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NUMERICAL EXPERIMENTS RELATED TO THE INFORMATION–THEORETIC SCHOTTKY AND TORELLI PROBLEMS

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Abstract. The Schottky Problem is to characterize the Jacobians in the space of principally polarized abelian varieties (PPAVs). Besides its intrinsic interest, it is related to deep questions in PDE and Physics. The perspective of Information Theory leads to the numerical investigation of properties of the distribution of the periods, especially in cases of relatively large genus. In turn, numerical results lead to conjectures that the periods of hyperelliptic curves are band-limited, and that the squared moduli of the periods of a general surface are Zipfian.

1. Introduction

1.1. **Introduction.** Interest in two classical problems from the study of compact Riemann Surfaces of genus at least two continues to grow as new interpretations and new computational tools arise. The first of these, the Schottky Problem, is to characterize the Jacobians in the space of principally polarized abelian varieties (PPAVs). Besides its intrinsic interest, its solutions involve such diverse areas as classical algebraic geometry and partial differential equations, especially the Korteweig–deVries (KdV) and the Kadomtsev–Petviashvili (KP) equations [9], which relate it to problems of physics. Additionally, one of the earliest modern solutions to the Schottky problem, that of Andreootti and Mayer [2], used the heat equation in an essential way.

The second of the two problems, the Torelli problem, seeks methods to determine properties of a Riemann Surface from its period matrix, which, in principle, is possible due to Torelli's Theorem [10], but which, in practice, has proven to be rather difficult, involving detailed properties of Riemann's theta function.

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More recently, *Information Theory* has become a prominent technique in many areas of mathematics and applications. For present purposes, the basic idea of information theory is to understand the minimum number of bits required for Alice to send a message to Bob. A simple but telling example comes from the idea of a Huffman code, which optimally determines the length of symbols used with respect to their frequency of use. In a Huffman code for ordinary English, the letters `T` and `E` are assigned the smallest number of bits, because they are used most often, while seldom–used letters like `Q` and `Z` are encoded with more bits.

This work describes numerical exploration of the Schottky and Torelli problems from the perspective of information theory. Alice wants to tell Bob about a Riemann surface, and, by Torelli's theorem, she can do so by giving him its period matrix. These experiments lead to conjectures about the information-theoretic nature of the period matrix of a general Riemann surface (a Schottkly problem), about distinguishing the period matrix of a hyperelliptic Riemann surface from that of a general Riemann surface (a Torelli problem), and about the effects of truncated arithmetic.

1.2. **Computational Perspective.** New computational tools now allow us to examine the Schottky and Torelli problems from a computational perspective. When Alice tells Bob about a Riemann surface using its period matrix,this message's length is $g(g+1)/2$ complex numbers, although in practice the information might be $g(g+1)b/2$ bits, where each complex number is approximated by a binary number with b or fewer bits.

However, since the moduli space of compact Riemann Surfaces of genus g has dimension $3g-3$ when $g > 1$, the message is intrinsically compressible, that is, it contains much less information than its naive length would indicate. The nature of the *Information-Theoretic Schottky Problem* is to exploit the compressibility to try to characterize period matrices in $\mathcal{H}_g$, and that of the *Information-Theoretic Torelli Problem* is to exploit properties of the signal to determine properties of the curve.

In a sense, the Information Theoretic Schottky Problem was anticipated by Rauch over 50 years ago [11]. He proved that if there is a set of g periods $\pi_{ij} = \int_{b_j} d\zeta_i$ on a non-hyperelliptic compact Riemann Surface W of genus g such that the products $d\zeta_{i_1} d\zeta_{i_2}$ formed a basis for the quadratic differentials (there being $3g-3$ such pairs), then any Riemann surface with periods π'_{ij}that agreed with π_{ij} at the associated indices is conformally equivalent to W. In other words, in some cases one can choose $3g-3$ elements of the period matrix as (local) moduli, or, put differently, Alice need only send these $3g-3$ periods to Bob.

