preserving the tiling be "substantial". We use the well known theory of quasiplatonic surfaces and symmetries of Riemann surfaces to propose a classification algorithm, entirely by finite group calculations. A comprehensive analysis of symmetries of quasiplatonic surfaces in low genus, by hand and by computer, is carried out, yielding triangulations of unoriented surfaces as a byproduct.

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1. Introduction

Let M be a closed, unorientable surface (Klein surface) which can be tiled by (l, m, n) triangles. We also desire that the group of automorphisms of M preserving the tiling be "substantial". We wish to determine all such surfaces, or at least a lot of them. Our idea will be to analyse the surface S which is the double (orientation) covering of M. We focus our attention on certain *quasiplatonic surfaces* (deferring definitions to later sections):

- The surface S has a triangular tiling by congruent (l, m, n) triangles.
- There is a finite, *triangular action group* G, acting by conformal automorphisms of S, that acts transitively on the tiles with the same orientation.
- There is a fixed point free symmetry ψ of S that preserves the tiling on S and normalizes the action of G.

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The tiling on S and the existence of ψ are completely determined by a *generating triple* (a, b, c) of G and the group structures of G and $\mathrm{Aut}(G)$. Working backwards, we will seek orientable surfaces S and symmetries ψ, satisfying the given conditions, and take $M = S/\langle\psi\rangle$. Here is an overview of the paper. The basics of surfaces with triangular tilings, the action group G, and the generating triples (a, b, c) are discussed in Section 2. Symmetries of S and their properties are discussed in Section 3. At the end of Section 3 we give the details of the program of finding triangulations of unorientable surfaces from surfaces satisfying the bullets above.

In the next two sections, we give a rich set of examples that can be computed by hand. In Section 4 we focus on the abelian and cyclic cases. In Section 5 we discuss in detail those triangular surfaces whose action group is a non-abelian group of order pq, where p and q are primes. These groups are small groups, amenable to hand calculation, but with more subtle properties than abelian groups. These groups provide interesting examples and counterexamples, for instance these surfaces have no Type I symmetries but do have Type II symmetries. In Section 6, for completeness, we discuss and give a few examples of surfaces with "extra" symmetries. Finally, in Section 7, we use the ideas and results of the first three sections to catalogue low genus, symmetric, quasiplatonic surfaces, which in turn can be used to produce unorientable surfaces with triangulations. We completed a computer search for symmetric, quasiplatonic surfaces with action group of order less or equal to 250, and have summarized the results in various tables in Section 7. The main take away is that there a numerous triangulated unorientable surfaces. The Magma code and detailed information the actions of each group are at this website [5].

2. Triangular surfaces and group actions

2.1. **Tilings by triangles.** Let (l, m, n) be a triple of integers all greater than or equal to 2. For each such triple there is a triangle Δ in one of the geometries $\widehat{\mathbb{C}}$, the Riemann sphere, $\mathbb{C}$, the complex plane, or $\mathbb{H}$, the hyperbolic plane, satisfying the following requirements.

- The triangle Δ is an (l, m, n) triangle, i.e., it has interior angles π/l, π/m, and π/n, in counterclockwise order.
- The triangle Δ is unique up to conformal isometry. In the Euclidean case, $\mathbb{C}$, this helps us avoid the ambiguity of similar triangles. In the other two geometries any two (l, m, n) triangles are automatically congruent.

Now pick a distinguished such triangle Δ_0, which we shall call the master tile (see Figure 1). Reflections in the sides of Δ_0 produce three anti-conformal isometric images of Δ_0 (tiles), joined along the sides of Δ_0. The first stage of this reflection process is illustrated in Figure 1. If we repeat the reflection process with the new triangles, and continue ad infinitum, we get a tiling $\mathcal{T}$ of $\widehat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{H}$ by (l, m, n) triangles, according to the Poincaré polygon theorem (for background, see [1]). In Figure 2, a partial diagram of a $(4, 4, 3)$ tiling of $\mathbb{H}$ is shown, using the unit disc model.

Remark 2.1. The diagram in Figure 1 is actually the image in the surface S, but is isometric to the corresponding diagram in the universal cover U. Seeking simplification, the notation for the geometric objects: vertices, edges (sides), tiles, and

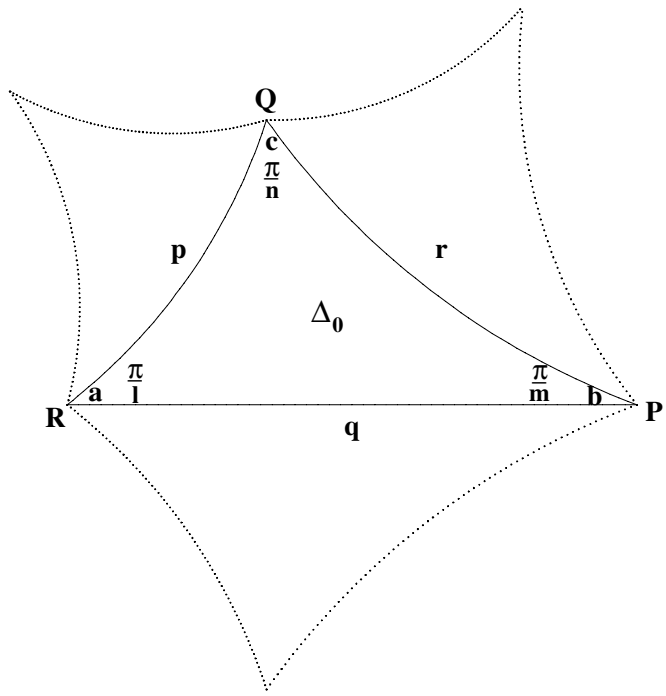

Figure 1. Mastertile and reflected tiles in S

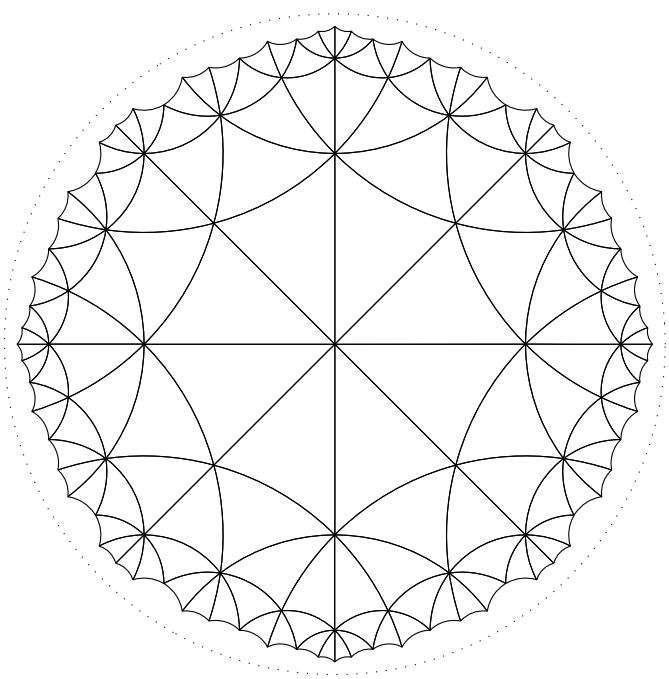

Figure 2. (4,4,3) tiling

angle measure can be the same in both U and S. However, the group information, introduced below, $A, B, C \in \mathrm{Aut}(U)$ and $a, b, c \in \mathrm{Aut}(S)$ must be distinguished.

2.2. **Algebra of tilings and triangular actions.** As in Figure 1, let P, Q, R denote the vertices of Δ_0, picking a specific counterclockwise ordering. (The constructions we shall make are dependent on the choice of the triangle, see the later paragraph on resolving ambiguity.) Let p, q, r denote the sides opposite to P, Q, R, respecting the ordering. We also assume that the interior angles at R, P, Q have radian measure π/l, π/m, and π/n, respecting order. Finally, we also let p, q and r denote the reflections

ψ_p, ψ_q, ψ_r in the sides p, q and r, respectively, and define these elements in $\mathrm{Aut}(U)$:

$$A = pq,\ B = qr,\ C = rp. \tag{1}$$

From the geometry we see that p, q, r have order 2, and A, B, C are counterclockwise rotations centred at R, P, and Q, respectively, and have orders l, m, n, respectively. The full and orientation-preserving isometry groups preserving the tiling $\mathcal{T}$ are called triangle groups and are denoted by $T^*_{l,m,n} = \langle p, q, r\rangle$ (orientation-preserving or not) and $T_{l,m,n} = \langle A, B, C\rangle$ (orientation preserving only). It is well known that these isometry groups, called *triangle groups*, have these presentations:

$$T^*_{l,m,n} = \left\langle p, q, r : p^2 = q^2 = r^2 = (pq)^l = (qr)^m = (rp)^n = 1\right\rangle \tag{2}$$

$$T_{l,m,n} = \left\langle A, B, C : A^l = B^m = C^n = ABC = 1\right\rangle \tag{3}$$

Remark 2.2. Later, we shall use the following conjugation formulas:

$$\begin{aligned} &pAp = A^{-1},\ pBp = AB^{-1}A^{-1},\ pCp = C^{-1},\\ &qAq = A^{-1},\ qBq = B^{-1},\ qCq = BC^{-1}B^{-1},\\ &rAr = CA^{-1}C^{-1},\ rBr = B^{-1},\ rCr = C^{-1}. \end{aligned}$$

Here are the details for p :

$$\begin{aligned} &pAp = ppqp = qp = A^{-1},\\ &pBp = pqrp = pqrqqp = AB^{-1}A^{-1},\\ &pCp = prpp = rp = C^{-1}. \end{aligned}$$

Triangular group actions (quasiplatonic). A closed surface S may be uniformized by a fixed point free group of conformal isometries Π, acting upon its universal covering space U, which is one of $\widehat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{H}$. We denote the universal cover by

$$\pi_S : U \to S. \tag{4}$$

If Π is a torsion-free subgroup of finite index in $T_{l,m,n}$, then the closed surface

$$S = U/\Pi. \tag{5}$$

will inherit a tiling $\overline{\mathcal{T}} = \mathcal{T}/\Pi$ by (l, m, n) triangles. If Π is normal then the finite group

$$G_S = T_{l,m,n}/\Pi, \tag{6}$$

acts naturally upon S as a group of conformal automorphisms that preserves the tiling $\overline{\mathcal{T}}$. We consider G_S as a subgroup of $\mathrm{Aut}(S)$, and call it the *rotation group,* as it is generated by local rotations.

In order to abstract the properties of G_S to an arbitrary isomorphic group G we call any monomorphism

$$\epsilon : G \to \mathrm{Aut}(S), \tag{7}$$

with $\epsilon(G) = G_S$, a *triangular G action on S*. We call G the *action group*, and (S, ϵ) an *action pair*. If there is no confusion we will identify $g \in G$ with $\epsilon(g) \in G_S$. The terms *quasi-platonic surface* and *quasiplatonic action* are also widely used. There has been an extensive enumeration of quasiplatonic surfaces, see for example [2] [13].

The triple (l, m, n) is called the *signature* of the action. The size of the group G, the genus σ, and the signature are related by the Riemann Hurwitz equation

$$\frac{2\sigma - 2}{|G|} = 1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}, \tag{8}$$

at least when the genus is not equal to 1.

Let $\eta : T_{l,m,n} \to G$ be the map, with kernel Π,

$$\eta : T_{l,m,n} \to T_{l,m,n}/\Pi \overset{\epsilon^{-1}}{\to} G, \tag{9}$$

called a *surface-kernel epimorphism.* Define

$$a = \eta(A), b = \eta(B), c = \eta(C). \tag{10}$$

Then the triple (a, b, c) satisfies

$$G = \langle a, b, c \rangle \tag{11}$$

$$a^l = b^m = c^n = abc = 1, \tag{12}$$

and we call (a, b, c) an (l, m, n) *generating triple*. The elements a, b, c define rotations of order l, m, n at R, P, Q in S (again see Figure 1).

Remark 2.3. Given any generating (l, m, n)-triple (a, b, c) in G there is an epimorphism $\eta : T_{l,m,n} \to G$, defined as in (10) with torsion free kernel $\Pi = \ker(\eta)$. Using the subgroup Π we may construct the surface $S = U/\Pi$, an (l, m, n) tiling $\overline{\mathcal{T}}$, and G-action on S defined by (10) and the action of $T_{l,m,n}/\Pi$ on U/Π. Therefore, triangular surfaces with triangular G actions may be studied using generating triples. By analogy, we shall also call $(S, (a, b, c))$ and action pair.

Resolving ambiguities. The construction of the objects $T_{l,m,n}$, Π, S, η, the G action ϵ, and the signature (l, m, n) has some ambiguity arising from the choice of the master tile. Suppose Δ_0' is a new, positively oriented (l, m, n) master tile. As previously mentioned, there is a unique isometry φ of $\mathbb{H}$ which maps Δ_0 to Δ_0', preserving the ordering of the vertices and hence (l, m, n). All of the new constructs will be denoted by a prime applied to the old construct. In particular, $T_{l,m,n} = \langle A, B, C \rangle$ and $T'_{l,m,n} = \langle A', B', C' \rangle$, where $A' = \varphi A \varphi^{-1}$, etc. The relationship between the new groups and surfaces and the old ones is summarized in these two commuting diagrams:

$$\begin{array}{ccccc} \Pi & \hookrightarrow & T_{l,m,n} & \overset{\eta}{\twoheadrightarrow} & G \\ \downarrow Ad_\varphi & & \downarrow Ad_\varphi & & \downarrow id \\ \Pi' & \hookrightarrow & T'_{l,m,n} & \overset{\eta'}{\twoheadrightarrow} & G \end{array}$$

$$\begin{array}{ccccc} \mathbb{H} & \twoheadrightarrow & \mathbb{H}/\Pi & = & S \\ \downarrow \varphi & & \downarrow \overline{\varphi} & & \downarrow \overline{\varphi} \\ \mathbb{H} & \twoheadrightarrow & \mathbb{H}/\Pi' & = & S' \end{array}$$

where

$$Ad_\varphi : L \to \varphi L \varphi^{-1},\ \overline{\varphi}(\Pi z) = \Pi' \varphi(z),$$

and we may compatibly define

$$\eta' = \eta \circ Ad_\varphi^{-1},\ \epsilon'(g) = \overline{\varphi}\epsilon(g)\overline{\varphi}^{-1}.$$

As $\eta'(A') = \eta(A)$, etc., then the generating triples are the same.

Next suppose that relabel the vertices with a cyclic permutation,

$$(A', B', C') = (B, C, A),\ (a', b', c') = (b, c, a),\ (l', m', n') = (m, n, l).$$

Note that we still have $A'B'C' = BCA = A^{-1}(ABC)A = 1$, and similarly $a'b'c' = 1$. Then $T'_{l,m,n} = T_{l',m',n'} = T_{l,m,n}$, $\Pi' = \Pi$, and $S' = S$. As maps, $\eta' = \eta$ and $\epsilon' = \epsilon$ and $G_{S'} = G_S$. Everything is the same at the surface level except for permuted signatures and generating triples. The other cyclic permutation works in the same way. The transposition $(l', m', n') = (m, l, n)$ corresponds to choosing, the lower tile in Figure 1 and these replacements

$$(A', B', C') = (B, A, BCB^{-1}),\ (a', b', c') = (b, c, bcb^{-1}),\ (l', m', n') = (m, l, n).$$

Again we get the same surface and the same G_S with permuted signature and suitably modified generating triple.
Further remarks. We finish this section with some further remarks that we use later in the paper.

