ALBANIAN JOURNAL OF MATHEMATICS
Volume 6, Number 2, Pages 87–102
ISSN: 1930-1235; (**2012**)

ON GENERALIZED GAUSSIAN NUMBERS

Esengül Saltürk
Yildiz Technical University
Department of Mathematics,
34210 stanbul-Turkey
Email: esalturk@yildiz.edu.tr

İrfan Şiap
Yildiz Technical University
Department of Mathematics,
34210 stanbul-Turkey
Email: isiap@yildiz.edu.tr

Abstract. We establish some new properties and identities of Generalized Gaussian Numbers (GGN) which are defined recently in [10, 11] parallel to those of Gaussian coefficients. We present generating functions and some properties which are very useful for GGN. We obtain some family of sequences which are unimodal and present the log-concavity property of GGN. Finally, we give a connection of GGN to the Rogers-Szego polynomials.

1. Introduction

In enumerative combinatorics, binomial coefficients and Gaussian coefficients are very important class of fields of studies. While binomial coefficients have interpretation in terms of subset selection, Gaussian coefficients have a classical interpretation related to counting subspaces of a finite vector space. Binomial and Gaussian coefficients are already well studied and well discussed in the past (see [3, 6, 7, 13]).

Recently, Generalized Gaussian Numbers (GGN) which in a special case give Gaussian coefficients are defined and some of their properties parallel to those of Gaussian coefficients are established. Moreover, GGN have an interpretation related to the counting of submodules of a finite module [10, 11]. In [10, 11], the

1991 *Mathematics Subject Classification.* Primary: 32A22 ; Secondary: 14J99.
Key words and phrases. Order of entire functions.

authors enumerate the codes over finite rings and define new number sequences by GGN and give some properties.

On the other hand, many integer sequences arising from enumerative combinatorics turn out to be unimodal, or even log-concave. Although proving these properties seem natural, it is sometimes difficult to overcome. A very good survey for these properties is given in [12]. Such as Gaussian coefficients, GGN play an important role in enumerative combinatorics since it presents the number of submodules of a finite module and in a special case GGN give binomial coefficients which are frequently involved in constructing properties of some special numbers.

We summarize some of the fundamental properties of binomial and Gaussian coefficients. Binomial coefficients, $\binom{n}{k}$, satisfy the well known relations: $\binom{n}{k} = \binom{n}{n-k}$, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ and (the identity) $\sum_{k-even} \binom{n}{k} = \sum_{k-odd} \binom{n}{k}$.

The construction of properties and identities of some special numbers is done by binomial coefficients. Hence binomial and Gaussian coefficients (or q-binomials) play an important role in Number Theory. From [3], Gaussian coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{q^{n-i+1}-1}{q^i-1}, \quad \begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1, \quad q \neq 1. \tag{1}$$

Unlike the binomial coefficients Gaussian coefficients have only a limited number of properties. The following properties of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are due to [3]. We have the triangular recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n \\ k-1 \end{bmatrix}_q, \tag{2}$$

the identity

$$\sum_{keven} \begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{kodd} \begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{3}$$

Some properties parallel to those Eq. (1) and Eq. (2) are given in [10] and [11].

The Rogers-Szego polynomial in a single variable, $H_n(t)$, is defined as

$$H_n(t) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k.$$

For nonnegative integers $k_1, k_2, \ldots, k_m$ such that $k_1+k_2+\ldots+k_m = n$, Gaussian multinomial coefficient (or q-multinomial coefficient) of length m is defined in [14] as

$$\begin{bmatrix} n \\ k_1, k_2, \ldots, k_m \end{bmatrix}_q = \frac{(q)_n}{(q)_{k_1}(q)_{k_2}\ldots(q)_{k_m}}$$

where $(q)_n = (1-q)(1-q^2)\ldots(1-q^n)$.

The homogeneous Rogers-Szego polynomial in m variables for $m \geq 2$, denoted by $\hat{H}_n(t_1, t_2, \ldots, t_m)$, is defined in [14] as

$$\hat{H}_n(t_1, t_2, \ldots, t_m) = \sum_{k_1+k_2+\ldots+k_m=n} \begin{bmatrix} n \\ k_1, k_2, \ldots, k_m \end{bmatrix}_q t_1^{k_1} \ldots t_m^{k_m} \tag{4}$$

and Rogers-Szego polynomial in $m-1$ variables, denoted $H_n(t_1, t_2, \ldots, t_{m-1})$, is defined as

$$H_n(t_1, t_2, \ldots, t_{m-1}) = \hat{H}_n(t_1, t_2, \ldots, t_{m-1}, 1).$$

For any $a, r \neq 1$ and $n \geq 1$, $(a, r)_n = (a; r)_n$ is defined in [14] by

$$(a, r)_n = (1-a)(1-ar)\ldots(1-ar^{n-1}),$$

and $(a, r)_0 = 1$.

When $a = r$, then,

$$(a, a)_n = (1-a)(1-a^2)\ldots(1-a^n).$$

Denote this number by $(a)_n$ shortly.

Rogers-Szego polynomials were first defined by Rogers [8] in terms of their generating function and some researchers have also studied [1, 2, 5] them. They have an important role in combinatorial number theory, symmetric function theory and orthogonal polynomials.

In this paper, we prove the log-concavity and unimodality of a sequence defined by GGN. Moreover, some more properties and identities of the GGN are given. Finally, we point out Rogers-Szego polynomial [8, 9, 14] and its relation to GGN.

2. Recurrence Relations and Generating Functions

In this section, some properties and generating functions of GGN are given. As we already know that the binomial coefficients stand for the number of subsets of a finite set of a particular size. Gaussian binomial coefficients stand for the number of vector spaces of a particular dimension of a finite dimensional space. Recently, a direct calculation of the number of submodules of a particular type of a finite module is introduced and these numbers are called Generalize Gaussian Numbers [11].

