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# PISOT DUAL TILINGS OF LOW DEGREE AND THEIR DISCONNECTEDNESS

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Abstract. We study the connectedness of the graph-directed self-affine tiles associated to $\beta-$expansions, called Pisot dual tilings. These tiles are examples of Rauzy fractals and play an important role in the study of $\beta$-expansion, substitution and symbolic dynamical system. Using the complete classification of the $\beta$-expansion of 1 for quartic Pisot units and the classification of the connected tilings given in [4] and [5], here we continue studying connectedness of Pisot dual tilings generated by a Pisot unit with integral minimal equation $x^4-ax^3-bx^2-cx-1=0$ in the special case when $a+c-2\lfloor\beta\rfloor=1$. It is shown that every tile is disconnected having infinitely many connected components.

## 1. Introduction

Let $\beta > 1$ be a real number which is not an integer. A *greedy expansion* of a positive real $x$ in base $\beta$ is an expansion of the form:

$$x=\sum_{i=N_0}^{\infty} a_{-i}\beta^{-i}=a_{-N_0},a_{-N_0-1},\cdots$$

with $a_i\in[0,\beta)\cap\mathbb{Z}$ and a *greedy condition*

$$0\le x-\sum_{N_0}^{N} a_{-i}\beta^{-i}<\beta^{-N} \qquad \forall N\ge N_0$$

Let $1=d_{-1}\beta^{-1}+d_{-2}\beta^{-2}+\cdots$ be an expansion of 1 defined by the algorithm

$$c_{-i}=\beta c_{-i+1}-\lfloor\beta c_{-i+1}\rfloor,\quad d_{-i}=\lfloor\beta c_{-i+1}\rfloor \tag{1}$$

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with $c_0 = 1$, where $\lfloor x \rfloor$ denotes the maximal integer not exceeding $x$. The sequence $d_\beta(1) = .d_{-1}, d_{-2}, \cdots$ is called $\beta-$*expansion of* 1.

Parry [12] has shown that a sequence $x = x_1, x_2, \cdots$ of nonnegative integers is realized as a $\beta-$expansion of some positive real number if and only if it satisfies the following lexicographical condition:

$$\forall p \geq 0, \quad \sigma^p(x) <_{\text{lex}} d^*(1)$$

with

$$d^*(1) = \begin{cases} d_\beta(1), & \text{if } d_\beta(1) \text{ is infinite;} \\ (d_{-1}, d_{-2}, \cdots, d_{-n+1}, (d_{-n} - 1), )^\omega, & \text{if } d_\beta(1) = d_{-1}, \cdots, d_{-n}, \end{cases}$$

where for a string of symbols $u$, $u^\omega$ represents the periodic expansion $u, u, \cdots$ and $\sigma$ is the shift defined by $\sigma((x_i)_{i \leq M}) = (x_{i-1})_{i \leq M}$. In this case this sequence $x = x_1, x_2, \cdots$ is called *admissible sequence*.

## 2. Tiling Construction

Let $\beta$ be a *Pisot number* which is an algebraic integer greater than 1 whose Galois conjugates other than itself have modulus smaller than 1. Let $\mathbb{Q}(\beta)_{\geq 0}$ be the nonnegative elements of the minimum field containing the rational numbers $\mathbb{Q}$ and $\beta$. We call a *Pisot unit* a Pisot number which is also a unit of the integer ring of $\mathbb{Q}(\beta)$.

The symbolic dynamical system attached to the $\beta$-expansion is sofic if and only if the $\beta-$expansion of 1 is eventually periodic. Especially when $\beta$ is a Pisot number it gives a sofic system. Thurston [15] introduced an idea to construct a self-affine tiling generated by a Pisot unit $\beta$ in connection to this sofic system. Akiyama [2] and Praggastis [13] studied in detail such self-affine tilings. G. Rauzy [14] already constructed this kind of tiling in a different approach closely related to substitutions. This tiling has a strong connection to the explicit construction of Markov partitions of dynamical systems, hopefully toral automorphisms; see also P. Arnoux-Sh. Ito [6].