Communication complexity depends on the distribution of the messages to be sent, so one is led to consider the distribution of the periods. When the genus is small, the periods have no statistical properties to speak of, but when the genus is large there is enough "data" to look for statistical properties. In practice this means to take the set of all periods, create a corresponding set of real numbers (by, *e.g.*, taking the modulus-squared, the argument, or imaginary part), and sorting that set. One thus obtains a distribution (in the sense of statistics). One can view this as a generalization of the approach of Buser and Sarnak [BS], who showed that the smallest period of a Riemann surface is smaller than expected.

2. General Conjectures

Numerical experiments, described below, lead to three conjectures. The first two are specific to the case of hyperelliptic surfaces.

If the arguments of the periods of a compact hyperelliptic Riemann surface were distributed uniformly over the unit circle, the expected value of the distance between arguments would be $R_g = \pi/g(g+1)$.

Conjecture 1. *The periods of a hyperelliptic Riemann surface are* band-limited*, that is, there exists an argument α and a radius $\varepsilon < R_g$ such that no arguments fall outside of the intervals $(\alpha - \varepsilon, \alpha + \varepsilon)$ and $(\alpha + \pi - \varepsilon, \alpha + \pi + \varepsilon)$. In other words, there is a large interval containing no argument of periods.*

Conjecture 2. *The distribution of the magnitudes of the periods of a hyperelliptic Riemann surface are characterized by large gaps.*

See below for more specifics about the nature of the gaps.

The third conjecture is more general.

Conjecture 3. *The distribution of the squared modulus of the periods of a non-hyperelliptic Riemann Surface is* Zipfian*, that is, there is a power $p < 1$ such that the n^{th} modulus squared grows like n^{-p}.*

3. Remarks on Truncated Arithmetic

While Alice can send Bob an exact representation of an irrational or transcendental period matrix entry (*e.g.*, $\frac{1}{2}+i\frac{\sqrt{5}}{2}$), it is important to be aware of the effect of truncation to something like $(.5+1.12i)$. Since the various loci of compact Riemann Surfaces with interesting properties are small, it seems likely that an approximation to the period matrix of a compact Riemann Surface with some interesting property will fail to reveal the property. For example, if a Riemann Surface has a non-trivial conformal automorphism, period matrices of Riemann Surfaces without non-trivial automorphisms are arbitrarily close.

This section presents some reassuring results in this regard. Consider *Bring's curve*, which is the unique compact Riemann surface of genus four with the full symmetric group S_5 of automorphisms. Its period matrix was determined by Riera and Rodriguez [12]. Their matrix depends on a parameter, that is, there is a 1−parameter family of such matrices in $\mathcal{H}_4$, one of which is actually the period matrix for Bring's curve. This value is transcendental, so any numerical investigation of the matrix will necessarily involve an approximation.

Using the techniques of Accola [1], it is possible to determine a large number of vanishings of Riemann's theta function at quarter periods of the Jacobian of Bring's curve from the various involutions in the automorphism group. Remarkably, an exhaustive search for vanishings of θ at the quarter periods of the principally polarized abelian variety constructed from an *approximation* to the period matrix of Riera and Rodríguez found the vanishings predicted by Accola'a method. In other words, while one must be cautious about using approximations, it may still be possible to determine interesting properties of a compact Riemann surface from an approximation to its period matrix

(The data files from this investigation are too large to appear in print; the author will gladly make them available on request.)

4. Notation

All Riemann surfaces to be considered are compact and and of genus $g > 1$. Choose a *symplectic homology basis*, that is, a basis $\{A_1, \ldots, A_g, B_1, \ldots, B_g\}$ for the singular homology group $H_1(W, \mathbb{Z})$. Let $A \cdot B$ denote the intersection product of the cycles A and B. In a symplectic basis, these are, by definition, as follows: for all i and j, $A_i \cdot A_j = B_i \cdot B_j = 0$, and $A_i \cdot B_j = \delta_{ij}$.