Theorem 2.1. [11] *The number of $\mathbb{Z}_q$-submodules of type $(k_1, k_2, \ldots, k_m)$ of the finite module $\mathbb{Z}_q^n$ where $q = p^m$ (p prime and m a positive integer) is*

$$\frac{\prod_{t=1}^{m} \prod_{i=0}^{k_t-1} ((p^{m-t+1})^n - (p^{m-t})^n \cdot p^{\sum_{j=0}^{t-1} k_j} \cdot p^i)}{\prod_{s=1}^{m} \prod_{r=0}^{k_s-1} \cdot A}$$

where

$$A = \prod_{a=1}^{s} (p^{m-s+1})^{k_a} \prod_{j=s+1}^{m} (p^{m-j+1})^{k_j} - (\prod_{a=1}^{s} (p^{m-s})^{k_a})(p^{m-s+1})^{k_{s+1}} \cdot \prod_{t=s+2}^{m} (p^{m-t+1})^{k_t} \cdot p^r$$

Denote the number given in the previous theorem by

$$N^q_{k_1,k_2,\ldots,k_m}(n) = \begin{bmatrix} n \\ k_1, k_2, \ldots, k_m \end{bmatrix}_{\mathbf{Z}_q}$$

where q is a prime power and m is the nilpotency of the generator matrix of the maximal ideal.

Definition 2.1. For $m = 2$ and $q = p^2$, we call the number of $\mathbb{Z}_q$-submodules of type (k_1, k_2) of the finite module$\mathbb{Z}_q^n$ as Generalized Gaussian Numbers (GGN) and denote it by $N^q_{k_1,k_2}(n) = \begin{bmatrix} n \\ k_1, k_2 \end{bmatrix}_{\mathbf{Z}_q}$.

Then, we write by [11]

$$N^q_{k_1,k_2}(n) = \begin{bmatrix} n \\ k_1, k_2 \end{bmatrix}_{\mathbf{Z}_q} =$$

(5)
$$\frac{(p^{2n} - p^n)\dots(p^{2n} - p^{n+k_1-1})(p^n - p^{k_1})\dots(p^n - p^{k_1+k_2-1})}{(p^{2k_1}p^{k_2} - p^{k_1+k_2})\dots(p^{2k_1}p^{k_2} - p^{k_1+k_2+k_1-1})(p^{k_1+k_2} - p^{k_1})\dots(p^{k_1+k_2} - p^{k_1+k_2-1})}.$$

In short, Eq. (5) is written as

$$\frac{p^{nk_1}\prod_{i=0}^{k_1-1}(p^n - p^i)\prod_{j=0}^{k_2-1}(p^n - p^{k_1+j})}{p^{k_1{}^2+2k_1k_2}\prod_{i=0}^{k_1-1}(p^{k_1} - p^i)\prod_{j=0}^{k_2-1}(p^{k_2} - p^j)}. \tag{6}$$

Theorem 2.2. *GGN satisfies the following triangular recurrence relation*

$$N^q_{n-(k-1),k}(n+1) = N^q_{n-(k-1),k-1}(n) + p^k N^q_{n-k,k}(n) \tag{7}$$

where $q = p^2$.

Proof. From Eq. (6), we obtain the following

$$N^q_{n-(k-1),k}(n+1) = \frac{p^{(n+1)(n-k+1)}\prod_{i=0}^{n-k}(p^{n+1} - p^i)\prod_{j=0}^{k-1}(p^{n+1} - p^{n-k+1+j})}{p^{(n-k+1)^2+2(n-k+1)k}\prod_{i=0}^{n-k}(p^{n-k+1} - p^i)\prod_{j=0}^{k-1}(p^k - p^j)} \tag{8}$$

(9)
$$N^q_{n-(k-1),k-1}(n) = \frac{p^{n^2-nk+n}\prod_{i=0}^{n-k}(p^n - p^i)\prod_{j=0}^{k-2}(p^n - p^{n-k+1+j})}{p^{(n-k+1)^2+2(n-k+1)(k-1)}\prod_{i=0}^{n-k}(p^{n-k+1} - p^i)\prod_{j=0}^{k-2}(p^{k-1} - p^j)}$$

and

$$N^q_{n-k,k}(n) = \frac{p^{n(n-k)}\prod_{i=0}^{n-k-1}(p^n - p^i)\prod_{j=0}^{k-1}(p^n - p^{n-k+j})}{p^{(n-k)^2+2(n-k)k}\prod_{i=0}^{n-k-1}(p^{n-k} - p^i)\prod_{j=0}^{k-1}(p^k - p^j)}. \tag{10}$$

In order to carry out the equality, we put the denominators of the equalities (9) and (10) into a common one similar to the denominator of the equality (8). Multiplying both numerator and denominator parts of equation (9) by

$$p^{-2(n-k+1)}p^{k-1}(p^k - 1),$$

we obtain

(11)
$$N^q_{n-(k-1),k-1}(n) = \frac{p^{n^2-nk+3n-k+1}(p^k - 1)\prod_{i=0}^{n-k}(p^n - p^i)\prod_{j=0}^{k-2}(p^n - p^{n-k+1+j})}{p^{(n-k+1)^2+2(n-k+1)k}\prod_{i=0}^{n-k}(p^{n-k+1} - p^i)\prod_{j=0}^{k-1}(p^k - p^j)}$$

Multiplying both numerator and denominator parts of equation (10) by

$$p^{2n+1}p^{n-k}(p^{n-k+1}),$$

we obtain

(12)

$$N^q_{n-k,k}(n+1) = \frac{p^{n^2-nk+1+3n-k}(p^{n-k+1}-1)\prod_{i=0}^{n-k-1}(p^n-p^i)\prod_{j=0}^{k-1}(p^n-p^{n-k+j})}{p^{(n-k+1)^2+2(n-k+1)k}\prod_{i=0}^{n-k}(p^{n-k+1}-p^i)\prod_{j=0}^{k-1}(p^k-p^j)}.$$

By multiplying (12) with p^k and then summing the result with (11) and by applying the necessary operations we obtain the result.

□

Next, we illustrate with a couple of examples.

