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RESTRICTED PARTITIONS AND SL_2-COHOMOLOGY

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Abstract. The aim of this paper is twofold. First, we study the number of partitions of a positive integer m into at most n parts in a given set A. We prove that such a number is bounded by the n-th Fibonacci number $F(n)$ for any m and some family of sets A including sets of powers of an integer. Then, in the second part of the paper, we provide new results in bounding the cohomology of the simple algebraic group SL_2 with coefficients in Weyl modules.

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1. Introduction

Let A be a subset of $\mathbb{Z}^+$ and m in $\mathbb{Z}^+$. A *restricted partition* of m with parts in A is a decomposition

$$m = \alpha_1 + \alpha_2 + \cdots + \alpha_t \tag{1}$$

where α_is are not necessarily distinct elements in A and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_t$. Each α_i is called a *part* of the partition, and t is the number of parts in the partition. For example, let $A_2 = \{2^i : i \in \mathbb{N}\}$ the set of all powers of 2. Then 5 has 4 restricted partitions in A_2: $1+1+1+1+1, 1+1+1+2, 1+2+2$, and $1+4$. These are called 2-ary partitions of 5. In this paper, we are interested in restricted partitions of the form (1) where t is at most n for some given positive integer n. We call them partitions of m into at most n parts from A and denote $P(m,n,A)$ the number of such partitions. Note that when $A = \mathbb{Z}^+$, i.e., there is no restriction for parts of partitions, computing $P(m,n,A)$ is a classical problem. Some first results on small values of n date back to the 19^{th} century by Herschel, Cayley, and Sylvester [WW]. Recently, some progress has been made by [K],[M], and [O]. In terms of an arbitrary set A, not much is known for $P(m,n,A)$. One may refer to [CW] and [A] for a discussion on the number of some related partitions. In Section 2 we show that for any set A satisfying a mild growth condition, $P(m,n,A)$ is bounded from above by the n-th Fibonacci number $F(n)$ for all $m,n \in \mathbb{Z}^+$. This bound is universal in the sense that it does not depend on the integer m. As a consequence, our result deduces an upper bound for the number of q-ary partitions of m into at most n parts for any positive integer $q \geq 2$.

Our work on restricted partitions is motivated by studying the cohomology of the simple algebraic group SL_2 defined over an algebraically closed field k of prime characteristic p. Note that, in general, the cohomology of algebraic groups is widely

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unknown and only a few cases have been computed explicitly, see [Jan] for a summary of this theory. Although the case of SL_2 has been extensively studied, there are still open problems. For example, determining a closed formula of the dimension of the cohomology for a simple (or indecomposable) module remains unknown. In fact, it is already a challenging task to find a sharp upper bound for this number. For the last few years, the third author has been interested in bounding the dimension of $\mathrm{H}^n(SL_2, V(m))$, the SL_2-cohomology for the Weyl module $V(m)$ of highest weight $m \in \mathbb{N}$. This problem is of independent interest and has its own history [EHP], [LNZ]. Together with Lux and Zhang, the third author was able to identify $\dim \mathrm{H}^n(SL_2, V(m))$ with the number of solutions to a system of linear equations, hence attaining a rough upper bound, see [LNZ, Section 4] for details. In Section 3 we use results from Section 2 to show that for any $m, n \in \mathbb{N}$

$$\dim \mathrm{H}^n(SL_2, V(m)) \leq \begin{cases} F(n+1) & \text{if } p \geq 5, \\ F(2n) & \text{if } p = 2, 3 \end{cases}$$

which significantly improves the bounds in [LNZ, Cor. 4.3, Prop. 4.4, and Thm. 4.6].

It is worth noting that there is a desire of finding explicit universal bounds (only depending on the degree n) for any simple algebraic group, see [1] et al. for a survey on this open problem. Currently, we are not able to generalize our results (for SL_2) to arbitrary algebraic groups. However, we suspect that there might be some connection between the dimension of cohomology of an algebraic group and the number of restricted vector partitions[1] (a generalization of restricted integer partitions). Hence, it is reasonable to ask whether there exists a universal upper bound for the latter. This would be an interesting problem for future research.