One can also choose a normalized basis ω_i for $H^{(1,0)}(W)$, the vector space of holomorphic 1-forms; normalization means that $\int_{A_i} \omega_j = \delta_{ij}$. The matrix

$$\Omega = \left[\int_{B_i} \omega_j \right]$$

is called the *period matrix*, and the columns of $[I, \Omega]$, where I is the $g \times g$ identity, define a lattice L_Ω in $\mathbb{C}^g$. The complex torus

$$\mathrm{Jac}W = \mathbb{C}^g / L_\Omega$$

is the *Jacobian* of W. Torelli's Theorem asserts that either Ω or, equivalently, Jac$W$ completely determines the conformal type of a Riemann surface W.

Now, let $\mathcal{H}_g$ denote the *Siegel upper half space* of symmetric $g \times g$ complex matrices with positive-definite imaginary part; every period matrix lies in the Siegel upper half space. The symplectic group $SP(2g, \mathbb{Z})$ acts on $\mathcal{H}_g$, and the quotient $\mathcal{A}_g = \mathcal{H}_g / SP(2g, \mathbb{Z})$ is a Hausdorff analytic space, providing a moduli space for PPAVs. The geometric meaning of this action is change-of-basis in the homology and cohomology groups.

Let $z \in \mathbb{C}^g$ and $\Omega \in \mathcal{H}_g$, and define *Riemann's theta function* by

$$\theta(z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp \pi i ({}^t n \Omega n + 2 {}^t n z);$$

here, ${}^t A$ denotes the transpose of the matrix A. The series converges absolutely and uniformly on compact subsets when $\Omega \in \mathcal{H}_g$.

It is important to keep in mind that most of the results presented here are *experimental*.

5. Hyperelliptic Period Matrices

Of particular interest are the periods of hyperelliptic Riemann Surfaces. In the communications scenario, the question becomes "If Alice sends Bob a period matrix, can an eavesdropper Eve determine whether it comes from a hyperelliptic surface?" In principle, Eve could apply the by–now classical theorems of Farkas [8] showing that certain vanishings of Riemann's Theta function at half-periods determine whether a surface is hyperelliptic. However, this calculation is daunting when the genus g is large, since there are 2^{2g} half periods to check.

Rauch's Theorem, mentioned in the previous section, does not apply in the hyperelliptic case.

Numerical experiments (using Maple) suggest that Eve might have an easier way to reject the hypothesis that the period matrix comes from a hyperelliptic curve. Note that if the arguments of the periods were distributed uniformly about the circle, the expected distance between an argument and its nearest neighbor would be $R_g = 4\pi/g(g+1)$. Deviations from uniformity indicate some special property of the associated Riemann Surface.

These conjectures only make sense when the genus is large.s

Previous results about the periods, such as the results of Bujalence, Costa, Gamboa, and Riera [3] on the periods of Accola–MacLachlan and Kulkarni Surfaces, depend on special properties of the surface. They did not examine the distribution of the arguments of the periods. These results apply to a specific family of hyperelliptic surfaces as well.

The rest of this paper will present some of the numerical evidence in support of the conjecture as well as some arguments supporting the idea that the distribution of hyperelliptic periods is somewhat intrinsic.

5.1. **Numerical Experiments.** Maple includes a package `algcurves` for computing period matrices, described in [De]. There is an intrinsic limitation that the coefficients must be Gaussian rationals, and there are extrinsic limitations in computing power. These experiments used Version 14 of Maple running on a 12 core 3.47GHz server, as well as on a variety of smaller machines.