Let us recall this tiling, which is called *dual tiling*, in the notion of [2]. Let

$$\beta = \beta^{(1)}, \beta^{(2)}, \cdots, \beta^{(r_1)} \text{ and } \beta^{(r_1+1)}, \overline{\beta^{(r_1+1)}}, \cdots, \beta^{(r_1+r_2)}, \overline{\beta^{(r_1+r_2)}}$$

be the real and the complex conjugates of $\beta$, respectively. We also denote by $x^{(j)}$ $(j = 1, 2, \cdots, r_1 + 2r_2)$ the corresponding conjugates of $x \in \mathbb{Q}(\beta)$.

Define a map

$$\Phi\colon \mathbb{Q}(\beta) \to \mathbb{R}^{\,r_1+2\,r_2-1},$$

by

$$\Phi(x) = \left(x^{(2)}, \cdots, x^{(r_1)}, \Re(x^{(r_1+1)}), \Im(x^{(r_1+1)}), \cdots, \Re(x^{(r_1+r_2)}), \Im(x^{(r_1+r_2)})\right).$$

Let $A = .a_{-1}, a_{-2}, \cdots$ be a greedy expansion in base $\beta$. Define $S_A$ to be the set of elements of $\mathbb{Z}[\beta]_{\geq 0}$ whose greedy expansion has the tail $A$. In other words we just classify all elements of $\mathbb{Z}[\beta]_{\geq 0}$ by their fractional part and map via $\Phi$ to have a protile $T_A = \overline{\Phi(S_A)}$. It is not so easy to show that these $T_A$ will give a non overlapping tiling of the space $\mathbb{R}^{r_1+2r_2-1}$. The finiteness condition described in [9] or its weaker version, namely *weakly finiteness* condition, described in [3] implies that these $T_A$ will give a non overlapping tiling of the space $\mathbb{R}^{r_1+2r_2-1}$; see also [2].

One of important aspects of the self-affine tiles is connectedness. Note that if a tile is connected then it must be arcwise connected. This is seen by Hata in [11].

The aim of this paper is to explore the disconnectedness problem of Pisot dual tilings of degree 4 with minimal equation $x^4 - ax^3 - bx^2 - cx - 1 = 0$ where $a, b, c \in \mathbb{Z}$. A general arcwise connectedness criterion for Pisot dual tilings is established in [4]. It is proved that each tile corresponding to a Pisot unit $\beta$ is arcwise connected if $d_\beta(1)$ terminates with 1.

To treat all Pisot units the above result is not enough since $\beta-$expansion of 1 is not finite in general. If $p(0) = 1$ then $\beta-$expansion of 1 can not be finite (see Proposition 1 of [1]). Even when $p(0) = -1$ there are many such cases. The main result of [4] that we want to extend here is:

*Let $\beta$ be a Pisot unit of degree 4 with minimal polynomial $p(x) = x^4 - ax^3 - bx^2 - cx - 1$. Then each tile is arcwise connected except for the following cases:*

$$\begin{cases} a \geq 5 \\ c = a - 3 \\ \dfrac{5-3a}{2} \leq b \leq -a \end{cases} \quad \begin{cases} a \geq 3 \\ c = a - 1 \\ \dfrac{1-a}{2} \leq b \leq -1 \end{cases} \quad \begin{cases} a \geq 3 \\ c = a + 1 \\ \dfrac{1+a}{2} \leq b \leq a-1 \end{cases} \quad \begin{cases} a \geq 1 \\ c = a + 3 \\ \dfrac{5+3a}{2} \leq b \leq 2a+2 \end{cases}$$

The above result was proved in [4] and [5] to be equivalent to:

*Let $\beta$ be a Pisot unit of degree 4 with minimal polynomial $p(x) = x^4 - ax^3 - bx^2 - cx - 1$. Then*

- $a + c - 2\lfloor\beta\rfloor \leq 1$,
- each tile is arcwise connected if and only if $a + c - 2\lfloor\beta\rfloor \leq 0$.