Remark 2.4. If ω is an automorphism of G then $\omega \circ \eta$ has the same kernel as η in (10). Thus, the triples (a, b, c) and $\omega \cdot (a, b, c) = (\omega(a), \omega(b), \omega(c))$ define the same kernel and the same tiled surface with G-action twisted by ω. The corresponding action monomorphisms are ϵ and $\epsilon \circ \omega^{-1}$, and we say that the actions are *algebraically equivalent*. So, we need only consider $\mathrm{Aut}(G)$ classes of generating triples, to construct the different surfaces. Observe that $\mathrm{Aut}(G)$ acts freely on generating triples. Though the number of triples for a given signature may be very large, the actual number of $\mathrm{Aut}(G)$ classes of generating triples is typically small.

Remark 2.5. The group G may not be the full automorphism group of S. Indeed, the full automorphism group is N/Π where $N = \mathrm{Nor}(\mathrm{Aut}(U), \Pi)$. However, N is a triangle group. See Singerman [19].

Remark 2.6. Let e be edge in the tiling $\overline{\mathcal{T}}$ and $\widetilde{e}$ an edge in $\mathcal{T}$ lying over e. The reflection $\psi_{\widetilde{e}} \in T^*_{l,m,n}$ descends to a globally defined reflection ψ_e on S if and only if $\Pi \lhd T^*_{l,m,n}$. The reflection ψ_e is locally defined, by which we mean that the ψ_e is defined at least for the two tiles that meet along the edge e. In the universal cover, for any $L \in T_{l,m,n}$ we have $L\psi_{\widetilde{e}}L^{-1} = \psi_{L\widetilde{e}}$. On the surface, for any $g \in G$ or isometry ϕ we have

$$g\psi_e g^{-1} = \psi_{ge} \tag{13}$$

$$\phi\psi_e\phi^{-1} = \psi_{\phi e} \tag{14}$$

at least locally, and if ψ_e is globally defined then the equation holds globally.

2.3. **Geometry of S/G and $U/T_{l,m,n}$ as $\widehat{\mathbb{C}}$.** To take advantage of the normalizing condition $\psi G \psi = G$ we will utilize a convenient model for S/G and the quotient map

$$\pi_G : S \to S/G. \tag{15}$$

By assumption S/G is a sphere and the quotient map π_G has three branch points on S/G, namely the images of the vertices of the triangles on S. For a model of S/G, we use the Riemann sphere $\widehat{\mathbb{C}}$ with branch points at $0, 1, \infty$. Our model of S/G is easily visualized as the complex plane, i.e., $\widehat{\mathbb{C}} - \{\infty\}$, with $0, 1, \infty$ lying on the projective real line $\widehat{\mathbb{R}}$. The points $0, 1, \infty$ break up $\widehat{\mathbb{R}}$ into three closed intervals $[\infty, 0]$, $[0, 1]$, $[1, \infty] \subset \widehat{\mathbb{R}}$ and $\widehat{\mathbb{R}}$ breaks up $\widehat{\mathbb{C}}$ into H^+ and H^- the upper half plane and lower half planes. This satem really is a tiling $\overline{\overline{\mathcal{T}}}$ on $\widehat{\mathbb{C}}$.

The quotient spaces S/G and $U/T_{l,m,n}$ are isomorphic. Indeed, using the sequence (9) and the covering map (4) we have $\pi(g \cdot z) = \eta(g) \cdot \pi(z)$ for all $g \in T_{l,m,n}$. It follows that the bijection $S/G \leftrightarrow U/T_{l,m,n}$ defined by the orbit equation $G \cdot \pi(z) = \pi(T_{l,m,n} \cdot z)$ is a biholomorphic map of quotient spaces. In Figure 1 a fundamental region for the G action is $\Delta_0 \cup q\Delta_0$. We may use the side pairing transformations a, b in the surface S and A, B in the universal cover U to obtain other models of the quotients S/G and $U/T_{l,m,n}$.

We may associate of vertices, sides, and triangles in the two different models as follows

(1) vertices: $R \leftrightarrow 0$, $P \leftrightarrow 1$, $Q \leftrightarrow \infty$;
(2) sides: $p, q \cdot p \leftrightarrow [\infty, 0]$, $q \leftrightarrow [0, 1]$, $r, q \cdot r \leftrightarrow [1, \infty]$, where $q \cdot p$, $q \cdot r$ denote the action of the *reflection* q upon the *sides* p and r, respectively; and
(3) triangles: $\Delta_0 \leftrightarrow$ upper half plane H^+ and $q\Delta_0 \leftrightarrow$ lower half plane H^-.

We can summarize this by saying that we have tile preserving maps

$$\pi_S : \mathcal{T} \to \overline{\mathcal{T}},\ \pi_G : \overline{\mathcal{T}} \to \overline{\overline{\mathcal{T}}},\ \pi_G \circ \pi_S : \mathcal{T} \to \overline{\overline{\mathcal{T}}}.$$

Remark 2.7. The map $\pi_G : S \to S/G$ is unramified over $\widehat{\mathbb{C}} - \{0, 1, \infty\}$, so $\pi_G : \pi_G^{-1}(\widehat{\mathbb{C}} - \{0, 1, \infty\}) \to \widehat{\mathbb{C}} - \{0, 1, \infty\}$ is a covering space and G acts simply transitively on the fibres. The subspace $\widehat{\mathbb{R}} - \{0, 1, \infty\}$ consists of three disjoint open intervals which lift by local homeomorphisms to the sides of tiles minus the end points. Thus there are three G orbits of sides, which can be visualized by colouring the edges. The set $\widehat{\mathbb{C}} - \widehat{\mathbb{R}}$ consists of two open sets, the upper and lower open half planes, H^+ and H^-, in $\mathbb{C}$. These open half planes lift to the interiors of tiles via local homeomorphisms. This can be visualized by colouring half the tiles white (upper) and the other half black (lower). The group G permutes, simple transitively, the tiles of the same colour. Using continuity and compactness arguments one can show that, for any closed tile Δ, π_G maps Δ homeomorphically onto a closed upper or lower half plane. The map is conformal in the interior of the tile, and extends conformally to a suitably small neighbourhod of interior points of the sides.

Remark 2.8. In Figure 1 the union $\Delta_0 \cup q\Delta_0$ is a *hyperbolic kite* symmetric about the bisecting edge q where the two tiles are joined. We call $\Delta_0 \cup q\Delta_0$ (and its translates) a q-kite. Similarly $\Delta_0 \cup p\Delta_0$ and $\Delta_0 \cup p\Delta_0$ (and their translates) are called a p-kites and r-kites, respectively. The action of G is simply transitive on the kites of the same type, indeed each kite is a fundamental region for the G action. The kites can also be defined by lifting from $\widehat{\mathbb{C}}$. To get the q-kites we cut out the

intervals $[\infty, 0]$ and $[1, \infty]$ from $\widehat{\mathbb{C}}$, what remains is a slit domain homeomorphic to a disc. The portions of S lying over the slit domain are the interiors of the q-kites and π_G is a homeomorphism of the interior of a q-kite to the slit domain. Similar remarks apply to p-kites and r-kites. You can visualize the kites in Figures 1 and 2 though the picture in Figure 1 is not exact.

Remark 2.9. By using a linear fractional transformation L of $\widehat{\mathbb{C}}$ we can move the branch points from $0, 1, \infty$ to any triple of points in $\widehat{\mathbb{C}}$, for instance $0, 1, -1$. The above analysis may be carried out word for word except that one of the three intervals has ∞ in its interior. If the three branch points are not all real, then $\widehat{\mathbb{R}}$ needs to be replaced by the unique circle passing through all three of the branch points.

3. Symmetries of surfaces

A symmetry ψ of a surface S an anti-conformal involution. Any such surface is called a *symmetric surface*. The fixed point set $\mathcal{M}_\psi$ of ψ, called the *mirror* of ψ, is a possibly empty, disjoint set of closed, simple, geodesic curves called *ovals* of the symmetry. The quotient surface $S/\langle\psi\rangle$ is a Klein surface, orientable if $S\backslash\mathcal{M}_\psi$ is disconnected (*separating symmetry*), and is unorientable otherwise (*non-separating symmetry*). Thus, if ψ is fixed point free, then $S/\langle\psi\rangle$ is an (unorientable) Klein surface without boundary. So, we shall search for fixed point free symmetries.

Here are some initial examples of symmetries.

Example 3.1. The prime example of a symmetry is given by a complex curve defined over $\mathbb{R}$. Suppose our surface S is given by $f(x, y) = 0$, suitably desingularized and compactified, and such that the coefficients of f are real. Then $f(\overline{x}, \overline{y}) = 0$ if and only if $f(x, y) = 0$ and the involution $(x, y) \leftrightarrow (\overline{x}, \overline{y})$ is a symmetry when restricted to S. The fixed points of the symmetry are the real solutions of $f(x, y) = 0$, namely a curve defined over $\mathbb{R}$.

Example 3.2. If, in the preceding example there are no real solutions to $f(x, y) = 0$ then the symmetry is fixed point free. Such a curve, empty of points over $\mathbb{R}$, is called an *imaginary curve*. The complex curve typically has points but does not meet $\mathbb{R}^2$ in $\mathbb{C}^2$, A simple example is the Fermat curve $x^n + y^n = -1$ with n even. But, note that $x^n + y^n = 1$ always has solutions.

Example 3.3. Let e be any edge in the tiling $\mathcal{T}$. Then the reflection ψ_e in e preserves the tiling and hence the triangle groups $T_{l,m,n}$ and $T^*_{l,m,n}$. If the subgroup $\Pi \subset T_{l,m,n}$ is normalized by ψ_e, then ψ_e descends to a reflection $\psi_{\overline{e}}$ in the image edge $\overline{e}$ in $\overline{\mathcal{T}}$.

3.1. Geometry of symmetries.

Assume now that the surface S has a triangular action of G, determined by the generating triple (a, b, c). We are interested in finding symmetries that normalize the action of G. In this case we say that the action is a *symmetric action*. Questions that are typically asked for symmetric actions are (see [11], [3], [6], [7], [21]):

(1) Are there any symmetries ψ of S normalizing the G action?
(2) If so, determine the conjugacy classes of symmetries.
(3) Is the mirror $\mathcal{M}_\psi$ non-empty, and if so, how many ovals are in the mirror of the symmetry?

(4) Is ψ a separating symmetry?
(5) Is the map $g \to \psi g \psi$ an inner or outer automorphism?

Obviously, questions 1, 2, 3 (empty mirror case) are important to our quest. Though less obvious, the answer to question 5 is also useful.

3.1.1. *The normalizing condition.* Since ψ is an isometry, then $\psi G \psi$ consists of conformal isometries, even if ψ does not normalize G. Thus $\psi G \psi \leq \mathrm{Aut}(S)$ and so G cannot be moved too far. Our observation implies that any symmetry of S automatically normalizes $\mathrm{Aut}(S)$, providing us with plenty of examples of normalizing symmetries, but not all. Now, $\mathrm{Aut}(S)$ is also a triangular group with signature (l', m', n'), and the strict inclusion $G < \mathrm{Aut}(S)$ induces a strict inclusion

$$T_{l,m,n} < T_{l',m',n'}. \tag{16}$$

These inclusions have been classified and form a part of Singerman's List [19]. We have recorded the information we need from the list in Table 5, Section 6. Thus, there are investigative tools for determining the pairs (G, ψ), $\psi G \psi \neq G$ However, there are there are plenty of pairs (G, ψ) where ψ is a normalizing symmetry and G is not the full automorphism group, so our normalizing condition is not unreasonable. For additional details on Singerman's list see [12].

An analysis of non-normalizing symmetries has been carried out in [14], though the exposition takes a different track, via NEC groups and dessins d'enfant. For some examples of non-normalizing symmetries see Example 3.4 and further discussion in Section 6.2.

We also have the possibility that the action of G on S is normalized by two symmetries ψ_1, ψ_2 that $\psi_2 \neq \psi_1 g$ for any $g \in G$. In this case there is a triangular overgroup $H \rhd G$, with normalizing symmetries, and a normal inclusion analogous to (16). In Section 6 we will investigate some examples of these "extra symmetries" when $G < \mathrm{Aut}(G)$, both normalizing and non-normalizing. We focus our main study on cases where there are no extra symmetries under consideration and formalize this situation in the following definition.

Definition 3.1. For a family of symmetries $\{\psi \in \Psi\}$, normalizing the action of G on a surface S, we say that the G action is *tightly normalized by the symmetries in* Ψ, if for any two symmetries $\psi_1, \psi_2 \in \Psi$, we have $\psi_1 \psi_2 \in G$. Otherwise, we say that the G action is *loosely normalized by the symmetries in* Ψ.

The proof of the following is easy and is left to the reader, once the notion of quotient symmetry $\overline{\psi}$ is introduced in Section 3.1.2.

Lemma 1. *For a surface S, let ψ_0 be a distinguished symmetry normalizing a triangular G action. Then, the family of symmetries*

$$\Psi = \left\{\psi : \psi G \psi = G, \overline{\psi} = \overline{\psi}_0\right\}$$

tightly normalizes the G action. The index of G in $\langle \psi, G \rangle$ is 2.

Example 3.4. The Klein quartic S is a symmetric surface of genus 3 whose full automorphism group is the Hurwitz group $\mathrm{Aut}(S) = PSL_2(7)$ with signature $(2, 3, 7)$. In Figure 3 we show a partial tiling of the plane by $(2, 3, 7)$ tiles. It turns out that each of the edges of the tiling induces a symmetry of S as described in Example 3.3. The group $\mathrm{Aut}(S)$ has a subgroup $G = \mathbb{Z}_3 \ltimes \mathbb{Z}_7$, but none of the symmetries of S induced by reflections in edges normalize G. However, the subgroup $\mathbb{Z}_7$ has normalizing symmetries induced by edges. In Figure 3 we can see how 24 of the small

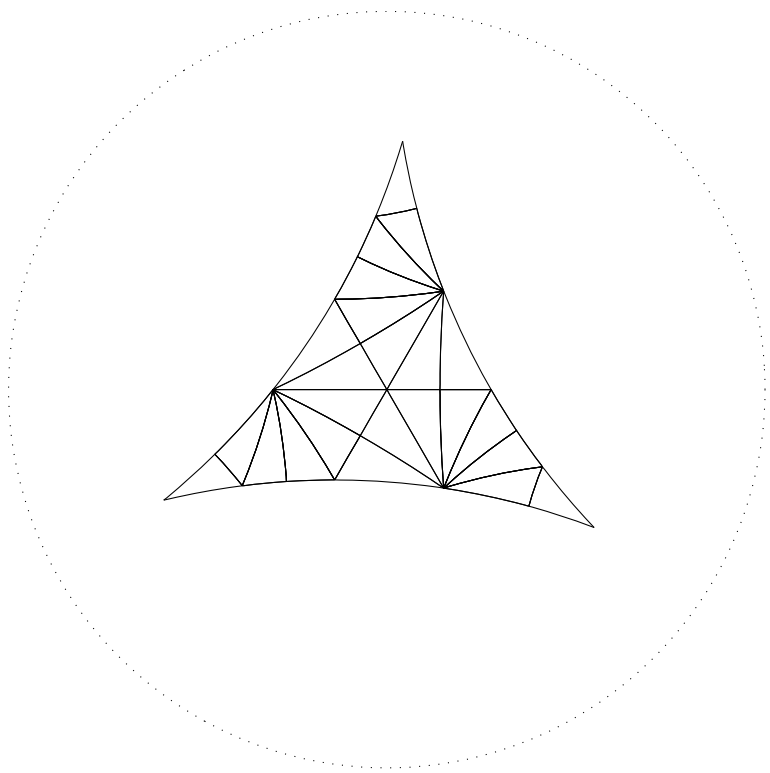

Figure 3. Portion of the (2,3,7) tiling

tiles can be arranged to create a $(7,7,7)$ tile and, hence, an inclusion $T_{7,7,7} \subset T_{2,3,7}$. In turn, we get an induced inclusion $\mathbb{Z}_7 \subset PSL_2(7)$. Since edges of the $(7,7,7)$ tile contain edges of some $(2,3,7)$ tiles, there are some symmetries of the surface that normalize $\mathbb{Z}_7$. None of the interior $(2,3,7)$ edges in $(7,7,7)$ tile define normalizing symmetries of $\mathbb{Z}_7$. On the other hand, we can see that the rotation of order 3 at the center of the figure is an automorphism of both tilings and so normalizes both $\mathbb{Z}_7$ and $PSL_2(7)$ This is consistent with the fact that the normalizer of $\mathbb{Z}_7$ in $PSL_2(7)$ is $\mathbb{Z}_3 \ltimes \mathbb{Z}_7$. If we could build a $(3,3,7)$ triangle out of eight of the $(2,3,7)$ tiles we could demonstrate geometrically that $\mathbb{Z}_3 \ltimes \mathbb{Z}_7$ has normalizing symmetries induced by edges. However, the construction just cannot be done. Later, in Section 5 we show algebraically that $\mathbb{Z}_3 \ltimes \mathbb{Z}_7$ cannot have any symmetries arising from reflection in edges of the $(2,3,7)$ tiling. For more details see [10].