Example 1. *For values $k=4, n=5, q=4$ we have*

$$N^4_{2,4}(6) = N^4_{2,3}(5) + 2^4 \cdot N^4_{1,4}(5).$$

$$651 = 155 + 2^4 31.$$

□

Example 2. *For values $k=5, n=6, q=4$ we have*

$$N^4_{2,5}(7) = N^4_{2,4}(6) + 2^5 \cdot N^4_{1,5}(6).$$

$$2667 = 651 + 2^5 \cdot 63.$$

□

Theorem 2.3. *GGN satisfies the following relation as a generating function*

$$(p-1)N^q_{k,1}(n) = (p^n-1)N^q_{k,0}(n-1) \tag{13}$$

where $q = p^2$.

Proof.

$$\begin{aligned}
(p-1)N^q_{k,1}(n) &= \frac{(p-1)p^{nk}(p^n-1)(p^n-p)\dots(p^n-p^{k-1})(p^n-p^k)}{p^{k^2+2k}(p^k-1)(p^k-p)\dots(p^k-p^{k-1})(p-1)} \\
&= \frac{p^{nk-2k}(p^n-1)p(p^{n-1}-1)\dots p(p^{n-1}-p^{k-2})p(p^n-p^{k-1})}{p^{k^2}(p^k-1)(p^k-p)\dots(p^k-p^{k-1})} \\
&= \frac{p^{nk-k}(p^n-1)(p^{n-1}-1)\dots(p^{n-1}-p^{k-2})(p^n-p^{k-1})}{p^{k^2}(p^k-1)(p^k-p)\dots(p^k-p^{k-1})} \\
&= \frac{(p^n-1)p^{(n-1)k}(p^{n-1}-1)(p^{n-1}-p)\dots(p^{n-1}-p^{k-1})}{p^{k^2}(p^k-1)(p^k-p)\dots(p^k-p^{k-1})} \\
&= (p^n-1)N^q_{k,0}(n-1).
\end{aligned}$$

This completes the proof. □

Example 3. $n=6, q=p^2=4:$

$$N^4_{3,1}(6) = (2^6-1)N^4_{3,0}(5) \qquad (k=3)$$

$$624960 = (2^6-1)9920,$$

$$N^4_{4,1}(6) = (2^6 - 1)N^4_{4,0}(5) \qquad (k = 4)$$

$$31248 = (2^6 - 1)496, \ \textit{and}$$

$$N^4_{5,1}(6) = (2^6 - 1)N^4_{5,0}(5) \qquad (k = 5)$$

$$63 = (2^6 - 1).$$

□

Example 4. $n = 4, q = p^2 = 9$:

$$2N^9_{3,1}(5) = (3^5 - 1)N^9_{3,0}(4) \qquad (k = 3)$$

$$2.130680 = (3^5 - 1)1080$$

$$2N^9_{4,1}(5) = (3^5 - 1)N^9_{4,0}(4) \qquad (k = 4)$$

$$2.121 = 3^5 - 1.$$

□

Theorem 2.4. [10] *Some of the properties of GGN which are used for generating Gaussian Numbers are given.* $(q = p^2)$

$$\begin{aligned} N^q_{0,k}(n) &= N^q_{0,n-k}(n), \quad N^q_{k,0}(n) = N^q_{n-k,0}(n) \\ N^q_{k_1,k_2}(n) &= N^q_{k_2,k_1}(n), \quad (k_1 + k_2 = n) \\ N^q_{0,n}(n) &= N^q_{n,0}(n) = 1 \\ N^q_{k,0}(n) &= (p^k)^{n-k} N^q_{0,k}(n) \\ N^q_{k_1,k_2}(n) &= N^q_{n-(k_1+k_2),k_2}(n). \end{aligned}$$

Proof. In order to prove the equalities above, we apply Eq. (6) to all of them similar to those we did in Theorem 2.2 and Theorem 2.3. □

$n/(0,k)$	$(\mathbf{0},\mathbf{1})$	$(\mathbf{0},\mathbf{2})$	$(\mathbf{0},\mathbf{3})$	$(\mathbf{0},\mathbf{4})$
1	$S_0(p)$			
2	$S_1(p)$	1		
3	$S_2(p)$	$p^2 + p + 1$	1	
4	$S_3(p)$	$p^4 + p^3 + 2p^2 + p + 1$	$p^3 + p^2 + p + 1$	1
5	$S_4(p)$	$p^6 + p^5 + 2p^4$ $+2p^3 + 2p^2 + p + 1$	$p^6 + p^5 + 2p^4$ $+2p^3 + 2p^2 + p + 1$	$p^4 + p^3$ $+p^2 + p + 1$

TABLE 1. Table of values for $N^{p^2}_{0,k}(n)$ and $S_k(p) = \sum_{i=0}^{k} p^i$.

$n/(k,1)$	$(\mathbf{0},\mathbf{1})$	$(\mathbf{1},\mathbf{1})$	$(\mathbf{2},\mathbf{1})$	$(\mathbf{3},\mathbf{1})$
1	$S_0(p)$			
2	$S_1(p)$	$p+1$		
3	$S_2(p)$	$p^4+2p^3+2p^2+p$	p^2+p+1	
4	$S_3(p)$	$p^7+2p^6+3p^5+$ $+3p^4+2p^3+p^2$	$p^7+2p^6+3p^5+$ $+3p^4+2p^3+p^2$	p^3+p^2+p+1
5	$S_4(p)$	$p^{10}+2p^9+3p^8$ $+4p^7+4p^6+$ $3p^5+2p^4+p^3$	$p^{12}+2p^{11}+4p^{10}$ $+5p^9+6p^8+5p^7$ $+4p^6+2p^5+p^4+$ $3p^3+2p^2+2p+1$	$p^{10}+2p^9+3p^8$ $+4p^7+4p^6+$ $3p^5+2p^4+p^3$

TABLE 2. Table of values for $N_{k,1}^{p^2}(n)$

Using the properties given by Eq. (7), Eq. (13) and the properties given in Theorem 2.4 we can quickly generate the values of $N_{k_1,k_2}^{q}(n)$ as shown in the following table where $q=p^2$.