2. Restricted partitions of m into at most n parts

In this section, we study the number $P(m, n, A)$ for various sets A. We aim to bound this number using the Fibonacci numbers. For the sake of our calculations, we first identify these partitions with integer solutions of a certain system of equations. In particular, we write $A = \{a_1, a_2, \ldots\}$ with $0 < a_1 < a_2 < \cdots$. Let $m \in \mathbb{Z}^+$ and a_r the largest number in A that is no more than m. Then each restricted partition of the form (1) can be rewritten as

$$m = x_1 a_1 + x_2 a_2 + \cdots + x_r a_r$$

where x_is are non-negative integers. Each x_i is called the *multiplicity* of a_i in the partition. As multiplicities are allowed to be zeros, every partition of m is uniquely determined by a sequence $\{x_i\}$ satisfying

$$m = \sum_{i=1}^{\infty} x_i a_i$$

where $x_i = 0$ for large enough i (such that $a_i > m$). Thus, the number of restricted partitions of m is equal to the number of such sequences $\{x_i\}$. Now, if we require

[1]A vector partition is a way of writing a vector with nonnegative integer entries as a sum of other vectors (with nonnegative integer entries) where the order of summands does not matter.

the number of parts in every partition of m to be at most n, then $P(m,n,A)$ is in fact equal to the number of sequences $\{x_i\}$ of non-negative integers such that

$$\begin{cases} \sum_{i=1}^{\infty} x_i a_i = m, \\ \sum_{i=1}^{\infty} x_i \leq n. \end{cases}$$

Solutions to the last system are essentially the same as tuples $(x_1, x_2, \dots, x_r)$ satisfying

$$\begin{cases} x_1 a_1 + x_2 a_2 + \dots + x_r a_r = m, \\ x_1 + x_2 + \dots + x_r \leq n \end{cases} \tag{2}$$

for sufficiently large r. From now on, we identify $P(m,n,A)$ with the number of solutions to the system (2).

We next recall the definition of Fibonacci sequence. Let

$$F(1) = 1, \quad F(2) = 1, \quad F(n) = F(n-1) + F(n-2),$$

for $n \geq 2$. We also assume $F(i) = 0$ for all integers $i \leq 0$. Our inductive proofs will use the following special property of the Fibonacci sequence.

Lemma 2.1. *For each positive integer n, we have*

$$\sum_{i=0}^{n} F(2i+1) = F(2n+2) \quad \text{and} \quad \sum_{i=0}^{n} F(2i) = F(2n+1) - 1.$$

Consequently, we always have

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} F(n-2i) \leq F(n+1).$$

Proof of the lemma is straightforward. We now prove the following

Theorem 2.2. *Let $A = \{a_1, a_2, \dots\} \subseteq \mathbb{Z}^+$ where all a_is are written in an increasing order and satisfy*

$$2a_{s-1} + 4a_{s-2} + 6a_{s-3} + \dots + 2(s-1)a_1 < a_s \tag{3}$$

for all $s \geq 2$. Then $P(m,n,A) \leq F(n)$ for all positive integers m, n.

Proof. We prove by induction on n. Obviously, $P(m,1,A) = 0$ or 1, so that it is true for $n = 1$. Let's take a look further to the case $n = 2$ [2]. We also have $P(m,2,A) \leq 1$ because for each m and a_s, the largest number in A such that $a_s \leq m$, the condition (3) guarantees that a_s is not equal to $2a_i$ nor $a_i + a_j$ for any $1 \leq i,j \leq s-1$. Assume that $P(m,t,A) \leq F(t)$ for any $t < n$ with $n \geq 3$. Now fix m and again let a_s be the largest number in A such that $a_s \leq m$. Since $2a_{s-1} + 4a_{s-2} + 6a_{s-3} + \dots + 2(s-1)a_1 < a_s \leq m$, we must have either one (or more) of the following

$$x_s \geq 1, x_{s-1} \geq 3, x_{s-2} \geq 5, \dots, x_1 \geq 2s-1.$$

[2] Since $F(1) = F(2) = 1$, it is necessary to show the base case with $n = 1, 2$, for otherwise we would have no idea whether $P(m,1,A)$ is bounded by $F(1)$ or $F(2)$. We will need to do the same for other inductive proofs in this paper.