3.1.2. *Type of a symmetry.* Since ψ normalizes G, ψ maps G-orbits to G-orbits. So, there an anti-conformal map $\overline{\psi}$ induced on the quotient S/G. By assumption S/G is a sphere and the quotient map π_G has three branch points in S/G, namely the images of the vertices of the triangles on S. The corresponding orbits are singular orbits, i.e., they have fewer than $|G|$ points. Indeed, the number of points in these singular orbits are $|G|/o(a)$, $|G|/o(b)$, and $|G|/o(c)$. These are the only singular orbits and ψ must permute these orbits among themselves. It follows that the

quotient $\overline{\psi}$ must permute the three branch points, $\{0, 1, \infty\}$, preserving branching order.

The symmetry $\overline{\psi}$ has the form $L(\overline{z})$ where L is a linear fractional transformation. Since we are specifying the L-images of the three points $\{0, 1, \infty\}$, the transformation L is unique. Since $\overline{\psi}$ has order 2, it fixes either 1 or 3 branch points. So we may identify $\overline{\psi}$ and define the *type of a symmetry* depending on the number and arrangement of the fixed points of $\overline{\psi}$. Let us make this precise in the following definition.

Definition 3.2. Let G have a triangular action on S and let ψ be a symmetry of S normalizing the action of G. Let $\pi_G : S \to \widehat{\mathbb{C}}$ be the quotient map, branched over $\{0, 1, \infty\}$, and mapping the vertices of the master tile as follows $R \to 0$, $P \to 1$, $Q \to \infty$. Then, the *type* of ψ with respect to this quotient map is the induced permutation of $\{0, 1, \infty\}$.

Now, let us describes the possibilities in more detail and affix labels to the various symmetry types.

(1) If $\overline{\psi}$ fixes all three branch points then we call ψ a symmetry of Type I. The signature can be any allowable signature. The fixed oval of $\overline{\psi}$ is the real projective line $\widehat{\mathbb{R}}$ and $\overline{\psi}$ is complex conjugation $z \to \overline{z}$.

(2) If $\overline{\psi}$ fixes one branch point and switches the other two, then we call ψ a symmetry of Type II. The branching orders of the switched pair must be the same so that the signature is $(l, l, n), (l, n, l), (n, l, l)$, or (l, l, l) with $l \neq n$. Assume that $\overline{\psi}$ switches 0 and 1 and fixes ∞. Then $\overline{\psi}$ is reflection in the line $\mathrm{Re}(z) = \frac{1}{2}$ with formula $z \to 1 - \overline{z}$. We call this a Type II.a symmetry. See the table below for the other two switches.

All the possible type of symmetries are given in Table 1 below, showing the allowed signatures, the formula for $\overline{\psi}$, and the fixed point set (oval) of $\overline{\psi}$. The last column *standard symmetries* will be explained in the next Section 3.1.3.

Type	Switch	Signatures	$\overline{\psi}(z)$	Oval of $\overline{\psi}$	Std Symm
I	none	(l, m, n)	$\overline{z}$	$\widehat{\mathbb{R}}$	ψ_p, ψ_q, ψ_r
II.a	$0 \leftrightarrow 1$	$(l, l, n), (l, l, l)$	$1 - \overline{z}$	$\mathrm{Re}(z) = \frac{1}{2}$	ψ_s
II.b	$0 \leftrightarrow \infty$	$(l, n, l), (l, l, l)$	$\frac{1}{\overline{z}}$	$z\overline{z} = 1$	ψ_t
II.c	$1 \leftrightarrow \infty$	$(n, l, l), (l, l, l)$	$\frac{\overline{z}}{\overline{z}-1}$	$(z-1)(\overline{z}-1) = 1$	ψ_u

Table 1. Geometry of $\overline{\psi}(z)$ and type of symmetries

Note that the three ovals meet at the points $\frac{1}{2} \pm \frac{\sqrt{3}}{2}$.

3.1.3. *Standard or boundary symmetries.* We shall define specific examples of Type I and II symmetries, called *standard symmetries*, that can be locally defined using the master tile. They are also called *boundary symmetries* since a part of the boundary of master tile is preserved.

Example 3.5. If globally defined, the reflection in any side of the tiling $\overline{\mathcal{T}}$ is a symmetry of Type I (recall Remark 2.6). Any such reflection is G conjugate to one of ψ_p, ψ_q, or ψ_r. We call ψ_p, ψ_q, or ψ_r the *standard symmetries of Type I.*

Example 3.6. If $l = m$ then $\Delta_0 \cup q\Delta_0$ (see Figure 1) is a (hyperbolic) rhombus with opposite angles of $\frac{2\pi}{l}$ and $\frac{\pi}{n}$. We call it the *master q-rhombus,* any translate

of this master rhombus is called a *q-rhombus*. We call the line segment, s, from Q to $q \cdot Q$ a *q-rhombic bisector*. Since $\Delta_0 \cup q\Delta_0$ is always a kite, even if Δ_0 is scalene, we call q the *q-kite bisector* to distinguish it from the rhombic bisector. In the isosceles case, the segments q and s are mutually perpendicular bisectors of each other. The local reflection in s, ψ_s, takes $\Delta_0 \cup q\Delta_0$ to itself, fixing Q and $q \cdot Q$ and exchanging R and P. If globally defined, ψ_s is called the *standard symmetry of Type II.a*. Observe that G acts simply transitively on the q-rhombi in $\overline{\mathcal{T}}$. The *q-rhombic bisectors*, which are G transforms of s, define a conjugacy class of reflections. They are all of Type II.a. In Figure 2 there are $l = 4$ q-rhombi in the central polygon.

In the case that $l = n$ we denote the rhombic bisector of $\Delta_0 \cup p\Delta_0$ by t, and if $m = n$ we denote the rhombic bisector of $\Delta_0 \cup r\Delta_0$ by u. The reflections ψ_s, ψ_t, ψ_u are the *standard symmetries of Type II.a, Type II.b and Type II.c*, respectively.

Remark 3.1. In the equilateral case ψ_s, ψ_t, ψ_u are all locally defined. Using formula (14), and the geometry of the master tile, we obtain the following Type II conjugation relations.

$$\begin{aligned} \psi_s\psi_t\psi_s = \psi_u,\ \psi_s\psi_u\psi_s = \psi_t, \\ \psi_t\psi_s\psi_t = \psi_u,\ \psi_t\psi_u\psi_t = \psi_s, \\ \psi_u\psi_s\psi_u = \psi_t,\ \psi_u\psi_t\psi_u = \psi_s, \end{aligned} \tag{17}$$

as well as these mixed-Type conjugation relations

$$\begin{aligned} \psi_s\psi_p\psi_s = \psi_r,\ \psi_s\psi_q\psi_s = \psi_q,\ \psi_s\psi_r\psi_s = \psi_p \\ \psi_t\psi_p\psi_t = \psi_p,\ \psi_t\psi_q\psi_t = \psi_r,\ \psi_t\psi_r\psi_t = \psi_q, \\ \psi_u\psi_p\psi_u = \psi_q,\ \psi_u\psi_q\psi_u = \psi_p,\ \psi_u\psi_r\psi_u = \psi_r. \end{aligned} \tag{18}$$

3.1.4. *Symmetries with nonempty mirrors.* The following lemma tells us when a symmetry has fixed points.

Lemma 2. *Let all notation and assumptions be as above. Then*

(1) *A surface S has a symmetry of Type I if and only if ψ_p, ψ_q, ψ_r, the local reflections in the sides of the master tile, extend to globally defined symmetries, and hence reflections in all the sides of the tiles are globally defined. A symmetry of Type I has fixed points if and only if it is conjugate by an element of G to one of ψ_p, ψ_q, or ψ_r.*

(2) *A surface S has a symmetry of Type II.a if and only if the reflection, ψ_s, in the rhombic bisector of the master rhombus $\Delta_0 \cup q\Delta_0$, is globally defined and hence the reflections in all the rhombic bisectors of q-rhombi are globally defined. A symmetry of Type II.a has fixed points if and only if it is conjugate, by an element of G, to the reflection ψ_s in the master rhombus $\Delta_0 \cup q\Delta_0$. Similar statements apply to symmetries of Type II.b and II.c, for allowable signatures.*

Proof. Type I symmetries. We first prove that ψ preserves the tiling $\overline{\mathcal{T}}$. To see this, we note that since ψ is a Type I symmetry then $\overline{\psi}$ is complex conjugation on $\widehat{\mathbb{C}} = S/G$. So $\overline{\psi}$ switches the upper and lower open half planes, and fixes $\widehat{\mathbb{R}}$. According to the discussion in Remark 2.7, and lifting to S we see that for each tile, Δ, $\psi\Delta$ is a tile of the opposite color and ψ preserves the colouring of the sides of the tile. Now $q\Delta_0$ and $\psi\Delta_0$ have the same colour and so there is a $g \in G$ mapping $\psi\Delta_0 \to q\Delta_0$ and so $g\psi$ maps $\Delta_0 \to q\Delta_0$. Thus $g\psi$ maps the q-kite $\Delta_0 \cup q\Delta_0$ to

itself and maps the edge q to itself. Using the fact $g\psi$ is an isometry or that $\overline{g\psi}$ is the identity on $(0,1) \subset \widehat{\mathbb{C}}$ we see that $g\psi$ fixes q pointwise.

Next, we lift $g\psi$ to some covering isometry $\widetilde{g\psi}$ on U. Arguing as before, any such $\widetilde{g\psi}$ preserves the tiling $\mathcal{T}$ and the system of q-kites. Let $\widetilde{\Delta_0 \cup q\Delta_0}$ denote some specific lift of the master q-kite. Since $T_{l,m,n}$ acts transitively on the q-kites there is an $L \in T_{l,m,n}$ such that $\widetilde{Lg\psi}$ maps $\widetilde{\Delta_0 \cup q\Delta_0}$ to itself. Finally, we let $\widetilde{q}$ denote the kite bisector of $\widetilde{\Delta_0 \cup q\Delta_0}$ and let $\psi_{\widetilde{q}}$ be the reflection in $\widetilde{q}$, which is globally defined on U. Using a previous argument, $\widetilde{Lg\psi}$ fixes $\widetilde{q}$ pointwise as does $\psi_{\widetilde{q}}$. Now $\psi_{\widetilde{q}}\widetilde{Lg\psi}$ is a conformal automorphism and an isometry of U that fixes all the points in an interval. It must therefore be the identity. So we have $\psi_{\widetilde{q}} = \widetilde{Lg\psi}$ on all of U. According to the Lemma 3 below $\widetilde{g\psi}$ normalizes Π and by assumption L normalizes Π because of (9). Thus $\psi_{\widetilde{q}}$ normalizes Π and ψ_q, the reflection in the edge q in the surface, is globally defined on S. Since $p = qa^{-1}$ and $r = qb$, p and r are globally defined. According to Remark 2.6 the reflection in any edge is globally defined and there are three G classes of them.

Now suppose that a Type I symmetry ψ has a non-empty mirror $\mathcal{M}_\psi$. Since $\overline{\psi}$ has fixed points only on $\widehat{\mathbb{R}}$ then $\mathcal{M}_\psi$ is contained in a union of edges. For any of the edges in $\mathcal{M}_\psi$ we have $\psi = \psi_e$. The conjugacy statement follows for the above discussion.

*Type II.a symmetries.*The case for Type II.a is similar. First let us describe the geometry of the tiling induced by $\overline{\psi}$. As we noted previously $\overline{\psi}$ has the formula $z \to 1 - \overline{z}$ and is the reflection in the line $\mathrm{Re}(z) = \frac{1}{2}$. This line breaks up $\widehat{\mathbb{C}}$ into a left and right half plane. As in the Type I case we colour the lifts of the left and right half plane in two colours black and white. The lift of the finite portion of the separating line is the interior of a rhombus bisector. Now there is a branch point in each half plane and so each half rhombus is replicated l times to create a regular polygon with l sides. The interior angles of this regular polygon have measure $\frac{\pi}{n}$ and so there are $2n$ polygons at a vertex. At a vertex the colours alternate between black and white. The tiling on the universal covering space can be obtained by dividing each (l,l,n) triangle into two $(2,l,2n)$ triangles using the rhombus bisector, The $(2,l,2n)$ triangle can be aggregated into the regular polygons just described. The edges of this new tiling are all the rhombus bisectors. The rest of the proof is similar to the Type I case. □

Lemma 3. *Let the group Π act on X, let $\pi : X \to X/\Pi = Y$ be the orbit space map, and suppose we have a bijective map $h : X \to X$.*

(1) *If h normalizes Π, then the map $\overline{h} : Y \to Y$, given by $\Pi x \to \Pi hx$ is well-defined, and the diagram below commutes*

$$\begin{array}{ccc} X & \xrightarrow{h} & X \\ \downarrow \pi & & \downarrow \pi \\ Y & \xrightarrow{\overline{h}} & Y \end{array} . \tag{19}$$

(2) *Now suppose that a map $\overline{h}$ exists so that the diagram 19 commutes, and the following additional hypotheses hold.*
- *X is a Baire space, e.g., a completely metrizable topological space or a locally compact Hausdorff space,*
- *Π is countable,*

- *there is at least one point* $x_0 \in X$ *such that the stabilizer* $\Pi_{x_0} = \{g \in \Pi : gx_0 = x_0\}$ *is trivial, and*
- *if an element of* $\langle h, \Pi\rangle$ *is the identity on an open set in* X*, then it equals the identity on all of* X*.*