One may try to obtain the numbers by different primes p and n using Table 1 or Table 2. They may be generalized to any n by Eq. (7), Eq. (13) and Theorem 2.4. Of course these tables are only two small examples of those sequences given by GGN. On the other hand, we can easily see some properties if we write the entries of (for example) Table 1 in the following way:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & p+1 & & 1 & & \\
 & 1 & & p^2+p+1 & & p^2+p+1 & & 1 & \\
1 & & p^3+p^2+p+1 & & p^4+p^3+2p^2+p+1 & & p^3+p^2+p+1 & & 1 \\
1 & p^4+p^3+p^2+p+1 & & p^6+p^5+2p^4+2p^3+2p^2+p+1 & & p^6+p^5+2p^4+2p^3+2p^2+p+1 & & p^4+p^3+p^2+p+1 & 1
\end{array}
$$

FIGURE 1. Pascal's Type triangle

The figure is analogous to Pascal's Triangle of the usual binomial coefficients. It has a symmetry like Pascal's Triangle. This figure is also the same as Gaussian coefficients. The properties are similar to those in Gaussian coefficients. However, the properties which are obtained by Table 2 are not exactly the same as the usual Gaussian numbers or binomial coeeficients since GGN is more general than the others.

3. Loq-Concavity, Unimodality of The Sequences and Rogers-Szego Polynomials

We prove that some of the number sequences which are obtained from GGN are log-concave. We present a family of sequences which are unimodal. We also generalize these sequences. Here q is always equal to p^2.

Definition 3.1. [12] A sequence is said to be unimodal if for some $0 \leq j \leq n$ we have $a_0 \leq a_1 \leq \ldots \leq a_j \geq a_{j+1} \geq \ldots \geq a_n$ and is said to be logarithmically concave (or log-concave for short) if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $1 \leq i \leq n-1$.

Theorem 3.1. *The sequence $N_{k,0}^q(n)$ satisfies the log-concavity property.*

Proof. In order to prove, we have to show that the fraction

$$\frac{(N_{k,0}^q(n))^2}{N_{k-1,0}^q(n)N_{k+1,0}^q(n)}$$

is larger than 1.

$$\begin{aligned}
\left(N_{k,0}^q(n)\right)^2 &= \left[\frac{(p^{2n}-p^n)(p^{2n}-p^{n+1})\ldots(p^{2n}-p^{n+k-1})}{(p^{2k}-p^k)(p^{2k}-p^{k+1})\ldots(p^{2k}-p^{k+k-1})}\right]^2 \\
N_{k-1,0}^q(n) &= \frac{(p^{2n}-p^n)(p^{2n}-p^{n+1})\ldots(p^{2n}-p^{n+k-1-1})}{(p^{2(k-1)}-p^{k-1})(p^{2(k-1)}-p^k)\ldots(p^{2k}-p^{k-1+k-1-1})} \qquad (14)\\
N_{k+1,0}^q(n) &= \frac{(p^{2n}-p^n)(p^{2n}-p^{n+1})\ldots(p^{2n}-p^{n+k+1-1})}{(p^{2(k+1)}-p^{k+1})(p^{2(k+1)}-p^{k+2})\ldots(p^{2(k+1)}-p^{k+1+k+1-1})}.
\end{aligned}$$

From the equalities given by Eq. (14), we write

$$\begin{aligned}
&\frac{(N_{k,0}^q(n))^2}{N_{k-1,0}^q(n)N_{k+1,0}^q(n)} = \\
&= \frac{\frac{(p^{2n}-p^n)(p^{2n}-p^{n+1})\ldots(p^{2n}-p^{n+k-1})}{(p^{2k}-p^k)(p^{2k}-p^{k+1})\ldots(p^{2k}-p^{k+k-1})}\frac{(p^{2n}-p^n)\ldots(p^{2n}-p^{n+k-1})}{(p^{2k}-p^k)(p^{2k}-p^{k+1})\ldots(p^{2k}-p^{k+k-1})}}{\frac{p^{2n}-p^{n+k-1}}{p^{2n}-p^{n+k-1}}\frac{(p^{2n}-p^n)(p^{2n}-p^{n+1})\ldots(p^{2n}-p^{n+k-1-1})}{(p^{2(k-1)}-p^{k-1})\ldots(p^{2k}-p^{k-1+k-1-1})}\frac{(p^{2n}-p^n)\ldots(p^{2n}-p^{n+k+1-1})}{(p^{2(k+1)}-p^{k+1})\ldots(p^{2(k+1)}-p^{k+1+k+1-1})}} \\
&= \frac{\frac{1}{(p^{2k}-p^k)(p^{2k}-p^{k+1})\ldots(p^{2k}-p^{2k-1})}}{\frac{p^{2n}-p^{n+k}}{(p^{2(k-1)}-p^{k-1})\ldots(p^{2(k-1)}-p^{2k-3})(p^{2(k+1)}-p^{k+1})\ldots(p^{2(k+1)}-p^{2k+1})(p^{2n}-p^{n+k-1})}} \\
&= \frac{1}{(p^{2k}-p^k)\ldots(p^k-p^{2k-1})(p^{2k}-p^k)\ldots(p^{2k}-p^{2k-1})(p^{2n}-p^{n+k})} \\
&\quad \cdots(p^{2(k-1)}-p^{k-1})(p^{2(k-1)}-p^k)\ldots(p^{2(k-1)}-p^{2k-3})(p^{2(k+1)}-p^{k+1})\ldots \\
&\quad \cdots(p^{2(k+1)}-p^{2k+1})(p^{2n}-p^{n+k-1})
\end{aligned}$$

We multiply both numerator and denominator parts of the last equality by k^2. Then the last equality is equal to

$$k^2\frac{(p^{2(k+1)}-p^{k+1})(p^{2n}-p^{n+k-1})}{(p^{2(k-1)}-p^{k-2})(p^{2n}-p^{n+k})}. \qquad (15)$$