Solutions of (2) combined with $x_s \geq 1$ are essentially solutions of

$$\begin{cases} x_1a_1 + x_2a_2 + \cdots + x_{s-1}a_{s-1} + x'_s a_s = m - a_s, \\ x_1 + x_2 + \cdots + x_{s-1} + x'_s \leq n-1 \end{cases}$$

by setting $x_s = 1 + x'_s$. The number of such solutions is $P(m - a_s, n-1, A)$. Similarly, numbers of solutions to (2) with other conditions are respectively $P(m - 3a_{s-1}, n-3, A), \ldots,$ and $P(m-(2s-1)a_1, n+1-2s, A)$. Hence, we obtain

$$\begin{aligned} P(m,n,A) \leq & P(m-a_s, n-1, A) + P(m-3a_{s-1}, n-3, A) + \\ & \cdots + P(m-(2s-1)a_1, n+1-2s, A). \end{aligned}$$

Next, applying our inductive hypothesis and Lemma 2.1 we have

$$P(m,n,A) \leq F(n-1) + F(n-3) + \cdots + F(n+1-2s) \leq F(n),$$

hence completing our inductive proof. □

We present some examples of the set A in the above theorem.

Theorem 2.3. *Let q be an integer greater than 3 and set $A_q = \{q^i : i \in \mathbb{N}\}$. Then A_q satisfies the condition (3). Consequently,*

$$P(m,n,A_q) \leq F(n).$$

Proof. It suffices to show that for any $s \geq 1$

$$2q^{s-1} + 4q^{s-2} + \cdots + 2(s-1)q + 2s < q^s.$$

This can be proven by induction on s. Indeed, it is easy to see that it's true for $s = 1$. Now to show that

$$2q^s + 4q^{s-1} + \cdots + 2sq + 2(s+1) < q^{s+1},$$

note that

$$\begin{aligned} 2q^s + & 4q^{s-1} + \cdots + 2sq + 2(s+1) \\ & = 2q^s + (2q^{s-1} + 4q^{s-2} + \cdots + 2(s-1)q + 2s) + (2q^{s-1} + 2q^{s-2} + \cdots + 2q + 2) \\ & < 2q^s + q^s + 2(q^{s-1} + q^{s-2} + \cdots + q + 1) \text{ by induction} \\ & \leq 3q^s + q^s - 1 < 4q^s \leq q^{s+1} \text{ using } q > 3 \text{ at multiple points.} \end{aligned}$$

This proves our induction proof. The remainder follows immediately from Theorem 2.2. □

Remark 2.4. The sets A_2 and A_3 do not satisfy (3). Moreover, the inequality in the above theorem doesn't hold for A_2 as we have

$$4 = 1 \cdot 2^2 = 2 \cdot 2 = 2 \cdot 1 + 1 \cdot 2.$$

Hence, $P(4, 3, A_2) = 3 > F(3)$.

We next modify the condition on the set A in the Theorem 2.2 so that A_2 and A_3 will satisfy it.

Theorem 2.5. *Let $A = \{a_1, a_2, \ldots\} \subseteq \mathbb{Z}^+$ where all a_is are written in an increasing order and satisfy*

$$a_{s-1} + 2a_{s-2} + 3a_{s-3} + 4a_{s-4} + \cdots + (s-1)a_1 < a_{s+1} \tag{4}$$

for all $s \geq 2$. Then $P(m,n,A) \leq F(2n-1)$ for all positive integers m, n.