Then h *normalizes* Π*.*

Proof. The first part of the Lemma is a standard fact about permutation groups. For the second part, the commutation relation $\overline{h} \circ \pi = \pi \circ h$ implies that for any $x \in X$, we have $\overline{h}(\pi(x)) = \pi(hx) = \Pi hx$. For any $g \in \Pi$ we have $\pi(hgx) = \overline{h}(\pi(gx)) = \overline{h}(\pi(x)) = \pi(hx)$ or $\Pi hgx = \Pi hx$. As this is true for all $g \in \Pi$, we get $\Pi hx = \Pi h \Pi x$ or

$$\Pi hx = h\Pi x \tag{20}$$

and hence

$$h^{-1}\Pi hx = \Pi x. \tag{21}$$

Now for the given $x_0 \in X$, (21) must hold, so for each $g \in \Pi$ there is a $g' \in \Pi$ such that $h^{-1}ghx_0 = g'x_0$ or $(g')^{-1} h^{-1}ghx_0 = x_0$. Now, it is possible that $(g')^{-1} h^{-1}gh \in \langle h, \Pi\rangle$ is not the identity but fixes x_0. This could only happen if $h^{-1}gh \notin \Pi$ for if $h^{-1}gh \in \Pi$ then $g^{'} = h^{-1}gh$ by the stabilizer condition $\Pi_{x_0} = \langle 1\rangle$. To remedy this situation we need to find a better x_0. For each pair g, g' such that $(g')^{-1} h^{-1}gh$ does not equal the identity, define the fixed point set

$$\Phi_{g,g'} = \left\{x \in X : (g')^{-1} h^{-1}gh = x\right\}.$$

By the last bulleted assumption $\Phi_{g,g'}$ is a closed set with empty interior. On the complement $X - \Phi_{g,g'}$ the element $(g')^{-1} h^{-1}gh$ has no fixed points. Consider the set

$$X^\circ = \bigcap_{g,g'} X - \Phi_{g,g'},$$

where intersection is taken over the pairs where $(g')^{-1} h^{-1}gh$ is not the identity. The set X° is a countable intersection of dense open sets in a Baire space and hence is dense in X. Now reselect the point x_0 so that it lies in X° and go through the argument again. It follows that for every choice of g the element $(g')^{-1} h^{-1}gh$ is the identity and $h^{-1}gh \in \Pi$. □

3.2. **Algebra of symmetries.** Assume that ψ is a symmetry and set $G^* = G^*_\psi = \langle \psi, G\rangle$, an isometry group of S, which we call the *tiling group*. The tiling group contains G as a subgroup of index 2 if and only if G is normalized by ψ. In this case, we write the automorphism of G induced by ψ as

$$\theta : g \to \psi g \psi = \psi \circ \epsilon(g) \circ \psi, \tag{22}$$

and call it the *symmetry automorphism* induced by ψ, writing θ_ψ if we need to be specific. Except in one case (see Remark 3.6 item 1), θ is an involution, so we also call it the *symmetry involution*. With the same exception noted above, the group G^*_ψ is isomorphic to $G^*_\theta = \langle\theta\rangle \ltimes G$, a subgroup of the holomorph of G. The elements of G^*_θ are of the form g or θg and have the commutation rule

$$g \cdot \theta = \theta \cdot \theta g \theta = \theta \cdot \theta(g). \tag{23}$$

We call G^*_θ the *symmetry holomorph* of the involution θ. There may be involutions

$\theta \in \mathrm{Aut}(G)$ for which there is no corresponding symmetry, though G^*_θ certainly exists.

The automorphisms of standard symmetries, called *standard automorphisms* or *standard involutions* are easily computed in terms of a generating triple, from the geometry of the master tile. Here is a table of formulas for standard symmetries. We are not claiming that the symmetries and automorphisms exist, but if they do, they must satisfy the given formulas. Moreover, the automorphisms are unique since they are specified on a generating triple.

(24)

Type	symmetry	$\theta(a), \theta(b), \theta(c)$
I	ψ_p	$a^{-1}, c^{-1}b^{-1}c, c^{-1}$
I	ψ_q	$a^{-1}, b^{-1}, a^{-1}c^{-1}a$
I	ψ_r	$b^{-1}a^{-1}b, b^{-1}, c^{-1}$
II.a	ψ_s	b^{-1}, a^{-1}, c^{-1}
II.b	ψ_t	c^{-1}, b^{-1}, a^{-1}
II.c	ψ_u	a^{-1}, c^{-1}, b^{-1}

Table 2. Formulas for standard involutions

3.2.1. *Companion Symmetries.* In the specific case that $\psi = \psi_q$, the reflection in the q edge of the master tile, $G^* = \langle q, G\rangle$ and contains the reflections $p = qa^{-1}$ and $r = qb$. More generally, assume that ψ is a symmetry normalizing G and that $\psi' \in G^* - G$. As G has index 2 in G^*, then $\psi' = \psi g$ for some $g \in G$ and ψ' is anticonformal. If ψ' is involutary then $1 = \psi g \psi g = \theta(g)g$ so that

$$\theta(g) = g^{-1}. \tag{25}$$

This allows us to identify all the symmetries in G^* once we know the associated automorphism θ. Any two symmetries in G^* have the same quotient symmetry so they are of the same type. On the other hand if two symmetries ψ, ψ' have the same type, then $\overline{\psi\psi'}$ is the identity and so $\psi' = \psi g$ for some $g \in G$. We formalize these observations in a definition.

Definition 3.3. Let G have a triangular action on S and suppose that ψ, ψ' are two symmetries normalizing the action of G. Then we say that ψ and ψ' are companion symmetries if they have the same type.

The following is easily proven, so we leave it to the reader.

Proposition 1. *Let G have a triangular action on S and suppose that ψ, ψ' are two symmetries normalizing the action of G. Then ψ and ψ' are companion symmetries if and only if either of the following hold*

(1)
$$\langle \psi, G\rangle = G^*_\psi = G^*_{\psi'} = \langle \psi', G\rangle$$

(2)
$$\psi' = \psi g \text{ for some } g \in G \text{ such that } \psi g \psi = g^{-1}.$$

Since equation (25) is important we are going to define two important quantities for an automorphism θ of G :

(1) *Inverted elements of* θ :

$$I_G(\theta) = \left\{ g \in G : \theta(g) = g^{-1} \right\}.$$

We call $I_G(\theta)$ the set of *inverted elements* of θ. We also have the related set.

(2) *Centralizer of* θ :

$$Z_G(\theta) = \{ g \in G : \theta(g) = g \}.$$

The centralizer $Z_G(\theta)$ is a subgroup of G though $I_G(\theta)$ typically is not. The following remark gives some properties of $I_G(\theta)$ and related sets.

Remark 3.2. For $\theta \in \mathrm{Aut}(G)$ and an integer e let

$$E(\theta, e) = \{ g \in G : \theta(g) = g^e \},$$

an analogy to eigenspaces. The set of inverted elements $I_G(\theta) = E(\theta, -1)$, and $Z_G(\theta) = E(\theta, 1)$. Though $E(\theta, e)$ need not be a subgroup unless $e = 1$, or G is abelian, it does have some interesting structure:

(1) If $g \in E(\theta, e)$ then $e^{o(\theta)} = 1 \bmod o(g)$.
(2) If $g \in E(\theta, e)$ then $g^t \in E(\theta, e)$ for all t, and $\langle \theta \rangle \ltimes \langle g \rangle$ forms group with presentation $\langle x, y : x^m = y^n = 1, y^x = y^e \rangle$ with $m = o(\theta)$, $n = o(g)$, and $e^m = 1 \bmod n$. If θ is involutary then $\langle \theta \rangle \ltimes \langle g \rangle$ is dihedral.
(3) If $g, h \in E(\theta, e)$ and g and h commute then $gh \in E(\theta, e)$.
(4) For $e = -1$, $\theta(h) g h^{-1} \in E(\theta, -1)$ for all $g \in E(\theta, -1)$ and $h \in G$.
(5) For $g \in E(\theta, -1)$, $\theta(g) g g^{-1} = g^{-1} g g^{-1} = g^{-1}$.
(6) For $g \in E(\theta, e)$, $h \in Z_G(\theta)$ we have $hgh^{-1} \in E(\theta, e)$.

We leave the proofs to the reader, we will only use $E(\theta, e)$ for $e = \pm 1$.

3.2.2. *Conjugacy classes of symmetries.* The group G acts by conjugation upon symmetries normalizing G, and preserves the type of the symmetry. To see this let ψ be a symmetry normalizing G and θ its automorphism. Any other symmetry, normalizing G, with the same type as ψ, has the form ψg for a $g \in G$ satisfying $\theta(g) = g^{-1}$. Then, for $h \in G$, $h \psi g h^{-1}$ certainly normalizes G, and

$$h \psi g h^{-1} = \psi \psi h \psi g h^{-1} = \psi \theta(h) g h^{-1}. \tag{26}$$

From item 4 of Remark 3.2, $\theta(h) g h^{-1} \in I_G(\theta)$, so that ψg and $h \psi g h^{-1}$ are companions of ψ, and all three symmetries have the same type.

If only θ is known the formula on $I_G(\theta)$ is

$$g \to \theta(h) g h^{-1} = h \cdot g \tag{27}$$

The map $(h, g) \to h \cdot g$ is a left action on G and, as noted, $g \in I_G(\theta) \Rightarrow h \cdot g \in I_G(\theta)$. We call the action θ-twisted conjugation. We summarize the forgoing as a proposition.

Proposition 2. *Let G have a triangular action on S. Then conjugation by G permutes the symmetries normalizing G of the same type among themselves. Let ψ be such a symmetry and θ its automorphism. Then there is a bijection between the G-conjugacy classes of the companions of ψ and the θ-twisted conjugacy classes on $I_G(\theta)$ induced by*

$$h \psi g h^{-1} \to \psi \theta(h) g h^{-1}. \tag{28}$$

Example 3.7. We look for a local criterion to guaranteeing conjugacy among p, q and r. From Remark 2.2 we see that $pq = a$, $q = pa$, and $pap = a^{-1}$. If a has odd order then $u^{-2} = a$ has a solution in $\langle a \rangle$. Applying the formulas above we see that

$$upu^{-1} = ppupu^{-1} = pu^{-1}u^{-1} = pa = q,$$

and so p and q are automatically conjugate. Similar remarks apply to b and c. Thus, for instance, for Hurwitz actions with signature $(2, 3, 7)$, q and r are conjugate and r and p are conjugate and hence so are p and q. So, there is only one class of Type I symmetries with fixed points.

Remark 3.3. We shall see later in Theorems 2 and 3, that the conjugacy classes of symmetries on S are in 1-1 correspondence with the G orbits in $I_G(\theta)$ under the action given in (27). In the Type I case, the element $1 = \mathrm{Id}(G)$ determines the standard symmetry ψ_q determined by the master q-kite. Hence the orbit $G \cdot 1$ corresponds to symmetries defined by q-kites. Likewise $G \cdot (a^{-1})$ and $G \cdot b$ determine the symmetries defined by p-kites and r-kites. Thus a Type I symmetry $\psi = \psi_q g$ has fixed points if and only if

$$g \in G \cdot 1 \cup G \cdot (a^{-1}) \cup G \cdot b.$$

In the Type II.a case ψ has fixed points if and only if $\psi = \psi_s g$ is conjugate to ψ_s, i.e., and only if

$$g \in G \cdot 1.$$

3.2.3. *$Symms(G)$ and inner automorphisms.* We may analyse potential symmetries and their conjugacy relations without actually knowing if there are symmetries normalizing the G-action! We construct the following subset of the holomorph $\mathrm{Aut}(G) \ltimes G$.

$$Symms(G) = \left\{(\theta, g) \in Aut(G) \times G : \theta^2 = 1, \theta(g) = g^{-1}\right\}. \tag{29}$$

For a fixed θ, the set $\left\{(\theta, g) : \theta(g) = g^{-1}\right\}$ is called the *θ-slice* of $Symms(G)$. The set $Symms(G)$ captures all normalizing symmetries on surfaces with G action, by means of the map $\psi g \to (\theta, g)$. Note that $\mathrm{Aut}(G)$ acts upon $Symms(G)$ via the formula $\omega \cdot (\theta, g) = (\omega\theta\omega^{-1}, \omega(g))$.

To be perfectly clear, we define the holomorph of G to be group of self maps of G generated by $\mathrm{Aut}(G)$ and the left translations $L_g : x \to gx$. The multiplication is determined by:

$$\begin{aligned}(\omega_1, g_1) \cdot (\omega_2, g_2) &= \omega_1 \circ L_{g_1} \circ \omega_2 \circ L_{g_2} \\ &= \omega_1 \circ \omega_2 \circ \omega_2^{-1} \circ L_{g_1} \circ \omega_2 \circ L_{g_2} \\ &= \omega_1 \circ \omega_2 \circ L_{\omega_2^{-1}(g_1)} \circ L_{g_2} \\ &= \left(\omega_1 \cdot \omega_2, \omega_2^{-1}(g_1) \cdot g_2\right).\end{aligned}$$

Inner automorphisms. We now state and prove a proposition on inner symmetry automorphisms which extends a proposition in [6] on Hurwitz groups whose symmetry automorphism is inner.

Proposition 3. *Suppose that G has trivial center and is not isomorphic to a dihedral group. Also suppose that G has a triangular action on a surface S that has a symmetry normalizing the G action. Suppose that the symmetry automorphism $\theta = Ad_v$ is inner, but not trivial. Then $v \in I_G(\theta)$ and corresponds to a fixed point free symmetry.*

Proof. Since $Ad_{v^2} = Ad_v^2 = \theta^2 = 1$ then v^2 lies in the center of G and so is trivial, forcing $v = v^{-1}$. Now $\theta(v) = v(v)v^{-1} = v = v^{-1}$, and so $v \in I_G(\theta)$. The G-orbit of v under the θ-twisted conjugation is:

$$G \cdot v = \{\theta(h)vh^{-1} : h \in G\} = \{vhv^{-1}vh^{-1} : h \in G\} = \{v\}.$$

By Remark 3.3, the companion symmetry corresponding v will have fixed points if an only if $v \in G \cdot 1 \cup G \cdot a^{-1} \cup G \cdot b$ in the case of a Type I symmetry and $v \in G \cdot 1$ in the case of a Type II symmetry. If $v \in G \cdot 1$ then $v = 1$ contradicting that θ is not trivial. If $v \in G \cdot a^{-1}$ then $a^{-1} = v$ and $G = \langle v, b\rangle$. But, as $vbv = b^{-1}$, G is dihedral, another contradiction. The case $v = b$ is similar. □

3.3. **Existence criteria for normalizing symmetries.** In this subsection we recall a theorem of Singerman [20] on the existence of symmetries derived from the group structure of G. First a remark on permuting the signature and generating triple.

Remark 3.4. Note that

(1) The triple (b, c, a) is an (m, n, l) generating triple for a G action and likewise for the pair (c, a, b) and (n, l, m). The permutations are achieved by relabelling the vertices of the master tile in an orientation preserving way. The constructed surfaces are the same.
(2) Also $(b, a, a^{-1}ca)$ is an (m, l, n) triple, $(c, b, b^{-1}ab)$ is an (n, m, l) triple, and $(a, c, c^{-1}bc)$ is an (n, l, m) triple for the same G action. These permutations are found by using one of three reflected neighbours of the master tile for a new master tile. The constructed surfaces are the same.

We will use these observations to shorten the statements of some theorems and to shorten the Magma search.

Here is the theorem of Singerman [20] written in the context of our paper.

Theorem 1. *Suppose that G has an (l, m, n) action on a surface S, defined by a generating triple (a, b, c) and the epimorphism (10). Then S has a symmetry normalizing the G action if and only if for some permutation of (l, m, n) and (a, b, c) as in Remark 3.4, at least one of the following occurs.*

(1) *There is an automorphism θ_1 of G that satisfies*

$$\theta_1(a) = a^{-1}, \theta_1(b) = b^{-1}. \tag{30}$$

In this case, there is a Type I standard symmetry ψ_1 on S, that induces θ_1 by conjugation: $\theta_1(g) = \psi_1 g \psi_1$. Moreover, the reflections in the edges p, q, and r are globally defined on S, in particular, $\psi_1 = \psi_q$. A symmetry ψ on S is a symmetry of Type I on S if and only if $\psi = \psi_1 g$ for some $g \in G$ such that $\theta_1(g) = g^{-1}$.