5.2. **Procedures.** To generate a "random" hyperelliptic curve, choose a degree d, number of terms r, and degrees $d_1, \ldots, d_{r-1}$. The coefficient c_j of x^{d_j} is formed by choosing four "random" integers a_{j_1}, b_{j_1}, a_{j_2}, b_{j_2} and setting

$$c_j = \frac{a_{j_1}}{b_{j_1}} + i\frac{a_{j_2}}{b_{j_2}}.$$

Maple then computes the period matrix of the curve $y^2 - \prod(x - \hat{c_j})$, where $\hat{c_j}$ denotes Maple's internal representation of c_j (the `algcurves` routine uses decimal representations of the coefficients of the curve).

Once the period matrix is computed, the complex argument and complex modulus functions are mapped onto the matrix. The moduli and arguments are extracted as a vector, whose entries are sorted in decreasing order of argument, and finally plotted.

The computational limitation appears to be on the size of d and on the number of terms r. Maple is unable to handle curves when either is large. For many of the cases here the clock time for generating the period matrix was approximately 1 minute, but for some the computation did not terminate in a reasonable amount of time.

5.3. **Results.** Figure 1 shows a scatterplot of the periods of a "random" hyperellliptic of genus 39. The horizontal axis is the argument; the vertical axis is the magnitude.

Notice that the two clusters differ by about π, *i.e.*, the periods are clustered around a line in $\mathbb{C}$. This occurred for all of the hyperelliptic curves tested.

To show that this pattern is not general, consider in Figure 2 the scatterplot of a "random" trigonal curve of similar genus.

While there is evidence of clustering, there are no large gaps between the arguments as there were in the hyperelliptic case.

For further contrast, Figure 3 is a scatterplot of the periods of the Fermat curve whose projective equation is $x^{10}+y^{10}+z^{10} = 0$. While there seems to be a preferred argument, the remaining arguments are well distributed around the circle.

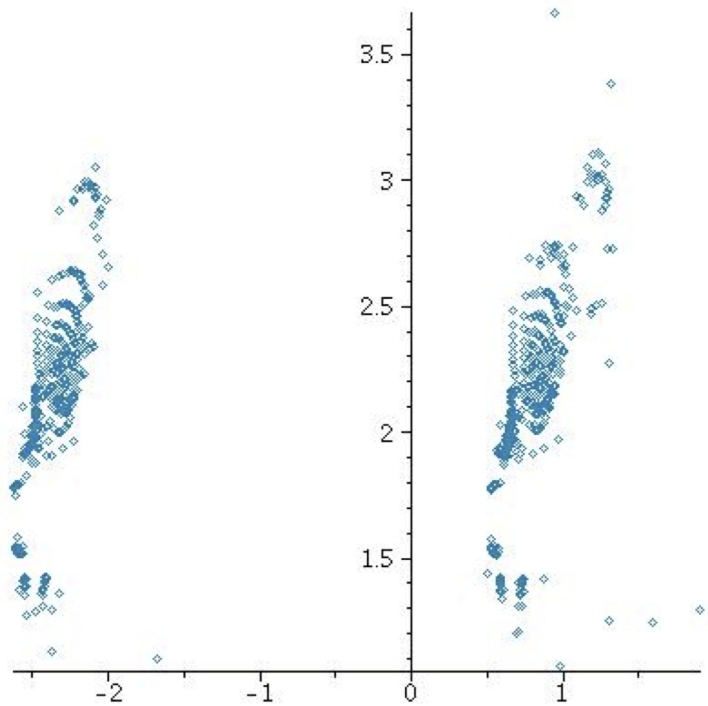

Figure 1. Periods of a Hyperelliptic Curve, Genus 39.

As further evidence that the distributions of periods of hyperelliptic curves have distinctive characteristics, consider the plot below of the *magnitudes* of the periods of a hyperelliptic curve $y^2 = f(x)$ where the degree of f is 96.

Notice that there are large "gaps" in the distribution of the magnitudes, which would be larger for the distribution of the squared magnitudes. All hyperelliptic curves investigated had such gaps, while no non-hyperelliptic curves investigated had comparable gaps. This is the evidence for 2.