Proof. It is straightforward to check that it is true for $n = 1, 2$. Assume that $P(m, t, A) \leq F(2t - 1)$ for any $t < n$ with $n \geq 3$. Now fix m and let a_{s+1} be the largest number in A such that $a_{s+1} \leq m$. From the condition (4), we must have either one (or more) of the following

$$x_{s+1} \geq 1, x_s \geq 1, x_{s-1} \geq 2, x_{s-2} \geq 3, \ldots, x_1 \geq s.$$

Now the argument goes very similar with that of Theorem 2.2. We then obtain

$$P(m, n, A) \leq P(m - a_{s+1}, n-1, A) + P(m - a_s, n-1, A) + \cdots + P(m - sa_1, n-s, A).$$

Next, applying our inductive hypothesis and Lemma 2.1 we have

$$\begin{aligned} P(m, n, A) &\leq F(2n-3) + F(2n-3) + F(2n-5) + \cdots + F(2n-2s-1) \\ &\leq F(2n-3) + F(2n-2) = F(2n-1), \end{aligned}$$

hence completing our inductive proof. □

Corollary 2.6. *For $m, n \in \mathbb{Z}^+$, we have $P(m, n, A_q) \leq F(2n-1)$ for $q \geq 2$.*

Proof. From the last theorem, it suffices to show that A_q satisfies the condition (4) with $q \geq 2$. Indeed, we claim that for any $r \in \mathbb{N}$,

$$\sum_{i=1}^{r+1} i \cdot q^{r+1-i} < q^{r+2}.$$

Proceeding by induction on r, the base case $r = 0$ is obviously true. Assume inductively that the inequality holds for some r. Now observe that

$$\sum_{i=1}^{r+2} i \cdot q^{r+2-i} = \sum_{i=1}^{r+1} i \cdot q^{r+1-i} + \sum_{j=0}^{r+1} q^{r+1-j} \leq \sum_{i=1}^{r+1} i \cdot q^{r+1-i} + q^{r+2} - 1.$$

By the inductive hypothesis, the last term is less than $q^{r+2} + q^{r+2} - 1 < q^{r+3}$; hence completing our inductive proof. □

Remark 2.7. Since $F(n) \leq F(2n-1)$, Theorem 2.3 immediately implies Corollary 2.6 for $q > 3$. It can also be observed that condition (4) is weaker than (3), i.e., that any set A satisfying condition (3) also satisfies condition (4). The weaker condition allows one to consider a larger collection of sets A, particularly A_2 and A_3, but at the expense of a bound on $P(m, n, A)$ that is not as good as the original.

We end this section with an interpretation of our results in terms of q-ary partitions, which are partitions of an integer into powers of q. It follows that $P(m, n, A_q)$ is the number of q-ary partitions of m into at most n parts. Hence, rephrasing the above results, we obtain the following

Corollary 2.8. *For $m, n \in \mathbb{Z}^+$, the number of q-ary partitions of m into at most n parts is no more than $F(n)$ if $q \geq 4$ or $F(2n-1)$ if $q = 2$ or 3.*

3. Cohomology of SL_2

The goal in this section is to bound the dimension of cohomology for Weyl modules over SL_2 using results from the previous section. For general background of rational cohomology of algebraic groups, the audience may refer to [Jan]. We only introduce here necessary material for our calculations. We use the same notation and conventions as in [LNZ]. In particular, let k be an algebraically closed field of prime characteristic $p > 0$. We fix $G = SL_2$ defined over k and a torus subgroup T

of G. Then the set of dominant weights associated with T can be identified with $\mathbb{N}$. Weyl modules (over SL_2) are indecomposable modules that are parametrized by dominant weights. Explicitly, we denote $V(m)$ the Weyl module of highest weight m for each $m \in \mathbb{N}$ with $V(0) = k$ the trivial module.