Multiplying the Eq. (15) by p^4, we obtain

$$\frac{p^4k^2(p^{2(k+1)}-p^{k+1})(p^{2n}-p^{n+k-1})}{(p^{2(k+1)}-p^{k+2})(p^{2n}-p^{n+k})} \tag{16}$$

Now it is obvious that Eq. (16) is greater than 1. □

Example 5. $q=4, n=4$*:*

$(N^4_{2,1}(4))^2 = 420^2, N^4_{1,1}(4) = 420, N^4_{3,1}(4) = 15, \qquad \frac{420^2}{420.15} = 28 > 1,$

$q=4, n=5$*:*

$(N^4_{1,3}(5))^2 = 930^2, N^4_{0,3}(5) = 155, N^4_{2,3}(5) = 155, \qquad \frac{930^2}{155.155} = 36 > 1,$

$q=4, n=6$*:*

$(N^4_{2,2}(6))^2 = 364560^2, N^4_{1,2}(6) = 78120, N^4_{0,2}(6) = 651, \qquad \frac{364560^2}{78120.651} \sim 2614 > 1.$ □

Example 6. $q=9, n=3$*:*

$(N^9_{1,1}(3))^2 = 156^2, N^4_{0,1}(3) = 13, N^4_{2,1}(3) = 13, \qquad \frac{156^2}{13.13} = 144 > 1$

$q=9, n=5$*:*

$(N^9_{2,0}(5))^2 = 882090^2, N^4_{1,0}(5) = 980, N^4_{3,0}(5) = 882090, \qquad \frac{882090^2}{980.882090} \sim 900 > 1.$ □

Lemma 3.2. *The sequence* $N^q_{k,1}(n)$ *is unimodal where* $k \leq n-1$.

Proof. We write below the sequence for all possible values of $k = 0, 1, 2, \ldots, n-1$ and any $q = p^2$:

$N^q_{0,1}(n) \qquad N^q_{1,1}(n) \qquad N^q_{2,1}(n) \qquad \ldots \qquad N^q_{n-2,1}(n) \qquad N^q_{n-1,1}(n)$

We apply Eq. (5) to the entries of the sequence:

(17)

$$N^q_{0,1}(n) = \frac{p^n - p^0}{p - p^0} = p^{n-1} + p^{n-2} + \ldots + p + 1,$$

$$\begin{aligned} N^q_{1,1}(n) &= \frac{(p^{2n} - p^n)(p^n - p)}{(p^3 - p^2)(p^2 - p)} = \frac{p^n(p^n - 1)p(p^{n-1} - 1)}{p^2(p-1)p(p-1)} \\ &= p^{n-2}(p^{n-1} + p^{n-2} + \ldots + 1)(p^{n-2} + p^{n-3} + \ldots + 1) \end{aligned}$$

$$\begin{aligned} N^q_{2,1}(n) &= \frac{(p^{2n} - p^n)(p^{2n} - p^{n+1})(p^n - p^2)}{(p^5 - p^3)(p^5 - p^4)(p^3 - p^2)} = \frac{p^n(p^n - 1)p^{n+1}(p^{n-1} - 1)p^2(p^{n-2} - 1)}{p^3(p^2 - 1)p^4(p-1)p^2(p-1)} \\ &= \frac{p^{2n-6}(p^{n-1} + p^{n-2} + \ldots + p + 1)(p^{n-2} + \ldots + p + 1)(p^{n-3} + \ldots + p + 1)}{p+1} \end{aligned}$$

$$\begin{aligned} N^q_{3,1}(n) &= \frac{(p^{2n} - p^n)(p^{2n} - p^{n+1})(p^{2n} - p^{n+2})(p^n - p^3)}{(p^7 - p^4)(p^7 - p^5)(p^7 - p^6)(p^4 - p^3)} \\ &= \frac{1}{(p+1)^2}\, p^{3n-12}(p^{n-1} + \ldots + p + 1)(p^{n-2} + \ldots + p + 1)(p^{n-3} + \ldots + p + 1)\cdot \\ &\quad \cdot (p^{n-4} + \ldots + p + 1) \end{aligned}$$

. . . . . .

$$N^q_{n-2,1}(n) = \frac{(p^{2n} - p^n)(p^{2n} - p^{n+1}) \ldots (p^{2n} - p^{2n-3})(p^n - p^{n-2})}{(p^{2n-3} - p^{n-1})(p^{2n-3} - p^n) \ldots (p^{2n-3} - p^{2n-4})(p^{n-1} - p^{n-2})}.$$

We multiply both numerator and denominator parts of the equality (17) by $p^{3(n-2)}$ and have

$$\begin{aligned} N^q_{n-2,1}(n) &= \frac{(p^{2n} - p^n)(p^{2n} - p^{n+1}) \ldots (p^{2n} - p^{2n-3})(p^n - p^{n-2})}{(p^{2n} - p^{n+2})(p^{2n} - p^{n+3}) \ldots (p^{2n} - p^{2n-1})(p^{n-1} - p^{n-2})} \\ &= \frac{p^n(p^n - 1)p^{n+1}(p^{n-1} - 1)(p^n - p^{n-2})p^{3n-6}}{p^{2n-2}(p^2 - 1)p^{2n-1}(p-1)(p^{n-1} - p^{n-2})} \\ &= p^{n-2}(p^{n-1} + \ldots + p + 1)(p^{n-2} + \ldots + p + 1). \end{aligned} \tag{18}$$

Finally, in a similar way we obtain

$$N^q_{n-1,1}(n) = p^{n-2} + p^{n-1} + \ldots + p + 1)(p^{n-3} + \ldots + p + 1. \tag{19}$$

If n is odd then the middle term, $\left(\frac{n+1}{2}\right)$–th term, is $N^q_{\frac{n+1}{2},1}(n)$. The sequence is increasing until the middle term and decreasing after it.

If n is even then we have two middle terms, $N^q_{\frac{n}{2},1}(n)$, $N^q_{\frac{n}{2}+1,1}(n)$, whose values are the same. This may be ensured by the equalities given by (17)-(18). Moreover, $5th$ equality of Theorem 2.3 guarantees this result.

Hence the sequence $N^q_{k,1}(n)$ is unimodal.