5.4. **Analytical Evidence.** The examples shown (and many similar examples) motivate the conjecture above. Further evidence comes from the following Proposition, which indicates that the distribution of the periods of hyperelliptic curves maintains its basic shape as the curve varies in moduli space; in other words, the distributions seen in the examples hold, at least qualitatively, for "nearby" curves.

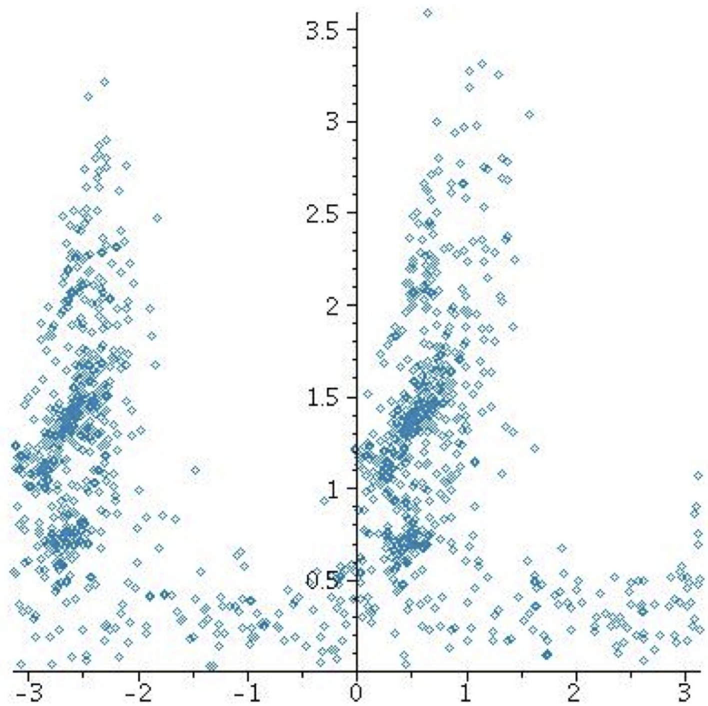

FIGURE 2. Periods of a Trigonal Curve, Genus 39.

Figure 4 shows the image of a canonical homology basis under the projection $(x, y) \mapsto x$ for a hyperelliptic curve X of genus 2 given by an equation $y^2 = f(x)$, where f has degree 6. The the branch cuts between the ramification points are black. The red (rectilinear) cycles represent the a–cycles, and the blue (curved) cycles represent the b–cycles. While some of the a–cycles appear to meet b–cycles twice, the second intersection does not occur on X itself. The images of the b–cycles cross the image of two branch cuts; this is necessary in order for the cycle to "jump" properly between sheets of X. Each cycle a_i encloses i branch cuts in such a way that the intersection products are correct.

The forms $\frac{x^a dx}{\sqrt{f}}$ $(0 \leq a \leq g-1)$ form a basis for $H^{(1,0)}(X)$. Normalize these to a basis $\{\omega_a\}$ such that $\int_{a_j} \omega_i = \delta_{ij}$.