For G-modules M and N, $\mathrm{Ext}^n_G(M, N)$ is the n-th degree extension space of M by N. When $M = k$, this space is called the n-th cohomology space of G with coefficients in N and denoted $\mathrm{H}^n(G, N)$. The notation $\dim \mathrm{H}^n(G, N)$ (or $\dim \mathrm{Ext}^n_G(M, N)$) denotes the dimension of the cohomology (or extension) as a vector space over k. We are interested in estimating an upper bound for these quantities.

We recall from [LNZ, Theorem 4.2] that if p is an odd prime, then for any integers $m, n \geq 0$ the dimension $\dim \mathrm{H}^n(SL_2, V(m))$ is equal to the number of solutions to the system

$$\begin{cases} 2\displaystyle\sum_{i=1}^{r} a_i + \sum_{j=1}^{r} b_j = n+1, \\ b_1 + \displaystyle\sum_{i=1}^{r-1}(a_i + b_{i+1})p^i + a_r p^r = \frac{m}{2} + 1, \end{cases} \tag{5}$$

where all a_is are in $\mathbb{N}$, b_i is either 0 or 1.[3] Here r is a sufficiently large integer in term of m. Note that the system (5) has no solutions if m is odd. Therefore, whenever considering the cohomology $\mathrm{H}^n(SL_2, V(m))$ we are only interested in the case when m is even.

Let $N(m, n)$ be the number of solutions to the system

$$\begin{cases} 2\displaystyle\sum_{i=1}^{r} a_i + \sum_{j=1}^{r} b_j = n, \\ b_1 + \displaystyle\sum_{i=1}^{r-1}(a_i + b_{i+1})p^i + a_r p^r = m. \end{cases} \tag{6}$$

Then we can deduce that $\dim \mathrm{H}^n(SL_2, V(m)) = N\left(\dfrac{m}{2} + 1, n+1\right)$ for $m, n \in \mathbb{N}$. We next prove the main result of this section, which strengthens [LNZ, Proposition 4.4] as we are now able to remove the condition $n \leq 2p - 3$.

Theorem 3.1. *Assume $p \geq 5$. For all integers $m, n \geq 0$, we have*

$$\dim \mathrm{H}^n(SL_2, V(m)) \leq F(n+1).$$

Proof. From earlier set up, we need to prove that $N\left(\frac{m}{2} + 1, n+1\right) \leq F(n+1)$ for every integer $n \geq 0$ and even integer $m \geq 0$. This is then reduced to showing that $N(m, n) \leq F(n)$ for all positive integers m, n (with m replacing $\frac{m}{2} + 1$ and n replacing $n+1$). We again proceed by an inductive argument on n. By [LNZ, Proposition 3.6], the last inequality holds for $n \leq 8$. For any positive $n \geq 9$, assume that the inequality holds up to $n - 1$. Since b_1 is either 0 or 1, we must have m is congruent to either 0 or 1 modulo p, for otherwise, $N(m, n) = 0$ for all n. If $m \equiv 1 \pmod{p}$, then $b_1 = 1$ and $N(m, n) = N(m-1, n-1) \leq F(n-1)$ by the

[3]Note that there is an abuse of notation here. All the a_is are now not elements of A as in the previous part of the paper. Instead, these a_is (and b_is) are variables in this context.

inductive hypothesis. Suppose that p divides m, so b_1 must be zero. Let s be the least integer such that $p^s \le m$. Then the system (6) is reduced to

$$\begin{cases} 2\sum_{i=1}^{s} a_i + \sum_{j=2}^{s+1} b_j = n, \\ \sum_{i=1}^{s} (a_i + b_{i+1}) p^i = m. \end{cases} \tag{7}$$

Let S be the set of solutions to this system. Set

$$U = \{(a,b) = (a_1,\dots,a_s,b_2\dots,b_{s+1}) : (a,b) \text{ satisfies } (7), \\ \exists i_0 \text{ so that } a_{i_0} \ge 1, b_{i_0+1} = 0\}.$$