For odd n

$$\begin{array}{ccccccccccccc} N^q_{0,1}(n) & & N^q_{1,1}(n) & & N^q_{2,1}(n) & \dots & N^q_{\frac{n+1}{2},1}(n) & \dots & N^q_{n-2,1}(n) & & N^q_{n-1,1}(n) \\ & \nearrow & & \nearrow & & \nearrow & & \searrow & \searrow & \searrow & \end{array}$$

For even n

$$\begin{array}{ccccccccccccc} N^q_{0,1}(n) & & N^q_{1,1}(n) & & N^q_{2,1}(n) & \dots & N^q_{\frac{n}{2},1}(n) & & N^q_{\frac{n}{2}+1,1}(n) & \dots & N^q_{n-2,1}(n) \\ & \nearrow & & \nearrow & & \nearrow & & \rightarrow & \searrow & \searrow & \searrow \end{array}$$

□

Example 7. $q = 4, n = 6$*:*

$$\begin{array}{cccccc} N^4_{0,1}(6) & N^4_{1,1}(6) & N^4_{2,1}(6) & N^4_{3,1}(6) & N^4_{4,1}(6) & N^4_{5,1}(6) \\ 63 & 31248 & 624960 & 624960 & 31248 & 63 \end{array}$$

$q = 4, n = 7$*:*

$$\begin{array}{ccccccc} N^4_{0,1}(7) & N^4_{1,1}(7) & N^4_{2,1}(7) & N^4_{3,1}(7) & N^4_{4,1}(7) & N^4_{5,1}(7) & N^4_{6,1}(7) \\ 127 & 256032 & 21165312 & 90708480 & 21165312 & 256032 & 127 \end{array}$$

$q = 9, n = 3$*:*

$$\begin{array}{ccc} N^9_{0,1}(3) & N^9_{1,1}(3) & N^9_{2,1}(3) \\ 13 & 156 & 13 \end{array}$$

$q = 9, n = 4$*:*

$$\begin{array}{cccc} N^9_{0,1}(4) & N^9_{1,1}(4) & N^9_{2,1}(4) & N^9_{3,1}(4) \\ 40 & 4680 & 4680 & 40. \end{array}$$

□

Lemma 3.3. *The sequence $N^q_{k,2}(n)$ is unimodal where $k \leq n-2$.*

Proof. We can give the proof in the similar way as in Lemma 3.2. We write below the sequence for all possible values of $k = 0, 1, 2, \dots, n-2$ and any $q = p^2$:

$$N^q_{0,2}(n) \quad N^q_{1,2}(n) \quad N^q_{2,2}(n) \quad \dots \quad N^q_{n-3,2}(n) \quad N^q_{n-2,2}(n)$$

We apply Eq. (5) to the entries of the sequence:

$$N^q_{0,2}(n) = \frac{(p^n-1)(p^n-p)}{(p^2-1)(p^2-p)} = \frac{(p^{n-1}+p^{n-2}+\dots+1)(p^{n-1}+p^{n-2}+\dots+1)}{p+1} \tag{20}$$

(21)
$$N^q_{1,2}(n) = \frac{(p^{2n}-p^n)(p^n-p)(p^n-p^2)}{(p^4-p^3)(p^3-p)(p^3-p^2)}$$
$$= \frac{p^{n-3}(p^{n-1}+p^{n-2}+\ldots+1)(p^{n-2}+p^{n-3}+\ldots+1)(p^{n-3}+p^{n-4}+\ldots+1)}{p+1}$$

(22)
$$N^q_{2,2}(n) = \frac{p^{2n-8}(\sum_{i=0}^{n-1} p^i)(\sum_{i=0}^{n-2} p^i)(\sum_{i=0}^{n-3} p^i)(\sum_{i=0}^{n-4} p^i)}{(p+1)^2}$$

(23)
$$N^q_{n-3,2}(n) = \frac{(p^{2n}-p^n)(p^{2n}-p^{n+1})\ldots(p^{2n}-p^{2n-4})(p^n-p^{n-3})(p^n-p^{n-2})}{(p^{2n-4}-p^{n-1})(p^{2n-4}-p^n)\ldots(p^{2n-4}-p^{2n-5})(p^{n-1}-p^{n-3})(p^{n-1}-p^{n-2})}$$
$$= \frac{p^{n-3}(p^{n-1}+p^{n-2}+\ldots+1)(p^{n-2}+p^{n-3}+\ldots+1)(p^{n-3}+p^{n-4}+\ldots+1)}{p+1}.$$

(24)
$$N^q_{n-2,2}(n) = \frac{(p^{2n}-p^n)(p^{2n}-p^{n+1})\ldots(p^{2n}-p^{2n-4})(p^n-p^{n-3})(p^n-p^{n-2})}{(p^{2n-4}-p^{n-1})(p^{2n-4}-p^n)\ldots(p^{2n-4}-p^{2n-5})(p^{n-1}-p^{n-3})(p^{n-1}-p^{n-2})}$$
$$= \frac{(p^{n-1}+p^{n-2}+\ldots+1)(p^{n-1}+p^{n-2}+\ldots+1)}{p+1}.$$

Hence, the expressions in the above equations have the following relations

$$Eq.\ (20) < Eq.\ (21) < Eq.\ (22) < \ldots$$

The inequality continues in the same way until the term $N^q_{\frac{n}{2},2}(n)$ for even n and $N^q_{\frac{n+1}{2},2}(n)$ (also $N^q_{\frac{n+1}{2}+1,2}(n)$) for odd n. After the terms $N^q_{\frac{n}{2},2}(n)$ and $N^q_{\frac{n+1}{2}+1,2}(n)$, the sequence is decreasing because of the equalities given by (20)-(24). This proves the Lemma. □

Example 8. $q = 4, n = 6$:

$N^4_{0,2}(6)$	$N^4_{1,2}(6)$	$N^4_{2,2}(6)$	$N^4_{3,2}(6)$	$N^4_{4,2}(6)$
651	78120	36456	78120	651

$q = 4, n = 7$:

$N^4_{0,2}(7)$	$N^4_{1,2}(7)$	$N^4_{2,2}(7)$	$N^4_{3,2}(7)$	$N^4_{4,2}(7)$	$N^4_{5,2}(7)$,
2667	1322832	26456640	26456640	1322832	2667

$q = 25, n = 3$:

$N^{25}_{0,2}(4)$	$N^{25}_{1,2}(4)$	$N^{25}_{2,2}(4)$
806	4030	806.