(2) *We have $m = l$ and there is an automorphism θ_2 of G that satisfies*

$$\theta_2(a) = b^{-1}, \theta_2(b) = a^{-1}. \tag{31}$$

In this case, there is a Type II.a symmetry ψ_2 that induces θ_2 by conjugation: $\theta_2(g) = \psi_2 g \psi_2$. The symmetry $\psi_2 = \psi_s$, where s is the bisector of the master rhombus. There are similar statements for symmetries of Type II.b and II.c

Remark 3.5. Here is a proof sketch for Type I symmetries, the proof for Type II symmetries is similar. Using covering space methods and the fundamental group $\pi_1(\widehat{\mathbb{C}} - \{0, 1, \infty\})$, we may lift the quotient symmetry $\overline{\psi}$ on $\widehat{\mathbb{C}}$ to a symmetry $\widetilde{\psi}$ of $\mathbb{H}$ such that

$$\widetilde{\psi} A \widetilde{\psi} = A^{-1} \text{ and } \widetilde{\psi} B \widetilde{\psi} = B^{-1}.$$

By hypothesis we have

$$\eta(\widetilde{\psi} A \widetilde{\psi}) = a^{-1} = \theta \circ \eta(A)$$
$$\eta(\widetilde{\psi} B \widetilde{\psi}) = b^{-1} = \theta \circ \eta(B)$$

and so $\theta \circ \eta = \eta \circ Ad_{\widetilde{\psi}}$. The kernels of η, $\theta \circ \eta$, and $\eta \circ Ad_{\widetilde{\psi}}$ all equal Π, and so $\widetilde{\psi}$ normalizes Π. Thus $\widetilde{\psi}$ descends to a symmetry ψ on S which covers $\overline{\psi}$.

Remark 3.6. Let all notation be as above. We note the following for future use.

(1) The standard automorphisms θ_1, θ_2 satisfy $\theta_1^2 = Id$, $\theta_2^2 = Id$. They are involutary, i.e., exact order 2, except in the case of the signature $(2, 2, n)$ for a dihedral action on the sphere. So, except in this case, we will call θ_1, θ_2 *standard involutions.*

(2) An involution $\theta \in \mathrm{Aut}(G)$ that satisfies $\theta(a) = b^{-1}$ automatically satisfies $\theta(b) = a^{-1}$.

(3) If $\omega \in \mathrm{Aut}(G)$ and we choose $(a', b', c') = (\omega(a), \omega(b), \omega(c))$ for our generating triple we get the same surface S, as noted in Remarks 2.3 and 2.4. The automorphisms in theorem θ_1 and θ_2 are then replaced by $\theta_1' = \omega\theta_1\omega^{-1}$ and $\theta_2' = \omega\theta_2\omega^{-1}$. They are the standard automorphisms with respect to (a', b', c'). If $\theta_1' = \theta_1$ or $\theta_2' = \theta_2$ then $\omega \in Cent(\mathrm{Aut}(G), \theta_1)$, $\omega \in Cent(\mathrm{Aut}(G), \theta_2)$, respectively.

(4) If $\overline{\psi}$, the symmetry on the quotient, lifts to a symmetry ψ on S then we get the same set of symmetry lifts of $\overline{\psi}$ on the common surface S determined for (a, b, c) and (a', b', c').

Remark 3.7. By items 1, 3, and 4 of the previous Remark we may find all symmetries of all triangular actions of G by selecting a representative θ from each conjugacy class of involutions in $\mathrm{Aut}(G)$ and then finding all generating triples that satisfy the symmetry equations (30) and (31). We may further have to select $\mathrm{Aut}(G)$ representatives of the generating triples found. We can skip $\theta = Id$ since this only give us dihedral actions on the sphere.

Lemma 4. *Let the hypotheses be as in Theorem 1, suppose that ψ is a symmetry of Type I, and that the automorphism $\theta = \theta_1$ is given as in (30). Then there is a $g \in G$ satisfying $\psi = \psi_q g$, with $\theta(g) = g^{-1}$. The mirror $\mathcal{M}_\psi$ is non-empty if and only if one of the following equations has a solution for one of $u, v, w \in G$.*

$$\theta(u)a^{-1}u^{-1} = g \tag{32}$$

$$\theta(v)v^{-1} = g \tag{33}$$

$$\theta(w)bw^{-1} = g \tag{34}$$

Again, let the hypotheses be as in Theorem 1, but now assume that ψ is a symmetry of Type II.a, and that the automorphism $\theta = \theta_2$ is given as in (31). Then there is a $g \in G$ satisfying $\psi = \psi_s g$, with $\theta(g) = g^{-1}$. The mirror $\mathcal{M}_\psi$ is non-empty

if and only if the following equation has a solution for $x \in G$,

$$\theta(x)x^{-1} = g. \tag{35}$$

We can get a stronger result on symmetries for odd order groups, which generalizes our observations about Fermat curves.

Proposition 4. *Let ψ be a symmetry normalizing the triangular action of an odd order group G. Then ψ must have fixed points.*

Proof. The symmetry ψ belongs to the tightly normalizing family of symmetries $\left\{\psi' : \overline{\psi'} = \overline{\psi}\right\}$. This family has a representative ψ_0 with fixed points and symmetry involution θ_0. Because the family is tight, $\psi = \psi_0 g$ for some $g \in I_G(\theta_0)$. For $h = g^e$,

$$h\psi h^{-1} = \psi_0\theta_0(g^e)gg^{-e} = \psi_0 g^{1-2e}.$$

Since G has odd order there is an e such that $1 = 2e \ mod \ o(g)$. For this e, $\psi = h^{-1}\psi_0 h$ and so ψ has fixed points. □

3.4. **Unorientable surfaces with (l, m, n) tilings.** As discussed in the introduction, it is well known that every unorientable surface, without boundary, has the form $M = S/\langle\psi\rangle$ where S is an orientable surface and ψ is a fixed point free symmetry. Since ψ is an isometry then, M inherits a constant curvature geometry, which will be hyperbolic in the cases of interest. As we will see in Sections 4 and 5 and the tables in Section 7, there are quite a few fixed point free symmetries.

Every symmetry normalizing the G action has a factorization $\psi_e \circ \epsilon(g)$ where ψ_e is a reflection in a segment e, which is either an edge of a tile or rhombus bisector. Both of the factors preserve the tiling on S so ψ must also preserve the tiling. Thus the tiling on S descends to a tiling on $S/\langle\psi\rangle$.

Now let us find the isometries of M. Each element $h \in Z_G(\psi) = Z_G(\theta)$ of the centralizer defines a quotient isometry $\overline{h}$ of M, so we know a piece of the isometry group of M. Computing $Z_G(\theta)$ is a step in the process of finding symmetries, so that it won't be an extra calculation. We can push this a bit further by looking at the full centralizer $Z_{full} = \{h \in \mathrm{Aut}(S) : h\psi = \psi h\}$ which could be bigger. By standard covering space arguments, if $\overline{h}$ is the identity then h is the identity. Thus Z_{full} embeds as a subgroup of the isometry group of M.

To get the full isometry group of M we work as follows. Any isometry on M lifts to an isometry on S, that commutes with ψ. This group is $\langle\psi, Z_{full}\rangle = \langle\psi\rangle \times Z_{full}$. Since ψ acts trivially on M then Z_{full} is the full isometry group of M.

4. Symmetries for abelian action groups

4.1. Abelian group examples.

Example 4.1. *Abelian Groups.* If ψ is a symmetry of Type I, then the automorphism θ induced by q must be the inversion map $g \to g^{-1}$. It follows that every element $\psi = \psi_q g, g \in G$ is a symmetry of Type I since (25) holds for all $g \in G$.

Now, let us look at Fermat curves where we have explicit equations for the surfaces.

Example 4.2. We examine the Fermat curve with equation $x^n + y^n = 1$. The group $G = \mathbb{Z}_n^2$ (additive) acts on S by

$$(u, v) \cdot (x, y) = \left(\exp\left(2\pi i\frac{u}{n}\right)x, \exp\left(2\pi i\frac{v}{n}\right)y\right). \tag{36}$$

The finite fixed points of the action are the n points $(0,\zeta)$, $\zeta^n = 1$ with stabilizer $\langle a\rangle$, $a = (1,0)$ and the n points $(\zeta, 0)$, $\zeta^n = 1$ with stablizer $\langle b\rangle$, $b = (0,1)$. At infinity, the points are the projective limits of $x^n + y^n = 0$, i.e., the lines $y = \vartheta x$ where $\vartheta^n = -1$, there are n of these. The element $c = (-1,-1)$ multiplies the x and y coordinates by the same number and so the lines and their projective limits are stabilized by $\langle c\rangle$. A generating triple for the action is (a,b,c), the signature is (n,n,n), and the genus of S is $\frac{(n-1)(n-3)}{2}$. Now define the symmetry $\psi : (x,y) \to (\overline{x},\overline{y})$. By direct calculation, the induced automorphism of G is $\theta : (u,v) \to (-u,-v)$, so ψ is of Type I.

Example 4.3. The same example as above, except that the symmetry is ψ : $(x,y) \to (\overline{y},\overline{x})$ and θ $(u,v) \to (-v,-u)$, and we have a Type II symmetry.

Example 4.4. Now pick a variant of the Fermat curve given by $x^n + y^n = -1$. For even n the real curve is empty. However, the two variants of the curve are isomorphic over $\mathbb{C}$ since the transformation $(x,y) \to (\lambda x, \lambda y)$ transforms the curve $x^n + y^n = 1$ to $x^n + y^n = \lambda^n$.

Example 4.5. Since the Fermat curve has both Type I and Type II symmetries then $G < \mathrm{Aut}(S)$. Indeed, G is a normal subgroup and $\mathrm{Aut}(S)/G \sim \Sigma_3$. The transformations $(x,y) \to (y,x)$ and $(x,y) \to \left(\frac{\vartheta x}{y}, \frac{1}{y}\right)$, $\vartheta^n = -1$, generate the action of Σ_3.

4.2. **Cyclic groups.** Suppose that G is cyclic and has a generating triple (a,b,c) with signature (l,m,n). Then, according to Harvey [15] this happens if and only if $|G| = \mathrm{lcm}(l,m,n) = \mathrm{lcm}(m,n) = \mathrm{lcm}(l,n) = \mathrm{lcm}(l,m)$ and the highest power of 2 dividing $|G|$ divides exactly two of l,m,n.

A cyclic triangular surface always has an affine defining equation of the form

$$y^t = x^{t_1}(x-1)^{t_2}(x+1)^{t_3}, \tag{37}$$

where

$$t = |G|\,, 1 \le t_1, t_2, t_3 < t, \tag{38}$$

$$t_1 + t_2 + t_3 = 0 \bmod t, \tag{39}$$

$$\gcd(t_1,t_2,t_3,t) = 1, \gcd(t_1,t) = l, \gcd(t_2,t) = m, \gcd(t_3,t) = n. \tag{40}$$

With this model the group action is generated by $(x,y) \to (x, \exp\left(\frac{2\pi i}{t}\right) y)$. The quotient map $S \to \widehat{\mathbb{C}}$ is defined by $(x,y) \to x$, after the affine curve is projectively completed and suitably desingularized. With this model, the quotient map is branched over $0, 1$ and -1 but not over infinity. However the action and exponents are easier to analyze in the finite plane (See Remark 2.9).

The defining equation (37) has real coefficients and so there is a symmetry induced by complex conjugation per Example 3.1. The action of this symmetry on G is $g \to g^{-1}$ so it is a Type I symmetry.

Example 4.6. *Type I symmetries* For cyclic groups there are always symmetries of Type I which must have the form $\psi = \psi_q g, g \in G$. According to Lemma 4 ψ will have fixed points if and only if we can find one of u,v,w satisfying

$$u^{-1}a^{-1}u^{-1} = \theta(u)a^{-1}u^{-1} = g$$
$$u^{-2} = ag$$

$$v^{-1}v^{-1} = \theta(v)v^{-1} = g$$
$$v^{-2} = g$$

$$w^{-1}a^{-1}w^{-1} = \theta(w)bw^{-1} = g$$
$$w^{-2} = b^{-1}g$$

If G has odd order then all three equations have solutions. The map $sq : G \to G$, $g \to g^{-2}$ is an isomorphism if G has odd order, otherwise it maps onto a subgroup G_2 of index 2. By the conditions on l, m, n exactly 2 of a, b, c lie in $G - G_2$, and hence at least one of a, b^{-1} lies in $G - G_2$. By considering images in G/G_2 it follows that least one of $ag, g, b^{-1}g$ lies in G_2, It follows then that ψ is reflection in some edge of the tiling and hence must have fixed points. This means that ψ is conjugate to at least one of ψ_p, ψ_q, or ψ_r.

We know that the surface has a defining equation (37) with real coefficients so complex conjugation is a symmetry. If t is odd then (37) has a solution or every value of x. If t is even, then at least one of t_1, t_2, t_3 is odd, so the righthand side of (37) changes sign at least once, and set of real points is non-empty.

In the next two examples we determine Type II symmetries.

Example 4.7. *Type II symmetries,* $G = \mathbb{Z}_{12}$. Now suppose that $l = m$. Using Harvey's observations we see that $|G| = l$, and n divides l. The genus of S is $\frac{1}{2}\left(l - \frac{l}{n}\right)$. We will start off with a detailed calculation of a small but illustrative example, and then take up the general case in Example 4.8 following. So assume that $G = \mathbb{Z}_{12}$ and consider the signatures $(12, 12, 6), (12, 12, 3)$ and $(12, 12, 2)$. There are no triples with $(12, 12, 12)$ and $(12, 12, 4)$ signatures. The elements a and b lie in $\{1, 5, 7, 11\}$, but we must throw out those triples with $a = b^{-1}$ resulting in 12 generating triples. The automorphism group of G is $\mathbb{Z}_{12}^* = \{1, 5, 7, 11\}$, acting by multiplication. Thus there are three classes, one for each signature.

For the remainder of the example we will use the additive notation for G. Below we have a table of signatures, generating triples, allowable symmetries of Type II, and the enabling automorphisms. The notation m_α indicates the operation of multiplication by α, $m_\alpha : z \to \alpha z$. The automorphism θ_2 is the automorphism of Type II.a in Theorem 1. The last two columns are explained below.