Then S is the union of the disjoint subsets U and $V = S \setminus U$. We now give an upper bound to each set. For each solution (a,b) in U, we choose the largest such i_0 and make a replacement $a_{i_0} \mapsto a_{i_0} - 1$ and $b_{i_0+1} \mapsto 1$. The resulting tuple is a solution of

$$\begin{cases} 2\sum_{i=1}^{s} a_i + \sum_{j=2}^{s+1} b_j = n-1, \\ \sum_{i=1}^{s} (a_i + b_{i+1}) p^i = m. \end{cases}$$

This replacement is a one-to-one mapping from U to the set of solutions to the system above. It follows from the inductive hypothesis that

$$|U| \le N(m, n-1) \le F(n-1).$$

Every solution (a,b) in V satisfies the condition that whenever $a_i > 0$, $b_{i+1} = 1$. Setting $d_i = a_i + b_{i+1}$ for all $1 \le i \le s$, we can see that each solution (a,b) in V is one-to-one corresponding [4] to a solution $(d_1,\dots,d_s)$ to the system

$$\begin{cases} \sum_{i=1}^{s} d_i \le n, \\ \sum_{i=1}^{s} d_i p^i = m, \end{cases} \tag{8}$$

where the inequality is obtained from rewriting the first equation of (7) to

$$\sum_{i=1}^{s} d_i = n - \sum_{i=1}^{s} a_i.$$

Consider the following cases.

- If $\sum_{i=1}^{s} a_i \le 2$ then $d_i \le a_i + 1 \le 3 < p$ for all $1 \le i \le s$ (recall that $p \ge 5$).

  Hence, the sum $\sum_{i=1}^{s} d_i p^i$ must be the p-adic expansion of m and so there is

  at most one solution to the system (8) for the case when $n-2 \le \sum_{i=1}^{s} d_i \le n$.

[4]The inverse is defined by setting $b_{i+1} = 1$ and $a_i = d_i - 1$ if $d_i > 0$, otherwise, $a_i = b_{i+1} = 0$.

- If $\sum_{i=1}^{s} a_i \geq 3$ then $\sum_{i=1}^{s} d_i = n - \sum_{i=1}^{s} a_i \leq n-3$. By Theorem 2.3, the number of solutions to (8), restricted to this case, is bounded by $F(n-3)$.

Summing up the two cases, we have $|V| \leq 1 + F(n-3) \leq F(n-2)$. Therefore, we obtain

$$N(m,n) = |S| = |U| + |V| \leq F(n-1) + F(n-2) = F(n),$$

which completes our inductive proof. □

We believe that the theorem also hold for $p = 3$. Unfortunately, our method does not work with this small prime. A different approach might be needed to tackle this case. Using the same idea as in the previous section, we can only prove the following

Proposition 3.2. *If $p = 3$, then we have for all integers $m \geq 0, n \geq 1$*

$$\dim \mathrm{H}^n(SL_2, V(m)) \leq F(2n).$$

Proof. Same argument as in the proof of the last theorem, we reduce to showing that $N(m,n) \leq F(2n-2)$ for all integers $m \geq 1, n \geq 2$. Recall that $N(m,n)$ is the number of solutions to the system (7). Again, by [LNZ, Proposition 3.6], the inequality holds for $n \leq 4$. For $n \geq 5$, we assume that the inequality holds up to $n-1$. For (7) to have solutions, we must have either one (or more) of the following conditions

$$a_s + b_{s+1} \geq 1, a_{s-1} + b_s \geq 2, a_{s-2} + b_{s-1} \geq p+1, a_{s-3} + b_{s-2} \geq p+2,$$
$$a_{s-4} + b_{s-3} \geq p^2 - p + 1, a_{s-5} + b_{s-4} \geq p^2 + 1, a_{s-6} + b_{s-5} \geq p^3 + 1,$$

and $a_{s-i} + b_{s-i+1} \geq p^{\lfloor \frac{i}{2} \rfloor} + 1$ for all $i \geq 6$. For otherwise, we would have

$$\sum_{i=1}^{s} (a_i + b_{i+1}) p^i \leq 2(p^{s-1} + p^{s-2} + \cdots + 1) \leq p^s - 1 < p^s \leq m.$$

Let N_i be the number of solutions to the system (7) restricted to each of these conditions respectively. Then it is easy to see that $N(m,n)$ is no more than the sum of all these N_is. We next consider each N_i.