In fact, here we prove that if $\deg \beta = 4$, $p(0) = -1$ and $a + c - 2[\beta] = 1$, each tile is disconnected having infinitely many connected components. As far as we know, no example of such type of disconnected Pisot dual tiles was known before. As the Pisot dual tiles are generated by consecutive integers, it was expected that they are always connected. Thus this result gives an unfortunate surprise that there exists a concrete family of Pisot units each tile of whose dual tiling is disconnected.

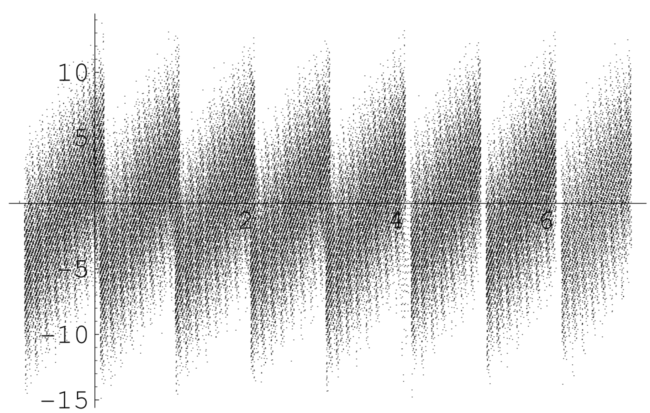


FIGURE 1. The projection of the central tile (disconnected) generated by the Pisot unit $\beta$ with minimal equation $x^4 - 7x^3 - 4x^2 - 8x - 1 = 0$

## 3. Disconnectedness

Let $\beta$ be a quartic Pisot unit of degree 4 with integral minimal polynomial

$$p(x) = x^4 - ax^3 - bx^2 - cx - 1.$$

From now on, we will assume that

$$a + c - 2\lfloor\beta\rfloor = 1. \tag{2}$$

Let $d_\beta(1) = .d_{-1}, d_{-2}, \cdots$ be the $\beta-$expansion of 1.

**Lemma 3.1.** *Suppose that $\xi = \xi_{-1}, \xi_{-2}, \cdots$ is an admissible expansion with $.\xi \geq .d_{-2}, d_{-3}, \cdots$. Then for every $m \in \{1, 2, \cdots, \xi_{-1}\}$ the following holds*

$$T_{.\xi} \cap (T. + \phi(\xi - m\beta^{-1}) = \emptyset$$

*or written in an equivalent way we can say that under Condition (2), if $T_{.e_1,e_2} \cap T_{.f_1,f_2} \neq \emptyset$ and $e_2 - f_2 \geq 1$ then $e_1 = \lfloor\beta\rfloor$ and $f_1 = 0$.*

*Proof.* First, recall that the condition $a+c-2\lfloor\beta\rfloor = 1$ is equivalent to the following 4 cases:

$$i)\ \begin{cases} a \geq 5 \\ c = a - 3 \\ \dfrac{5-3a}{2} \leq b \leq -a \end{cases}$$

$$ii)\ \begin{cases} a \geq 3 \\ c = a - 1 \\ \dfrac{1-a}{2} \leq b \leq -1 \end{cases}$$

$$iii)\ \begin{cases} a \geq 3 \\ c = a + 1 \\ \dfrac{1+a}{2} \leq b \leq a-1 \end{cases}$$

$$iv)\ \begin{cases} a \geq 1 \\ c = a + 3 \\ \dfrac{5+3a}{2} \leq b \leq 2a+2 \end{cases}$$

Let $\gamma$ be the negative root of the equation $x^2 - \lfloor\beta\rfloor x - 1$. Let us show that in 4 possible cases we have that $p(\gamma) > 0$.