Signature	(a, b, c)	Genus of S	Type II?	θ_2	$G_{\alpha-1}$	$G^{ann}_{\alpha+1}$
$(12, 12, 6)$	$(1, 1, 10)$	5	yes	m_{11}	$2\mathbb{Z}_{12}$	$\mathbb{Z}_{12}$
$(12, 12, 3)$	$(1, 7, 4)$	4	yes	m_5	$4\mathbb{Z}_{12}$	$2\mathbb{Z}_{12}$
$(12, 12, 2)$	$(1, 5, 6)$	3	yes	m_7	$6\mathbb{Z}_{12}$	$3\mathbb{Z}_{12}$

Table 3: Type II symmetries for $\mathbb{Z}_{12}$

Let us analyze $Symms(G)$. The operations of the holomorph of G may be realized as affine linear maps $z \to \alpha z + \beta$ where $\alpha \in \mathbb{Z}_{12}^*$ and $\beta \in \mathbb{Z}_{12}$. For elements of $Symms(G)$ we must have:

$$\alpha^2 = 1, \ (\alpha + 1)\beta = 0. \tag{41}$$

A conjugation operation by an $x \in G$ has the form $z \to \alpha(z - x) + \beta + x = \alpha z + \beta - (\alpha - 1)x$. Since we wish to transform $z \to \alpha z + \beta$ to $z \to \alpha z$ we need to

solve

$$(\alpha-1)x=\beta,\ \text{given}\ (\alpha+1)\beta=0 \tag{42}$$

In the last two columns we record the two subgroups

$$G_{\alpha-1}=\{(\alpha-1)x : x\in G\} \tag{43}$$

$$G^{ann}_{\alpha+1}=\{\beta\in G : (\alpha+1)\beta=0\} \tag{44}$$

In all cases $G_{\alpha-1}\subset G^{ann}_{\alpha+1}$. It is not hard to show that the conjugacy classes of Type II symmetries correspond to the cosets $G^{ann}_{\alpha+1}/G_{\alpha-1}$. There are two cosets, and so there is one class of companion symmetries, one with ovals and the other without ovals. Note that the three different values of α correspond to symmetries on surfaces of different genera.

Now onto the general cyclic case.

Example 4.8. *Type II symmetries - general case.* We can extend the analysis to the general case $G=\mathbb{Z}_t$ as follows. As noted in the previous example, the signatures are of the form (l,l,n) where $l=t=|G|$ and n divides l. The numbers a,b are generators and so lie in $\mathbb{Z}_t^*$ and there $\phi(t)$ choices for each. The number $c=(ab)^{-1}$ cannot be trivial so the selection $b=a^{-1}$ is not allowed. This leaves us with $\phi(t)(\phi(t)-1)$ triples. Since $\mathrm{Aut}(G)=\mathbb{Z}_t^*$ acts without fixed points on the generating triples, and $|\mathrm{Aut}(G)|=\phi(t)$, there are $\phi(t)-1$ classes.

As in the last example we switch to additive notation for G and affine linear maps $z\to\alpha z+\beta$ to represent holomorph operations. The maps must satisfy (41). Now suppose that $\alpha\in\mathbb{Z}_t^*-\{1\}$, satisfies $\alpha^2=1$ and consider the generating triple $(1,-\alpha,\alpha-1)$. The effect of m_α is $(1,-\alpha,\alpha-1)\to(\alpha,-\alpha^2,\alpha(\alpha-1))=(\alpha,-1,1-\alpha)$. This satisfies the equations (31) for a Type II.a symmetry. Note that every generating triple is $\mathrm{Aut}(G)$-equivalent to one of the form $(1,-\alpha,\alpha-1)$, where $\alpha\in\mathbb{Z}_t^*-\{1\}$. Thus we have captured all of the possible Type II symmetry automorphisms and generating triples.

The elements $\alpha\in\mathbb{Z}_t^*$ satisfying $\alpha^2=1$ form an elementary abelian 2-group. Thus the number surfaces with Type II.a symmetry is 2^k-1 for some k. Since $|\mathbb{Z}_t^*|=\phi(t)$ is even, then there is at least non-trivial involution, $2^k-1>0$, and at least one surface has a Type II.a symmetry.

To analyze companion symmetries, we proceed as in the previous example, and define the subgroups $G^{ann}_{\alpha+1}$ and $G_{\alpha-1}$. We have $(\alpha+1)(\alpha-1)x=(\alpha^2-1)x=0$, since α is an involution, and so $G_{\alpha-1}\subseteq G^{ann}_{\alpha+1}$. The group $G^{ann}_{\alpha+1}/G_{\alpha-1}$ is a subquotient of a cyclic group and so is also cyclic, we shall show that its order ie 2 or 1. 2. If $\beta\in G^{ann}_{\alpha+1}$ then

$$2\beta=(\alpha+1)\beta-(\alpha-1)\beta=-(\alpha-1)\beta\in G_{\alpha-1}.$$

It follows that 2 annihilates the cyclic group $G^{ann}_{\alpha+1}/G_{\alpha-1}$, so its order must be 1 or 2.

If G is odd then $G^{ann}_{\alpha+1}/G_{\alpha-1}$ has odd order, and so must be trivial since it is annihilated by the invertible element 2. It follows that in the odd case for a given involution α there is a single conjugacy class of Type II.a symmetries and they all have fixed points.

The even case is more complex. Let us show how to directly compute $|G^{ann}_{\alpha+1}/G_{\alpha-1}|$. We need the following for $\gamma\in\mathbb{Z}_t$:

- $|G_\gamma| = \frac{t}{\gcd(t,\gamma)}$, and
- $|G_\gamma^{ann}| = \gcd(t,\gamma)$.

We then get:

$$\left|G_{\alpha+1}^{ann}/G_{\alpha-1}\right| = \frac{\gcd(\alpha+1,t)\gcd(\alpha-1,t)}{t}. \tag{45}$$

The number of standard and fixed point free Type I symmetries for various cyclic groups are given in Table 4 below.

The example above is worth stating as a Proposition, whose proof is already given.

Proposition 5. *Let $G \simeq \mathbb{Z}_t$ be a cyclic group, and identify its holomorph with $\mathbb{Z}_t^* \ltimes \mathbb{Z}_t$ the group of affine linear transformations*

$$\mathbb{Z}_t^* \ltimes \mathbb{Z}_t \sim \{(\alpha,\beta) \in \mathbb{Z}_t^* \ltimes \mathbb{Z}_t : z \to \alpha z + \beta\}.$$

Suppose that α, β satisfy $\alpha^2 = 1, \alpha \neq 1, \alpha\beta = -\beta$. Then:

- *The triple $(1, -\alpha, \alpha - 1)$ is a generating triple for a triangular G action on a surface S.*
- *The signature of the action is (l, l, n) where $l = |G|$, $n = \gcd(t, \alpha - 1)$ and S has genus $\frac{1}{2}\left(l + \frac{l}{n}\right)$.*
- *The automorphism $m_\alpha : z \to \alpha z$ is a standard Type II.a automorphism for the generating triple $(1, -\alpha, \alpha - 1)$.*
- *For $t > 2$, the number of involutions is $2^k - 1$ for some integer $k > 0$.*
- *The number of G-conjugacy classes of Type II.a symmetries equals $\left|G_{\alpha+1}^{ann}/G_{\alpha-1}\right|$ given in (45) and must equal 1 or 2.*
- *If $\left|G_{\alpha+1}^{ann}/G_{\alpha-1}\right| = 1$ then there is exactly one class of Type II.a symmetries, which must have ovals.*
- *If $\left|G_{\alpha+1}^{ann}/G_{\alpha-1}\right| = 2$ then there are two classes of Type II.a symmetries, one with ovals and the other without ovals.*
- *If t is odd then $\left|G_{\alpha+1}^{ann}/G_{\alpha-1}\right| = 1$*
- *If t is even then $\left|G_{\alpha+1}^{ann}/G_{\alpha-1}\right| = 1$ or 2. Both numbers are possible.*
- *There are no other Type II.a symmetries.*

In Table 4, the column #SS is the number of standard symmetries which also is the number of involutions. The column #FPF is the number of fixed point free symmetries, and the column "total" is the sum of the preceding two columns. Undoubtedly, the general case can be analysed in term of the prime power factorization of t. For instance, it appears that the integer k is the number of different primes in the factorization. Though an interesting topic, this problem is beyond the scope of this paper.

$\lvert G\rvert$	#SS	#FPF	total
$4 = 2^2$	1	1	2
$8 = 2^3$	3	1	4
$16 = 2^4$	3	1	4
$6 = 2 \cdot 3$	1	1	2
$12 = 2^2 \cdot 3$	3	3	6
$24 = 2^3 \cdot 3$	7	3	10
$48 = 2^4 \cdot 3$	7	3	10
$30 = 2 \cdot 3 \cdot 5$	3	3	6
$60 = 2^2 \cdot 3 \cdot 5$	7	7	14
$120 = 2^3 \cdot 3 \cdot 5$	15	7	22
$240 = 2^4 \cdot 3 \cdot 5$	15	7	22

Table 4. Type II.a symmetries for cyclic groups

5. Non-abelian groups of order pq

We give a detailed analysis of triangular actions and symmetries of non-abelian groups of order pq. We study these examples since they are uncomplicated examples of non-abelian group actions. The groups were extensively studied by Wolfart and Streit in [22] in the context of dessins d'enfant. In [8] they were studied as automorphism groups of surfaces S whose quotient is a torus with one branch point. The criterion for these groups to have such an action is identical to the criterion which prevents these groups from having Type 1 symmetries. Though these groups have no Type I symmetries the do have Type II symmetries.

5.1. **Definition and properties of pq groups.** Let $p < q$ be two primes such that p divides $q - 1$. Most of our results are intended for $p > 2$. It is well known that there is exactly one isomorphism class of non-abelian groups of order pq, all isomorphic to $\mathbb{Z}_p \ltimes \mathbb{Z}_q$. We shall call such groups *(non-abelian) pq groups.* A pq group G has a presentation

$$G = \langle x, y : x^p = y^q = 1, y^x = y^r \rangle \,, \tag{46}$$

where $1 < r < q$ and $r^p = 1 \mod q$. The non-trivial elements comprise $(p-1)q$ elements of order p and $q-1$ elements of order q. The subgroup $\langle y \rangle$ is a characteristic subgroup.

The order of the automorphism group $\mathrm{Aut}(G)$ is $q(q-1)$, consisting of the products $U_u V_v$, $0 \le u < q$, $1 \le v < q$, where $U_u : x \to x^{y^u} = xy^{u-ru}$, $y \to y$ and $V_v : x \to x, y \to y^v$. This is easily established once it is shown that $\mathrm{Aut}(G)$ acts trivially on $\overline{G} = G/\langle y \rangle$, according to Lemma 5 following.

Lemma 5. *Let G be a non-abelian pq group as given (46). Then, $Aut(G)$ acts trivially on $\overline{G} = G/\langle y \rangle$.*

Proof. Suppose $\omega \in \mathrm{Aut}(G)$ satisfies $\omega(x) = x^t y^k$, $\omega(y) = y^s$ where t and s have multiplicative inverses $\mod p$ and q respectively and $t \neq 1 \mod p$. Then as

$\omega(y^x) = \omega(y^r)$ we must have;

$$\begin{aligned} y^{-k}x^{-t}y^s x^t y^k &= y^{rs}, \\ (x^{-t}yx^t)^s &= (y^r)^s, \\ y^{r^t} &= y^r. \end{aligned}$$

Thus $r^t = r \bmod q$ and since $r \neq 0$, then $r^{t-1} = 1 \bmod q$. If $t > 1$ then, $1 = gcd(t-1, p) = u(t-1) + vp$ for some integers u and v, and $r = r^{u(t-1)+vp} = \left(r^{(t-1)}\right)^u (r^p)^v = 1 \bmod q$. But, $r \neq 1 \bmod q$, a contradiction. □

5.2. **Triangular actions and signatures of pq groups.** We now determine all the triangular actions of G. If (a, b, c) is a generating triple for G then the quotient triple $(\overline{a}, \overline{b}, \overline{c})$ for $\overline{G}$ must have or two or three elements of order p. This implies the (l, m, n) must be some permutation of (p, p, p) or (p, p, q).

Example 5.1. *Actions with signature* (p, p, q). A general (p, p, q) triple has the form $(x^{u_1}y^{v_1}, x^{u_2}y^{v_2}, y^{v_3})$, where $u_i \neq 0 \mod p$ and $v_3 \neq 0 \bmod q$. By considering the quotient triple $(\overline{a}, \overline{b}, \overline{c})$ we see that $u_2 = -u_1 \bmod p$. Noting that

$$y^v x^u = x^u y^{r^u v}, \tag{47}$$

we have:

$$\begin{aligned} 1 = abc &= x^{u_1}y^{v_1}x^{-u_1}y^{v_2}y^{v_3} \\ &= x^{u_1}x^{-u_1}y^{v_1 r^{-u_1}}y^{v_2}y^{v_3} \end{aligned} \tag{48}$$

$$= y^{v_1 r^{-u_1}}y^{v_2}y^{v_3}. \tag{49}$$

Thus, $r^{-u_1}v_1 + v_2 + v_3 = 0 \bmod q$. There are $p-1$ selections for u_1, and once u_1 is chosen, v_2 and v_3 may be chosen in $q(q-1)$ ways. Then, $v_1 = -r^{u_1}(v_2 + v_3)$ is uniquely determined, and the total number of generating triple classes is $(p-1)q(q-1)/q(q-1) = (p-1)$.

Example 5.2. *Actions with signature* (p, p, p). A general (p, p, p) triple has the form $(x^{u_1}y^{v_1}, x^{u_2}y^{v_2}, x^{u_3}y^{v_3})$, where $u_i \neq 0 \mod p$. A calculation similar to (48) shows that

$$\begin{aligned} 1 = abc &= x^{u_1}y^{v_1}x^{u_2}y^{v_2}x^{u_3}y^{v_3} \\ &= x^{u_1}y^{v_1}x^{u_2+u_3}y^{r^{u_3}v_2}y^{v_3} \\ &= x^{u_1+u_1+u_3}y^{r^{(u_2+u_3)}v_1}y^{r^{u_3}v_2}y^{v_3} \end{aligned} \tag{50}$$

Thus

$$u_1 + u_2 + u_3 = 0 \bmod p, \tag{51}$$

and

$$r^{(u_2+u_3)}v_1 + r^{u_3}v_2 + v_3 = 0 \bmod q. \tag{52}$$

There are $(p-1)^2$ selections for u_1, u_2 except that we have to remove the selections where $u_2 = -u_1 \bmod p$ which are $p-1$ in number. We end up with $(p-1)(p-2)$ solutions for (u_1, u_2, u_3). We may freely pick v_1 and v_2, but then v_3 is fixed. Thus we have a total of $(p-1)(p-2)q^2$ triples, though not all triples generate G. If (a, b, c) fails to generate G then it generates one of the q Sylow p subgroups of G. By a previous calculation, each of the Sylow subgroups has $(p-1)(p-2)$

generating triples with signature (p,p,p) and so we must subtract $(p-1)(p-2)q$ from $(p-1)(p-2)q^2$ to get $(p-1)(p-2)q(q-1)$ generating triples. This give us $(p-1)(p-2)$ $\mathrm{Aut}(G)$ classes of triples.

5.3. **Involutions in** $\mathrm{Aut}(G)$**.** For our work on symmetries we need to determine the involutions in $\mathrm{Aut}(G)$. We state the result as a Lemma.

Lemma 6. *Let G be a non-abelian pq group as given (46). Then the involutions $\theta \in \mathrm{Aut}(G)$ have the form*

$$\theta(x) = xy^t, \theta(y) = y^{-1}, \tag{53}$$

where $0 \le t < q$. There are q such involutions, and they form a single conjugacy class in $Aut(G)$. The centralizer of the involution, of order $q-1$, is the group $\{V_v, 1 \le v < q\}$, where $V_v : x \to x, y \to y^v$.