□

Theorem 3.4. *The sequence $N^q_{k_1,k_2}(n)$ is unimodal where $k_1 + k_2 \leq n$ and $q = p^2$.*

Proof. By Theorem 2.4, Lemma 3.2 and Lemma 3.3 we obtain the result. □

In the following theorem, we give a connection between GGN and Rogers-Szego polynomials.

Prior to the theorem, we first write (6) as

$$\left[\begin{array}{c} n \\ k_1, k_2 \end{array}\right]_{\mathcal{Z}_q} = \frac{\overbrace{p^{nk_1}\prod_{i=0}^{k_1-1}(p^n-p^i)}^{A}\,\overbrace{p^{k_1k_2}\prod_{j=0}^{k_2-1}(p^{n-k_1}-p^j)}^{B}}{\underbrace{p^{{k_1}^2+k_1k_2}\prod_{l=0}^{k_1-1}(p^{k_1}-p^l)}_{C}\,\underbrace{p^{k_1k_2}\prod_{m=0}^{k_2-1}(p^{k_2}-p^m)}_{D}}. \tag{25}$$

Theorem 3.5. *The following equality holds:*

$$A\cdot\hat{H}_n(t_1,t_2,t_3) = \sum_{k_1+k_2+k_3=n} N\cdot t_1^{k_1}t_2^{k_2}t_3^{k_3} \tag{26}$$

where $A = p^{2nk_1+nk_2-2{k_1}^2-{k_2}^2-2k_1k_2}$ *and* $N = \left[\begin{array}{c} n \\ k_1, k_2 \end{array}\right]_{\mathcal{Z}_q}$ $(q=p^2)$.

Proof. By using the notation introduced in Eq. (25), part A can be rewritten as

$$p^{nk_1}\prod_{i=0}^{k_1-1}(p^n-p^i) = p^{nk_1}p^{nk_1}(1-(\frac{1}{p})^n)(1-(\frac{1}{p})^{n-1})\dots(1-(\frac{1}{p})^{n-k_1+1})$$

$$A = p^{2nk_1}\frac{(1-(\frac{1}{p}))(1-(\frac{1}{p})^2)\dots(1-(\frac{1}{p})^n)}{(1-(\frac{1}{p}))(1-(\frac{1}{p})^2)\dots(1-(\frac{1}{p})^{n-k_1})} = p^{2nk_1}\frac{(\frac{1}{p},\frac{1}{p})_n}{(\frac{1}{p},\frac{1}{p})_{n-k_1}}, \tag{27}$$

similarly part B can be rewritten as

$$p^{k_1k_2}\prod_{j=0}^{k_2-1}(p^{n-k_1}-p^j) =$$
$$p^{k_1k_2}p^{(n-k_1)k_2}(1-(\frac{1}{p})^{n-k_1})(1-(\frac{1}{p})^{n-k_1-1})\dots(1-(\frac{1}{p})^{n-k_1-k_2+1})$$

$$B = p^{nk_2}\frac{(1-(\frac{1}{p}))(1-(\frac{1}{p})^2)\dots(1-(\frac{1}{p})^{n-k_1})}{(1-(\frac{1}{p}))(1-(\frac{1}{p})^2)\dots(1-(\frac{1}{p})^{n-k_1-k_2})} = p^{nk_2}\frac{(\frac{1}{p},\frac{1}{p})_{n-k_1}}{(\frac{1}{p},\frac{1}{p})_{n-k_1-k_2}}, \tag{28}$$

part C can be rewritten as

$$p^{{k_1}^2+k_1k_2}\prod_{l=0}^{k_1-1}(p^{k_1}-p^l) =$$
$$p^{{k_1}^2+k_1k_2}p^{{k_1}^2}(1-(\frac{1}{p})^{k_1})(1-(\frac{1}{p})^{k_1-1})\dots(1-(\frac{1}{p})^{k_1-k_1+1})$$

$$C = p^{2{k_1}^2+k_1k_2}(\frac{1}{p},\frac{1}{p})_{k_1}, \tag{29}$$

and finally part D can be rewritten as

$$p^{k_1k_2}\prod_{m=0}^{k_2-1}(p^{k_1}-p^m) = p^{k_1k_2}p^{{k_2}^2}(1-(\frac{1}{p})^{k_2})(1-(\frac{1}{p})^{k_2-1})\dots(1-(\frac{1}{p})^{k_2-k_2+1})$$

$$D = p^{k_1k_2+{k_2}^2}(\frac{1}{p},\frac{1}{p})_{k_2}. \tag{30}$$

Then, by using the new expressions of A, B, C, D in (25)

$$\begin{bmatrix} n \\ k_1, k_2 \end{bmatrix}_{\mathcal{Z}_q} = \frac{p^{2nk_1}\frac{(\frac{1}{p},\frac{1}{p})_n}{(\frac{1}{p},\frac{1}{p})_{n-k_1}}p^{nk_2}\frac{(\frac{1}{p},\frac{1}{p})_{n-k_1}}{(\frac{1}{p},\frac{1}{p})_{n-k_1-k_2}}}{p^{2{k_1}^2+k_1k_2}(\frac{1}{p},\frac{1}{p})_{k_1}p^{k_1k_2+{k_2}^2}(\frac{1}{p},\frac{1}{p})_{k_2}} \tag{31}$$

Take $\frac{1}{p} = q$:

$$\begin{bmatrix} n \\ k_1, k_2 \end{bmatrix}_{\mathcal{Z}_q} = p^{2nk_1+nk_2-2{k_1}^2-{k_2}^2-2k_1k_2}\frac{(q,q)_n(q,q)_{n-k_1}}{(q,q)_{n-k_1}(q,q)_{k_1}(q,q)_{n-k_1-k_2}(q,q)_{k_2}} \tag{32}$$