If $a_s + b_{s+1} \geq 1$, then there are 2 cases:

- $b_{s+1} = 0$ and $a_s \geq 1$. Then the system (7) can be rewritten to

$$\begin{cases} 2\sum_{i=1}^{s-1} a_i + \sum_{j=2}^{s} b_j = n - 2a_s, \\ \sum_{i=1}^{s-1} (a_i + b_{i+1}) p^i = m - a_s p^s. \end{cases}$$

  Hence, by the inductive hypothesis the number of solutions in this case is

$$\sum_{a_s=1}^{\lceil \frac{m}{p^s} \rceil} N(m - a_s p^s, n - 2a_s) \leq \sum_{a_s=1}^{\lceil \frac{m}{p^s} \rceil} F(2n - 4a_s - 2).$$

- $b_{s+1} = 1$ and $a_s \geq 0$. Then the system (7) can be rewritten to

$$\begin{cases} 2\sum_{i=1}^{s-1} a_i + \sum_{j=2}^{s} b_j = n - 1 - 2a_s, \\ \sum_{i=1}^{s-1} (a_i + b_{i+1})p^i = m - p^s - a_s p^s. \end{cases}$$

Again, by the inductive hypothesis the number of solutions in this case is

$$\sum_{a_s=0}^{\lceil \frac{m}{p^s} \rceil} N(m - p^s - a_s p^s, n - 1 - 2a_s) \leq \sum_{a_s=0}^{\lceil \frac{m}{p^s} \rceil} F(2n - 4a_s - 4).$$

Now summing up theses 2 cases and using Lemma 2.1, we have

$$N_1 \leq \sum_{i=2}^{\lceil \frac{m}{p^s} \rceil} F(2n - 2i) \leq F(2n - 3).$$

If $a_{s-1} + b_s \geq 2$ then $a_{s-1} \geq 1$. Now replacing a_{s-1} by $a'_{s-1} + 1$ with $a'_{s-1} \geq 0$ we have (7) rewritten to

$$\begin{cases} 2\sum_{i=1}^{s-2} a_i + 2a'_{s-1} + 2a_s + \sum_{j=2}^{s+1} b_j = n - 2, \\ \sum_{i=1}^{s-2} (a_i + b_{i+1})p^i + (a'_{s-1} + b_s)p^{s-1} + (a_s + b_{s+1})p^s = m - p^{s-1}. \end{cases}$$

Hence, there are $N_2 = N(m - p^{s-1}, n - 2)$ solutions in this case. Similar argument can be applied to obtain

- $N_3 = N(m - p^{s-1}, n - 2p)$
- $N_4 = N(m - p^{s-2} - p^{s-3}, n - 2p - 2) \cdots$
- $N_s = N(m - p^{\lfloor \frac{s}{2} \rfloor}, n - 2p^{\lfloor \frac{s}{2} \rfloor})$.