Proof. Let $\theta \in \mathrm{Aut}(G)$ be an involution. By Lemma 5 we must have $\theta(x) = xy^t$, for some t satisfying $0 \le t < q$. Since $\langle y \rangle$ is characteristic in G, then $\theta(y) = y^e$, for some e satisfying $1 \le e < q$. We obtain

$$\begin{aligned} x &= \theta^2(x) = \theta(xy^t) = xy^t y^{et} = xy^{t(1+e)} \\ y &= \theta^2(y) = \theta(y^e) = y^{e^2}. \end{aligned}$$

From the first equation, $t(1+e) = 0 \bmod q$, so one of $t = 0 \bmod q$ or $e = -1 \bmod q$ must hold. By the second equation $e^2 = 1 \bmod q$ and thus $e = \pm 1 \bmod q$. If $e = 1$, then $t = 0$ and we get the identity. Otherwise $e = -1 \bmod q$. The proposed automorphisms preserve the relations in G :

$$\begin{aligned} \theta(x)^p &= \left(xy^t\right)^p = 1, \\ \theta(y)^q &= \left(y^{-1}\right)^q = 1, \\ \theta(x^{-1}yx) &= y^{-t}x^{-1}y^{-1}xy^t = y^{-t}y^{-r}y^t = y^{-r} = \theta(y^r) \end{aligned}$$

Since t can be arbitrary, we have q involutions. To prove the statement on the involutions forming a conjugacy class we use conjugation by $y^v : g \to y^{-v}gy^v$, namely we consider the automorphism $g \to y^v\theta(y^{-v}gy^v)y^{-v}$. We compute

$$\begin{aligned} y^v\theta(y^{-v}xy^v)y^{-v} &= y^v y^v xy^t y^{-v} y^{-v} = xy^{2rv+t-2v} \\ y^v\theta(y^{-v}yy^v)y^{-v} &= y^{-1} \end{aligned}$$

As we cycle v through $0 \le v < q$ the new exponent $t + 2(r-1)v$ cycles through the same range. The centralizer statement is easily proven by direct computation. The centralizer size comes from this calculation:

$$q(q-1) = |\mathrm{Aut}(G)| = \left|\theta^{\mathrm{Aut}(G)}\right| |Cent(\theta, \mathrm{Aut}(G))|,$$

valid for any $\theta \in \mathrm{Aut}(G)$. The posited centralizer has the right size and is easily verified to lie in the centralizer by direct computation. □

5.4. **Symmetries for** pq **groups.** We first note that there are no Type I symmetries for pq groups, $p > 2$. Indeed, if there are Type I symmetries then the symmetry automorphism must be the map $g \to g^{-1}$ on $G/\langle y \rangle$. But this contradicts Lemma 5 unless $p = 2$.

So now we search for Type II.a symmetries. By Remark 3.7 and Lemma 6 we need only consider the involution $\theta : x \to x, y \to y^{-1}$. Using previous calculations, a general generating triple has the form $(a,b,c) = (x^{u_1}y^{v_1}, x^{u_2}y^{v_2}, x^{u_3}y^{v_3})$ with

the restrictions given in (51) and (52). The θ transform of (a, b, c) is $\theta \cdot (a, b, c) = (x^{u_1}y^{-v_1}, x^{u_2}y^{-v_2}, x^{u_3}y^{-v_3})$. Thus, we must have:

$$x^{u_1}y^{-v_1} = \theta(a) = b^{-1} = y^{v_2}x^{-u_2} = x^{-u_2}y^{v_2r^{-u_1}}.$$

It follows that $u_2 = -u_1$, $u_3 = 0$, and $v_1 = -r^{u_2}v_2$, and that the signature is (p, p, q). Putting the first two restrictions into (52) we get

$$r^{u_2}v_1 + v_2 + v_3 = 0 \bmod q,$$
$$(1 - r^{2u_2})v_2 + v_3 = 0 \bmod q.$$
$$v_2 = -(1 - r^{2u_2})^{-1}v_3$$

As long as $r^{2u_2} \neq 1 \bmod q$ and $v_3 \neq 0 \bmod q$ we will have a θ compatible generating triple. Since r has order $p \bmod q$ then we only need $u_2 \neq 0 \bmod p$. There are $(p-1)(q-1)$ such solutions

$$(a, b, c) = (x^{-u_2}y^{w_1}, x^{u_2}y^{w_2}, y^{v_3})$$

where $w_2 = -(1 - r^{2u_2})^{-1}v_3$, $w_1 = r^{u_2}(1 - r^{2u_2})^{-1}v_3$. Applying the centralizer of θ to the collection of (a, b, c), we see that each class can be identified by u_2 and we can that $v_3 = 1$. Thus every (p, p, q) action has at least one Type II.a symmetry.

6. Extra symmetries when $G \neq \mathrm{Aut}(S)$

Throughout most of this paper we analyzed one tightly normalizing families of symmetries, normalizing a triangular action. We now consider two different cases of extra symmetries. We shall just give examples and not perform a comprehensive analysis. As discussed previously, we will need a portion of Singerman's list, in a format suitable for the context of this paper, see Table 4 below. In the table, the Case column indicates a normal inclusion with and N (first three) and non-normal inclusion by NN (last 11). The column labeled divisible will be discussed later. There are restrictions on d, e to yield hyperbolic surfaces, they are given in Singerman's paper [19].

Case	(l, m, n)	(l', m', n')	index	divisible
N1	(d, d, d)	$(2, 3, 2d)$	6	yes
N2	(d, d, d)	$(3, 3, d)$	3	no
N3	(d, d, e)	$(2, d, 2e)$	2	yes
NN1	$(7, 7, 7)$	$(2, 3, 7)$	24	yes
NN2	$(4, 8, 8)$	$(2, 3, 8)$	12	yes
NN3	$(9, 9, 9)$	$(2, 3, 9)$	12	no
NN4	$(3, 8, 8)$	$(2, 3, 8)$	10	no
NN5	$(2, 7, 7)$	$(2, 3, 7)$	9	no
NN6	$(3, 3, 7)$	$(2, 3, 7)$	8	no
NN7	$(4, 4, 5)$	$(2, 4, 5)$	6	no
NN8	$(d, 4d, 4d)$	$(2, 3, 4d)$	6	yes
NN9	$(d, 2d, 2d)$	$(2, 4, 2d)$	4	yes
NN10	$(3, d, 3d)$	$(2, 3, 3d)$	4	yes
NN11	$(2, d, 2d)$	$(2, 3, 2d)$	3	yes

Table 5 - Triangle group inclusions

6.1. **Actions with multiple types of normalizing symmetries.** Suppose that the action of G on S is normalized by two or more symmetries $\psi_1, \ldots, \psi_k$ such that $\psi_i\psi_j \notin G$ for $i \neq j$. (We will only need $k = 2, 3$.) Then, $\psi_i\psi_j$ is a conformal automorphism of S normalizing G. It is easily shown that

$$\langle G, \psi_1, \ldots, \psi_k\rangle \cap \mathrm{Aut(S)} = \langle G, \psi_1\psi_2, \psi_2\psi_3, \ldots, \psi_{k-1}\psi_k\rangle\,,$$

and a normal inclusion of triangle groups, analogous to (16), is determined by

$$G \lhd \langle G, \psi_1\psi_2, \psi_2\psi_3, \ldots, \psi_{k-1}\psi_k\rangle = H.$$

According to Singerman's list in [19], the possible normal inclusions of triangle groups are the first three cases in Table 5, namely:

$$\begin{aligned} T_{l,l,n} &\lhd T_{2,l,2n} \text{ (index 2)},\\ T_{l,l,l} &\lhd T_{3,3,l} \text{ (index 3)},\\ T_{l,l,l} &\lhd T_{2,3,2l} \text{ (index 6)}. \end{aligned} \tag{54}$$

Example 6.1. As an example of the first case of inclusions we shall take $\psi_1 = \psi_s$ and $\psi_2 = \psi_q$. The geometry is described in Example 3.6. Since ψ_s, ψ_q are reflections in perpendicular line segments, they commute, and hence $x = \psi_s\psi_q$ is a half turn with a fixed point at $q \cap s$. So $H = \langle G, x\rangle$, and we may write

$$H = \langle x\rangle \ltimes G, \text{ where } xax = b, xbx = a, xcx = bcb^{-1}.$$

using the known action of ψ_s and ψ_q. The rhombic bisector, s, splits the master tile into two $(2, l, 2n)$ triangles, one of which has bounding symmetries (ψ_s, ψ_q, ψ_r). By the standard construction of a generating triple from a tile, we see that $(x, b, (xb)^{-1})$ is a generating triple for H.

The remaining examples have two different Type II boundary symmetries and so the triangles must be equilateral.

Example 6.2. In the second case in (54) we choose $\psi_1 = \psi_s$ and $\psi_2 = \psi_t$, boundary symmetries of Type II.a and II.b. From (17) we see that the Type II.c symmetry ψ_u also normalizes G. Now, ψ_s, ψ_t are reflections in line segments meeting at an acute angle of $\pi/3$, and hence $x = \psi_t\psi_s$ is a counter-clockwise rotation of order 3 with fixed point at $C_0 = t \cap s$, the center of the master tile. So $H = \langle G, x\rangle$, with semi-direct product structure:

$$H = \langle x\rangle \ltimes G, \text{ where } xax^{-1} = b, xbx^{-1} = c, xcx^{-1} = a.$$

The conjugation action follows from the equations (18).

The union of the rhombus bisectors creates a tiling $\overline{\mathcal{U}}$ on S, dual to $\overline{\mathcal{T}}$, which is invariant under the group $H^* = \langle \psi_s, \psi_t, \psi_u\rangle$. A fundamental region for this reflection group is the triangle bounded by these segments, that may be drawn in Figure 1:

- the segment from vertex $P \in \mathbf{\Delta}_0$ to the center C_0, this segment is a portion of the bisector t;
- the segment from C_0 and qC_0, the center of $q\mathbf{\Delta}_0$, this segment is a portion of the bisector s; and
- the segment from qC_0 to the vertex P, this segment is a portion of the bisector $\psi_q(t)$.

The reflections in the sides of this $(3,3,l)$ triangle, in the order given are ψ_t, ψ_s, and $\psi_q\psi_t\psi_q$. Using the relations (18), the resulting generating triple may be computed: $(x, x^{-1}b^{-1}, b)$,

Example 6.3. For the third inclusion in (54) let us select three symmetries: $\psi_1 = \psi_q$ and $\psi_2 = \psi_s$, $\psi_3 = \psi_u$, boundary symmetries of Types I and II.a and II.c From (17) and (18) we see that all six boundary symmetries are globally defined, and all normalize G. From previous discussion $x = \psi_q\psi_s$ is a half turn with fixed point $= q \cap s$, and $y = \psi_s\psi_u$ is a counter-clockwise rotation of order 3 with fixed point at $C_0 = s \cap u$. So $H = \langle G, x, y\rangle$, with these relations:

$$xax = b, xbx = a, xcx = bcb^{-1}, \tag{55}$$

$$yay^{-1} = b, yby^{-1} = c, ycy^{-1} = a. \tag{56}$$

The union of the edges of the two tilings $\overline{\mathcal{T}}$ and $\overline{\mathcal{U}}$, creates $\overline{\mathcal{V}}$, a tiling by $(2,3,l)$ triangles which is invariant under the group $H^* = \langle \psi_q, \psi_s, \psi_u\rangle$. A fundamental region for this reflection group is the triangle bounded by these segments:

- the segment from vertex $R \in \mathbf{\Delta}_0$ to the midpoint of q,
- the segment from the midpoint of q to C_0, and
- the segment from C_0 back to R.

The reflections in the sides of this $(2,3,l)$ triangle, in the order given are ψ_q, ψ_s, and ψ_u, and the resulting generating triple is $(x, y, y^{-1}x^{-1})$, using the relations (18). The triple does generate H since

$$(xy)^2 = (\psi_q\psi_u)(\psi_q\psi_u) = \psi_q(\psi_u\psi_q\psi_u) = \psi_q\psi_p = a^{-1}$$

and the action of x and y on G given in (55) and (56). Note that the tiling $\overline{\mathcal{V}}$ refines both of $\overline{\mathcal{T}}$ and $\overline{\mathcal{U}}$

Example 6.4. Let us give a concrete example of the previous general example using $G = \mathbb{Z}_n \times \mathbb{Z}_n$, considered as an additive group, and with automorphism group $GL_2(\mathbb{Z}_n)$. For our generating triple we pick $a = (1,0), b = (0,1), c = (-1,-1)$. We need to find matrices $\theta_q, \theta_s, \theta_u$ that act upon (a,b,c) as required by Table 2 (24).The three matrices are:

$$\theta_q = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \ \theta_s = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \ \theta_u = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Of course this example is none other than the Fermat curve reduced to group computations.

6.2. **Actions with non-normalizing symmetries.** Suppose that ψ is a symmetry of S that does not normalize the G action. Then $G < H = \mathrm{Aut}(S)$ and the signature of H must come from the third column of Table 4. Except in the case of $N2$ and $N3$ with $d = 2e$, if the parameters d, e are chosen so that the signature of G is hyperbolic, then the triangle for H must be scalene. Thus, with these two possible exceptions ψ, must be Type I.

In Example 3.4 we produced a pair $G < H$ corresponding to case $NN1$ where some symmetries S normalized the G action and some did not. We also examined a tiling of the plane by $(2,3,7)$ triangles that tiled a $(7,7,7)$ triangle. In this case we say that the $(7,7,7)$ tiling is *divisible*. All such divisible pairs of triangles were found in [10], including pictures of the tile divisions. The fourth column of Table 5 indicates which pairs have divisible tilings (Type I only!). An analysis of each

divisible tiling pair, similar to Example 3.4 could be carried out. Though the topic is of interest a full analysis of all the cases is beyond the scope of this paper.

7. Search for orientable and unorientable surfaces with (l, m, n) tilings

7.1. **Symmetry search.** We are going to answer questions 1, 2 and 3 (empty mirrors) at the beginning of Section 3.1 for low genus surfaces. As a byproduct, this will allow us to find unorientable surfaces with (l, m, n) tilings. Specifically, when we classify the symmetries of surfaces with (l, m, n) actions of a small group G, the fixed point free symmetries yield unorientable surfaces with (l, m, n) tilings. So we will solve these problems: classify surfaces with small (l, m, n) action groups, that have normalizing symmetries, classify the normalizing symmetries of these surfaces, and build a collection of unorientable surfaces with (l, m, n) tilings.

In our search, instead of taking the genus σ to be a crude top level classifier, we will use the order $|G|$ of the action group G. Indeed, from the Riemann Hurwitz equation we obtain

$$\sigma = 1 + \frac{|G|}{2}\left(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}\right) \tag{57}$$

and

$$|G| = \frac{2\sigma - 2}{\left(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}\right)}. \tag{58}$$

For hyperbolic surfaces we have $\frac{1}{42} \leq \left(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}\right) < 1$, the lower bound achieved for $(l, m, n) = (2, 3, 7)$ (Hurwitz bound) and the upper bound comes from taking the limit as $l, m, n \to \infty$. We get the bounds below for hyperbolic surfaces with triangular action group, showing that σ and $|G|$ are comparable crude classifiers:

$$1 + \frac{|G|}{84} \leq \sigma < 1 + \frac{|G|}{2},$$
$$2(\sigma - 1) < |G| < 84(\sigma - 1).$$

In our "G first" approach to classification, we will adopt the following hierarchy of classification, which will dictate the steps of our algorithms. We will make substantial use of the $Symms(G)$ construct.

(1) $|G|$,
(2) G itself,
(3) conjugacy classes of involutions in $\mathrm{Aut}(G)$,
(4) two separate processes for Type I and Type II symmetries,
(5) generating triples corresponding to an involution,
(6) analysis of companions including signature, surface genus, and fixpoint-free status.

We are going to rephrase our previous work to set up our "G first" approach to classification. We split our statement into two theorems one for Type I symmetries and another for Type II symmetries.