For $(q) = (q,q)_n$ in (32),

$$\begin{bmatrix} n \\ k_1, k_2 \end{bmatrix}_{\mathcal{Z}_q} = p^{2nk_1+nk_2-2{k_1}^2-{k_2}^2-2k_1k_2}\frac{(q)_n(q)_{n-k_1}}{(q)_{n-k_1}(q)_{k_1}(q)_{n-k_1-k_2}(q)_{k_2}} \tag{33}$$

The following two equalities are well known:

$\begin{bmatrix} n \\ k_1 \end{bmatrix}_q = \frac{(q)_n}{(q)_{n-k_1}(q)_{k_1}}$, $\begin{bmatrix} n-k_1 \\ k_2 \end{bmatrix}_q = \frac{(q)_{n-k_1}}{(q)_{n-k_1-k_2}(q)_{k_2}}$.

Then, (33) is equal to the number

$$p^{2nk_1+nk_2-2{k_1}^2-{k_2}^2-2k_1k_2}\begin{bmatrix} n \\ k_1 \end{bmatrix}_q\begin{bmatrix} n-k_1 \\ k_2 \end{bmatrix}_q. \tag{34}$$

Moreover, if we do some abbreviation in (33), we obtain (35):

$$p^{2nk_1+nk_2-2{k_1}^2-{k_2}^2-2k_1k_2}\frac{(q)_n}{(q)_{k_1}(q)_{k_2}(q)_{n-k_1-k_2}} \tag{35}$$

$$= p^{2nk_1+nk_2-2{k_1}^2-{k_2}^2-2k_1k_2} \underbrace{\begin{bmatrix} n \\ k_1, k_2, n-k_1-k_2 \end{bmatrix}_q} \tag{36}$$

$(k_3 = n - k_1 - k_2, \quad A = p^{2nk_1+nk_2-2{k_1}^2-{k_2}^2-2k_1k_2})$

$$\begin{bmatrix} n \\ k_1, k_2 \end{bmatrix}_{\mathcal{Z}_q} = A \underbrace{\begin{bmatrix} n \\ k_1, k_2, k_3 \end{bmatrix}_q}_{q-multinomial \quad coefficient} \tag{37}$$

We let $N = \begin{bmatrix} n \\ k_1, k_2 \end{bmatrix}_{\mathcal{Z}_q}$. Then $\begin{bmatrix} n \\ k_1, k_2, k_3 \end{bmatrix}_q = \frac{N}{A}$.

The underlined expression in (36) and (37) is due to the definition given in [14].

The homogeneous Rogers-Szego polynomial in 3 variables is the following:

$$\hat{H}_n(t_1, t_2, t_3) = \sum_{k_1+k_2+k_3=n} \underbrace{\begin{bmatrix} n \\ k_1, k_2, k_3 \end{bmatrix}_q}_{\frac{N}{A}} t_1^{k_1} t_2^{k_2} t_3^{k_3},$$

where $N = \begin{bmatrix} n \\ k_1, k_2 \end{bmatrix}_q$ is GGN and A is equal to the number

$$A = p^{2nk_1+nk_2-2{k_1}^2-{k_2}^2-2k_1k_2}.$$

Then the result is obvious:

$$A \cdot \hat{H}_n(t_1, t_2, t_3) = \sum_{k_1+k_2+k_3=n} N \cdot t_1^{k_1} t_2^{k_2} t_3^{k_3} \tag{38}$$

□

4. Conclusion

In this paper, we have developed some functions and properties for Generalized Gaussian Numbers which are related to the number of submodules of a finite module. We present that some families of sequences which are obtained via GGN are log-concave and unimodal and we give some examples. As a future work, these studies may be generalized to GGN for any m and some further relations may be obtained between GGN and Rogers-Szego polynomials.

Acknowledgment: The authors wish to thank the referees for their valuable remarks. This research is supported by Yildiz Technical University Research Support Unit. (2011-03-DOP01)

References

[1] G. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications, Addison-Wesley Reading, Mass.-London-Amsterdam, (1976).

[2] W. Y. C. Chen, H. L. Saad and L. H. Sun, *The bivariate Rogers-Szego polynomials*, J. Phys. **A 40**, (2007), no. 23, 6071–6084.

[3] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht (1974).

[4] R.B. Corcino, *On p, q-Binomial Coeficients*, Electronic Journal of Combinatorial Number Theory **8**, (2008).

[5] N. J. Fine, *Basic Hypergeometric Series and Applications*, Mathematical Surveys and Monographs 27, American Mathematical Society, Providence, RI, (1988).
[6] R. Graham, Knuth D., and Patashnik O., *Concrete Mathematics: A Foundation for Computer Science*, Addison-Wesley, Reading, MA (1989).
[7] J. Riordan, *An Introduction to Combinatorial Analysis*, *Wiley, New York* (1958).
[8] L. J. Rogers, *On a three-fold symmetry in the elements of Heine's series*, Proc. London Math. Soc. **24**, (1893), 171–179.
[9] L. J. Rogers, *On the expansion of some infinite products*, Proc. London Math. Soc. **24**, (1893), 337–352.
[10] Salturk E., Siap I., *On The Number of Linear Codes over* $\mathcal{Z}_{p^m}$, submitted.
[11] E. Salturk and I. Siap, *Generalized Gaussian Numbers Related to Linear Codes over Galois Rings*, European Journal of Pure and Applied Mathematics **5**, (2012), 250-259.
[12] R.P., Stanley, *Log-concave and unimodal sequences in algebra, combinatorics and geometry*, Ann. New York Acad. Sci. **576** (1989), 500-535.
[13] R. P. Stanley, *Enumerative Combinatorics*, Wadsworth Brooks-Cole, Belmont, CA. (1986).
[14] C. R. Vinroot, *Multivariate Rogers-Szego polynomials and flags in finite vector spaces*, preprint, (2010).