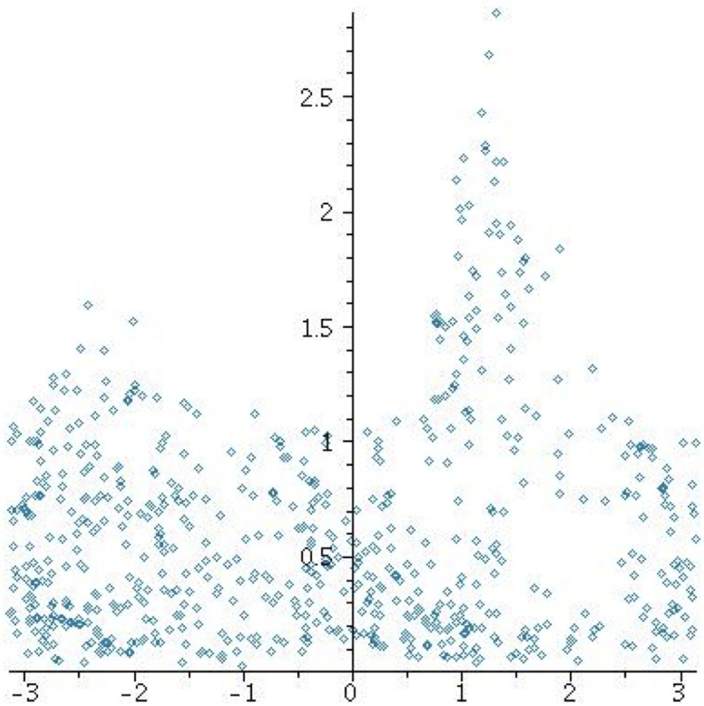

FIGURE 3. Periods of Fermat Curve of Degree 10

Lemma 1. *There is a canonical homology basis such that one of the branch points is not in the interior of any of the images of the a– or b–cycles in the x–plane. Call this the* free ramification point*.*

Proof. Examining Figure 4 shows that one of the branch points is not enclosed by any of the a– or b–cycles. This is because the number of branch cuts is one more than the genus, so one branch cut is not enclosed by any b-cycle, and since the image of each a–cycle encloses two points, two of the $2g+2$ branch points are not enclosed by such a cycle. □

The point labelled s in Figure 4 is a free ramification point. Clearly the choice of free ramification point depends on the choice of canonical homology basis.

Rewrite the equation for X as

$$y^2 = (x-s)f_\infty(x);$$

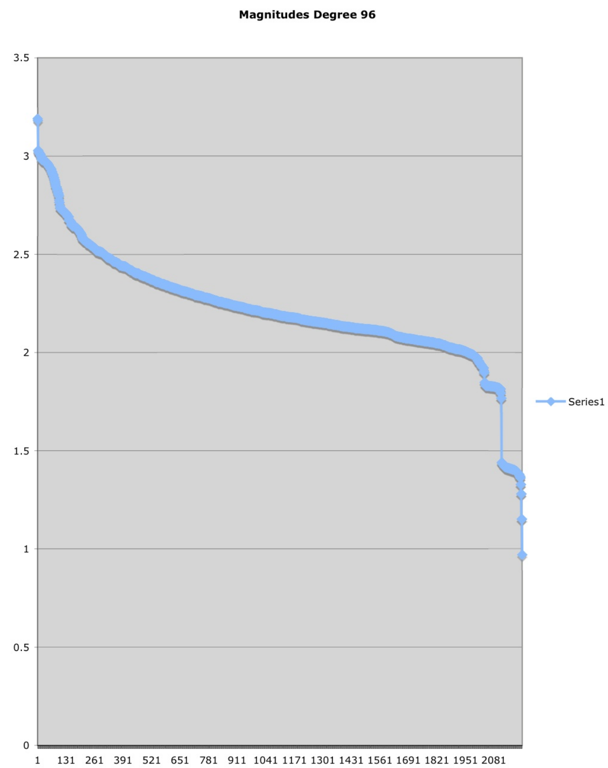

FIGURE 4. Magnitudes Periods of Hyperelliptic Curve of Degree 96

where f_∞ has degree $2g-1$. The curve $y^2 = f_\infty(x)$ has genus g but is ramified at infinity. Let X_s denote the curve $y^2 = (x-s)f_\infty(x)$, and X_∞ denote the curve $y^2 = f_\infty(x)$.

Proposition 1. *When the free ramification point on a hyperelliptic curve is large the period matrix is approximately that of X_∞.*

Proof. Let s denote the free ramification point. The key to the proof is to notice that the canonical homology basis stays the same as s becomes large, although the basis for $H^{(1,0)}(X)$ depends on s. For convenience, let $\sigma = \sqrt{x-s}$, so the equation for X_s is $y^2 = \sigma^2 f_\infty(x)$.