Theorem 2. *Let G be a finite group and let θ be an involution in $Aut(G)$.*

(1) *The involution θ is a symmetry automorphism of some Type I symmetry ψ on a surface S of genus σ if and only if there is a generating triple (a, b, c) in G with signature (l, m, n) satisfying*

$$\theta(a) = a^{-1}, \theta(b) = b^{-1}. \tag{59}$$

(2) *The surface S may be constructed as in Remark 2.3. The signature of the action is $(l, m, n) = (o(a), o(b), o(c))$ and the genus of S is given by is given by (57).*
(3) *A second generating triple (a', b', c') satisfying the analogous condition to (59) defines an algebraically equivalent action of G on the same surface S if and only if $(a', b', c') = \omega \cdot (a, b, c)$ for some $\omega \in \mathrm{Cent}(\mathrm{Aut}(G), \theta)$.*
(4) *As in Remark 3.2, let $I_G(\theta) = \left\{g \in G : \theta(g) = g^{-1}\right\}$. Then generating triples satisfying (59) are*

$$\left\{(a, b, c) : a, b \in I_G(\theta), c = (ab)^{-1} \neq 1, G = \langle a, b\rangle\right\}.$$

(5) *The G action on S is tightly normalized by the family of Type I symmetries of S normalizing the G action.*
(6) *Let ψ_0 be a Type I symmetry inducing θ. Every Type I symmetry ψ of S, normalizing the action of G lies in $\langle \psi_0, G\rangle$ and has a factorization of the form $\psi = \psi_0 g$ where $g \in I_G(\theta)$. Furthermore, if $g \in I_G(\theta)$ then $\psi_0 g$ is a Type I symmetry.*
(7) *For every $h \in G$, and $g \in I_G(\theta)$ we have $\theta(h)gh^{-1} \in I_G(\theta)$. The map $(h, g) \to h \cdot g = \theta(h)gh^{-1}$ is a left action of G upon itself that maps $I_G(\theta)$ to itself. The G conjugacy classes of Type I symmetries are in 1-1 correspondence to the orbits of this action upon $I_G(\theta)$.*

The next Theorem applies to Type II symmetries.

Theorem 3. *Let G be a finite group and let θ be an involution in $Aut(G)$.*

(1) *The involution θ is a symmetry automorphism of some Type II symmetry ψ on a surface S of genus σ if and only if there is a generating triple (a, b, c) in G with signature (l, m, n) satisfying*

$$\theta(a) = b^{-1}, \theta(b) = a^{-1} \text{ and } l = m, \tag{60}$$

$$\theta(a) = c^{-1}, \theta(c) = a^{-1} \text{ and } l = n, \tag{61}$$

$$\theta(b) = c^{-1}, \theta(c) = b^{-1} \text{ and } m = n. \tag{62}$$

(2) *Same as previous theorem.*
(3) *Same as previous theorem.*
(4) *If the signature is (l, l, n) with $l \neq n$, then the generating triples satisfying (60) are*

$$\left\{(a, \theta(a^{-1}), c) : a \neq 1, c = \theta(a)a^{-1} \neq 1, G = \langle a, \theta(a)\rangle\right\}.$$

There are similar constructions for the cases (61) and (62).
(5) *In the isosceles, non-equilateral case, the G action on S is tightly normalized by the family of Type II symmetries of S normalizing the G action.*
(6) *Same as the previous theorem, replacing Type I with Type II.*
(7) *Same as the previous theorem.*

7.2. **Magma search for (l, m, n) tilings of orientable and unorientable surfaces.** Here are the steps of our search. We describe the steps in more detail after listing the steps.

(1) Pick G and find $\mathrm{Aut}(G)$.
(2) Find conjugacy classes $[\theta]$ of involutions θ in $\mathrm{Aut}(G)$. We also include the case $[\theta] = Id$ as this picks up dihedral actions on the sphere.
(3) For each class $[\theta]$ of involutions pick a representative θ and then compute the inverted elements $I_G(\theta) = \left\{g \in G : \theta(g) = g^{-1}\right\}$ and the centralizer $Z_G(\theta)$.
(4) Compute X_θ, the family of θ-compatible generating triples.
   (a) For Type I symmetries use elements of $a, b \in I_G(\theta)$, set $c = (ab)^{-1}$, and test that $a, b, c \neq 1$ and $G = \langle a, b\rangle$.
   (b) For Type II symmetries use elements from G. Pick $a \in G - \langle 1\rangle$, define $b = \theta(a^{-1})$, $c = (ab)^{-1}$, and test that $c \neq 1$ and $G = \langle a, b\rangle$.
(5) Compute the orbits of $Z_G(\theta)$ on X_θ in both cases above.
(6) For each representative (a, b, c) of an orbit found in the step above, compute the signature $(l, m, n) = (o(a), o(b), o(c))$ and the genus of S, using formula (57). In the Type I case, only record those triples for which (l, m, n) is properly ordered.
(7) Compute the orbits of G on $I_G(\theta)$ under the action $h \cdot g = \theta(h)gh^{-1}$.

Notes on the steps.

(1) For a comprehensive search, each group G should be selected from the SmallGroup database. The automorphism group $\mathrm{Aut}(G)$ can be automatically found by Magma. It is easier to perform computations for $\mathrm{Aut}(G)$ using a Magma supplied faithful permutation representation $\rho : \mathrm{Aut}(G) \to \Sigma_w$, for some suitable symmetric group.
(2) Use the permutation representation above to quickly find conjugacy classes of involutions, using standard Magma commands.
(3) See Remark 7.1 below for selecting a unique representative from a set $X \subseteq G$. The subgroup $Z_G(\theta)$ can be stored efficiently as a Magma group, but the inverted elements $I_G(\theta)$ are just a set, which is typically small.
(4) For each triple (a, b, c) constructed, θ is the standard Type I or Type II.a automorphism for the given triple. Moreover, all generating triples whose action has a normalizing symmetry are captured in this is way. For, if (a', b', c') defines an action with a normalizing symmetry then there is a standard involution θ' satisfying a "primed" version of (59) or (60). There is an $\omega \in \mathrm{Aut}(G)$ such that $\theta = \omega\theta'\omega^{-1}$ equals a unique chosen representative θ from our list, and θ is a standard automorphism with respect to $(a, b, c) = \omega \cdot (a', b', c')$. The two generating triples define the same surface. By the discussion on resolving ambiguities, we lose nothing by retaining only those triples with properly ordered signatures.

   The triples (a, b, c) and (a', b', c') are called *θ-related* if a relation such as the above holds for symmetries of the same type.
(5) If two triples (a, b, c) and (a', b', c') are constructed for the same θ and type, and $(a, b, c) = \omega \cdot (a', b', c')$ then $\omega \in Z_G(\theta)$. So, by choosing a unique triple (a, b, c) in the $Z_G(\theta)$ orbit of generating triples with standard involution θ, we are choosing a unique triple in $\mathrm{Aut}(G) \cdot (a', b', c')$.

(6) As noted in the preceding point we have captured each action pair $(S, (a, b, c))$ with ordered signature exactly once, up to algebraic equivalence. We compute signature and genus for completeness. Again, use Remark 7.1 below for selecting a unique representative generating triple from a set of triples $X \subseteq G^3$.
(7) Even though our calculations give a unique representative for each $\mathrm{Aut}(G)$ action class there is still some redundancy in the isosceles and equilateral cases. Namely, two algebraically inequivalent actions may yield conformally equivalent surfaces. See Example 7.1. Most of the redundancy has been eliminated by ordering the signature in the Type I cases and imposing $l = m$ in the Type II cases. To eliminate all redundancy, we would need to allow for permuting vertices with the same order. Since the goal to catalogue each symmetry at least once, eliminating isosceles and equilateral redundancies is more trouble than it is worth.
(8) We have not recorded any action class for which there are no symmetries. Our direct construction method form $Symms(G)$ leaves these actions out. We did not attempt to account for the lost actions.
(9) We refer to Remark 3.3 for determining the conjugacy classes of companion symmetries via the θ-twisted conjugation by G on $I_G(\theta)$.

Remark 7.1. Let X be a subset of the Cartesian product G^d. For a finite group G, Magma provides a numbering map $f : G \to I = \{1, \ldots, |G|\}$ which depends on the specific representation of G and its generating set. Extend $f : G^d \to I^d$ by $f(g_1, \ldots, g_d) = (f(g_1), \ldots, f(g_d))$. We can order the elements of G^d by lexicographically ordering their f values. A unique element of X can be chosen to be the minimum under this ordering. We will only consider sets $X \subseteq G$ and sets of generating triples $X \subset G^3$.

Example 7.1. As given in Table 2 in Section 3.2 the local q-action and r-action on (l, m, n) triples are

$$\theta_q(a, b, c) = (a^{-1}, b^{-1}, bc^{-1}b^{-1}),$$
$$\theta_r(a, b, c) = (b^{-1}a^{-1}b, b^{-1}, c^{-1}).$$

Now suppose that $\theta_q = \omega\theta_r\omega^{-1}$, then $(\omega(b), \omega(c), \omega(a))$ is a generating (m, n, l) triple (see Remark 3.4). We have

$$\begin{aligned}\theta_q(\omega(b), \omega(c), \omega(a)) &= \omega\theta_r(b, c, a)\\ &= \omega \cdot (b^{-1}, c^{-1}, b^{-1}ab)\\ &= (\omega(b)^{-1}, \omega(c)^{-1}, \omega(b^{-1}ab))\end{aligned}$$

It follows that $(\omega(b), \omega(c), \omega(a))$ is an (m, n, l) triple that satisfies the $\theta = \theta_q$ compatibility relations. Thus the triples (a, b, c) and $(\omega(b), \omega(c), \omega(a))$ will both be found in the Type I triple search. If the triangle is not equilateral then at most one of the triples will be recorded. If the triangle is equilateral then may be recorded even though they both yield conformally equivalent surfaces.

7.3. **Search results.** All our search results may be found at the website [5]. We present our Magma search results in aggregated form in Tables 5 and 6. The search was performed in 5 tranches of about 50 groups each to break up the computation and to show growth with group size. The range of group orders is recorded in the first column of both tables.

Each group has its own separate, appropriately named log file – yes almost 7000 files. Each file gives some detail of all the steps of the computation. Included in the log files are: all the basic group data; for each conjugacy class of involutions and each type, an accounting of the search for θ compatible generating triples; the signature and surface genus for each class of triples found; and an analysis of the θ slice of $Symms(G)$. The analysis of the $Symms(G)$ slices is the part of the calculation that allows us to find fixed point free symmetries. Throughout the calculations of each tranche, counters for data given in the tables are updated and a summary file is posted on the website for each tranche. There are more details in these files.

The Magma code also has a provision for analyzing a single group with more detail given, such as the formula for θ and the actual generating vector representatives. There is no restriction on the size of the group but the calculation tends to bog down for a group of size of $10,000$ or greater. The largest group in the log files is $Alt(9)$ with group order $181,440$ and about 2000 classes of actions with typical surface genus size in the range $20,000$ to $60,000$ range. The computation did however complete in a respectable amount of time. The larger groups show behaviours not found in smaller groups.

Many groups have no quasiplatonic actions or symmetries. The rank of the abelianization G_{ab} of G must satisfy $rk\, G_{ab} < 3$. Also $|\mathrm{Aut}(G)| \leq |G|^2$, since $\mathrm{Aut}(G)$ acts freely on the generating vectors. These conditions are tested before computing and, if found, no computation is done and "none" is appended to the title of the log file.

In Table 5 the "# groups" column is the number of groups tested, not the number of groups with symmetries. The "Type I" and "Type II" columns are the number of actions that have a θ compatible triple that satisfies equation (59) or (60), respectively. No attempt was made to account for actions satisfying both equations.

$\lvert G\rvert$	# groups	Type I	Type II	max genus
2-50	256	1126	318	24
51-100	791	3613	714	49
101-150	2843	6359	947	74
151-200	2183	9156	1321	99
201-250	895	10803	1229	124
total	6959	31057	4529	

Table 5. Surfaces (actions) with symmetries

In Table 6, the number of G conjugacy classes of symmetries is recorded. The columns headed by "#SS" are *standard symmetry* classes (with ovals) and the columns headed "#FPF" are *fixed point free symmetry* classes (no ovals).

	SS symms		FPF symms	
$\lvert G\rvert$	Type I	Type II	Type I	Type II
2-50	1918	318	159	172
51-100	6358	714	656	399
101-150	11429	947	1295	570
151-200	16529	1321	1948	800
201-250	18523	1229	1662	762
total	54748	4529	6720	2703

Table 6. Classes of symmetries

7.4. **Some observations and questions.** We conclude our paper with some observations and comments gleaned from an initial examination of the log files.

(1) For completeness, the search picks up symmetric actions on genus 0 and genus 1 surfaces. They are a small percentage of the total number of symmetric actions

(2) The number of elements in $I_g(\theta)$ tends to be small except for Type I symmetries for abelian groups. Likewise, the number of G orbits on $I_g(\theta)$ tends to be small, even in the abelian case. If there are fixed point free symmetries there is typically only one class. For $Alt(9)$ there are symmetries with two different classes of fixed point free symmetries.

(3) Typically, for a given symmetry automorphism, the number of fixed point free symmetry classes is the same no matter what the generating vector is. However for $Alt(8)$ and $Alt(9)$ there are symmetry automorphisms for which the number of fixed point free symmetry classes are different for different generating vectors. For example, there are two $(4, 4, 6)$ actions of Alt(8) on surfaces of genus 3361, where one action has no fixed point free actions and the other has one class of fixed point free actions. We made no attempt to find a minimal case.

(4) The search flags whether a symmetry automorphism is inner or not. If an automorphism is inner then, except in dihedral case, there will be a fixed point free symmetry. See Proposition 3. However there are plenty of cases where the outer automorphisms have fixed point free symmetries.

(5) The isometry groups of the unoriented surfaces tend to be much smaller than action groups of the orientation covering surface. For instance $PSL(2, 8)$ has an involution θ for which all of the 46 classes of actions have fixed point free symmetries. The subgroup $Z_G(\theta) \hookrightarrow \mathrm{Aut}(M)$ for all of these surfaces has size 24 even though the action group size is 504. The index of $Z_G(\theta)$ in G is 21. Again we have not taken into account redundant actions and "extra" automorphisms.

(6) Hurwitz surfaces have a $(2, 3, 7)$ action group, which must be the entire automorphism of the surface and is the maximal size for a symmetry group. These surfaces must have an invariant $(2, 3, 7)$ tiling whether or not the surface is symmetric. Because the isometry group of an unorientable surface is much smaller than the corresponding action group, then using maximal group size to define Hurwitz unorientable surfaces does not wok well. So, we could call an unorientable surface a Hurwitz surface if it has a $(2, 3, 7)$ tiling? There is extensive work on Hurwitz $PSL(2, q)$ groups, see [6] for

example. The smallest $PSL(2,q)$ action group with signature $(2,3,7)$ and with a fixed point free symmetry is $PSL(2,8)$ with a Type I symmetry on a surface of genus 7, the next is $PSL(2,13)$ acting on a surface of genus 14.

(7) In all the orientable cases a q-kite is a fundamental region for the G action. In fact the union of a black and a white triangle is a fundamental region. Therefore there are $2\,|G|$ triangles which are identified in two colour pairs to form $|G|$ triangles on the quotient. Therefore, in the unorientable case we need

$$\frac{|G|}{|Z_G(\theta)|}$$

monochrome triangles for form a fundamental region. An interesting question is to draw such a fundamental region in the plane. In the $PSL(2,8)$ Hurwitz case $504/24 = 21$ such triangles are needed.

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