Now using the inductive hypothesis and Lemma 2.1, we obtain

$$\begin{aligned} N(m, n) &\leq F(2n - 3) + F(2n - 6) + F(2n - 4p - 2) + \cdots + F(2n - 4p^{\lfloor \frac{s}{2} \rfloor} - 2) \\ &\leq F(2n - 2), \end{aligned}$$

completing our inductive proof. □

Remark 3.3. Theorem 3.1 does not hold for $p = 2$. Indeed, from [EHP, Corollary 3.2.2], the dimension of $\mathrm{H}^n(SL_2, V(m))$, for any $m, n \in \mathbb{N}$, is equal to the number of solutions $(a_1, \ldots, a_r) \in \mathbb{N}^r$ of the system

$$\begin{cases} a_1 + a_2 + \cdots + a_r = n + 1, \\ a_1 2 + a_2 2^2 + \cdots + a_r 2^r = \frac{m}{2} + 1. \end{cases} \tag{9}$$

In the case when $m = 286, n = 4$, there are 6 solutions to the system (9) as follows

- $1 \cdot 2^4 + 4 \cdot 2^5 = 144$
- $3 \cdot 2^4 + 1 \cdot 2^5 + 1 \cdot 2^6 = 144$
- $2 \cdot 2^3 + 2 \cdot 2^5 + 1 \cdot 2^6 = 144$
- $2 \cdot 2^2 + 1 \cdot 2^3 + 2 \cdot 2^6 = 144$
- $4 \cdot 2^2 + 1 \cdot 2^7 = 144$
- $2 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3 + 1 \cdot 2^7 = 144$.

Hence, $\dim \mathrm{H}^4(SL_2, V(286)) = 6 > F(5)$. Instead, we can have the same bound as for the case $p = 3$ as follows.

Corollary 3.4. *If $p = 2$, then for all integers $m \geq 0, n \geq 1$*

$$\dim \mathrm{H}^n(SL_2, V(m)) \leq F(2n).$$

Proof. By Corollary 2.6, it's straightforward to have

$$\dim \mathrm{H}^n(SL_2, V(m)) \leq P\left(\frac{m}{2} + 1, n+1, A_2\right) \leq F(2n+1).$$

In fact, Corollary 2.6 implies that

$$\sum_{i=0}^{n} \dim \mathrm{H}^n(SL_2, V(m)) \leq F(2n+1).$$

It is possible to lower the bound for $\dim \mathrm{H}^n(SL_2, V(m))$ to $F(2n)$. Indeed, let $M(\frac{m}{2}+1, n+1)$ be the number of solutions to the system (9). Similar inductive argument as in the proof of Theorem 2.5 may be used to establish

$$M(m, n) \leq F(2n-2)$$

for all integers $m \geq 1, n \geq 2$, which is sufficient to show the corollary. □

Remark 3.5. Using exact same proof for [LNZ, Theorem 5.4] with Theorem 3.1 replacing [LNZ, Proposition 4.4] (following that the condition $n \leq 2p-3$ can be removed), we can prove that for $p \geq 5$

$$\dim \mathrm{Ext}^n_{SL_2}(V(m_2), V(m_1)) \leq F(n+1) + (s-1)F(n).$$

for $m_1, m_2, n \in \mathbb{N}$, and s the least positive integer such that $m_2 < p^s$. This is not a significant bound as it is not sharp even for the low degree n. For example, we have

$$\dim \mathrm{Ext}^n_{SL_2}(V(m_2), V(m_1)) \leq n$$

for $n \leq 3$, see [LNZ, Section 5.1] for details. Finding a sharp bound, for large values of n, of these extension spaces is still an open problem. We propose the following

Conjecture 3.6. *For $m_1, m_2 \in \mathbb{N}$, and $p \geq 3$, we have*

$$\dim \mathrm{Ext}^n_{SL_2}(V(m_2), V(m_1)) \leq F(n+1).$$

Remark 3.7. The same arguments as in [LNZ, Section 6.2] can show, in the case when $p \geq 5$, that both $\dim \mathrm{H}^n(SL_2, L)$ and $\dim \mathrm{H}^n(SL_2(\mathbb{F}_{p^s}), L')$ are bounded by $(2n+7)F(n)$, where L (resp. L') is any simple module over SL_2 (resp. the finite group of Lie type $SL_2(\mathbb{F}_{p^s})$ for any $s \geq 1$). Again, this is an improvement of results in [LNZ, Section 6.2], but it is not a sharp upper bound even with small values of n.

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