When s is large, σ is approximately constant, so the unnormalized period matrix is approximately $\frac{1}{\sigma}$ times the period matrix for X_∞. Tracing through the normalization computation leads immediately to the result. $\square$

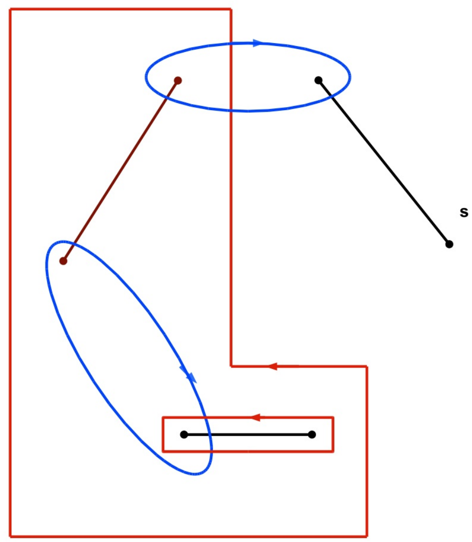

FIGURE 5. Canonical Homology Basis.

6. The General Case

Using numerical procedures similar to those described above, one can examine Maple-generated period matrices for arbitrary (within Maple's limitations) plane curves. These results deal with the modulus (absolute value) of the periods, rather than the arguments. This list was sorted in descending order.

Notice that it is possible to construct elements of $\mathcal{H}_g$ with any distribution by choice of the entries. In particular, the distribution may be concave down, concave up, have apparent discontinuities, or even be linear. But compare these possibilities with a typical result, the periods of the Fermat curve of degree 11, which appears below.

All of the period matrices of non-hyperelliptic surfaces examined have had a similar shape to the distribution, that is, generally concave up (which is consistent with a Zipfian distribution), hence conjecture 3.

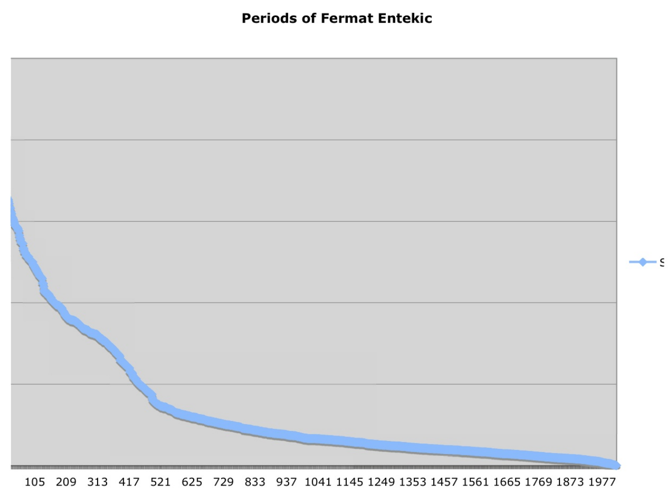

FIGURE 6. Periods of the Fermat Curve of degree 11

From the perspective of signal processing, this conjecture would imply that the period matrix of a Riemann surface has a fairly small number of *large* coefficients, while most are *small* – in effect, noise.

7. Conclusions

The numerical experiments outlined here demonstrate that the information-theoretic perspective leads to interesting new questions about the period matrices of compact Riemann surfaces.

In addition to proving or refuting the conjectures presented here, further work should explore the application of ideas from *Compressed Sensing* [6] [5], a technique from signal processing which can often find exact reconstruction of sparse signals from small samples. Sparsity occurs in natural signals such as image. A period matrix, whose size is $O(g^2)$ but which depends on only $O(g)$ parameters, is also a sparse signal. Rauch's 1954 result mentioned above suggests that this approach could be fruitful.

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