i) If $c = a-3$ then $\lfloor\beta\rfloor = a-2$ and $b \leq -a$. So

$$p(\gamma) \geq \gamma^4 - a\gamma^3 + a\gamma^2 - (a-3)\gamma - 1 = -\gamma^3 + 2\gamma^2 > 0$$

ii) If $c = a-1$ then $\lfloor\beta\rfloor = a-1$ and $b \leq -1$. So

$$p(\gamma) \geq \gamma^4 - a\gamma^3 + \gamma^2 - (a-1)\gamma - 1 = \gamma^4 - a\gamma^3 > 0$$

iii) If $c = a+1$ then $\lfloor\beta\rfloor = a$ and $b \leq a-1$. So

$$p(\gamma) \geq \gamma^4 - a\gamma^3 - (a-1)\gamma^2 - (a+1)\gamma - 1 = -\gamma^3 + \gamma^2 > 0$$

iv) If $c = a+3$ then $\lfloor\beta\rfloor = a+1$ and $b \leq 2a+2$. So

$$p(\gamma) \geq \gamma^4 - a\gamma^3 - (2a+2)\gamma^2 - (a+3)\gamma - 1 = -\gamma^3 + 2\gamma^2 > 0$$

 

Let $\theta$ be the biggest among the negative roots of the polynomial $p(x) = x^4 - ax^3 - bx^2 - cx - 1$. The existence of such root is implied from the fact that $p(-1) > 0$ and $p(0) = 1$. Since $x \in (\theta, 0) \Rightarrow p(x) < 0$ then $\gamma < \theta < 0$. So $\theta^2 - \lfloor\beta\rfloor\theta - 1 < 0$.

If we suppose that $\exists m \in \{1, 2, \cdots, \xi_{-1}\}$ such that

$$T_{.\xi} \cap (T_{.} + \phi(\xi - m\beta^{-1}) \neq \emptyset$$

then exists an expansion of 0 in base $\theta$

$$m\theta^{-1} + c_0 + \sum_{i=1}^{\infty} c_i \theta^i$$

such that $1 \leq m \leq \xi_{-1}$, $\forall i \in \{0, 1, 2, \cdots\}$ we have that $c_i \in \mathbb{Z} \cap [-\beta, \beta]$ and $c_0 \leq \beta - 1$. So we have that

$$0 = m\theta^{-1} + c_0 + \sum_{i=1}^{\infty} c_i\theta^i \leq \theta^{-1} + \lfloor\beta\rfloor - 1 - \frac{\lfloor\beta\rfloor\theta}{1+\theta} = \frac{\theta^2 - \lfloor\beta\rfloor\theta - 1}{-\theta(1+\theta)} < 0.$$

This contradiction ends the proof of the current lemma. □

For a Pisot number of degree $d$ let $G_{-1}$ be the natural map defined by the following commutative diagram:

(3)
$$\begin{array}{ccc} \mathbb{Q}(\beta) & \xrightarrow{\times\beta} & \mathbb{Q}(\beta) \\ {\scriptstyle \Phi}\big\downarrow & & \big\downarrow{\scriptstyle \Phi} \\ \mathbb{R}^{d-1} & \xrightarrow[G_{-1}]{} & \mathbb{R}^{d-1}. \end{array}$$

where, we denote by $a \oplus b$ the concatenation of words $a, b$. Then $G_{-1}$ is contractive since $\beta$ is a Pisot number. The set equations are given in this form:

(4)
$$T_{.A} = \bigcup_{.i\oplus A} G_{-1}(T_{.i\oplus A}),$$

where the summation is taken over all possible $i \in [0, \beta) \cap \mathbb{Z}$ such that $i \oplus A$ is admissible (see [3]). Note that we identify $.i \oplus A$ with the corresponding $\beta-$expansion to realize it as a non negative real number. For the $\beta-$expansion of 1 that appear in the following lemmas see [10].

**Lemma 3.2.** *Let $d_\beta(1) = .\lfloor\beta\rfloor, d_{-2}, \cdots$ be the $\beta-$expansion of 1. If $d_{-2} < \lfloor\beta\rfloor$ then each tile is disconnected.*

*Proof.* Let $T_{.\omega}$ be a tile, which means that $0 \leq .\omega < .d_{-1}, d_{-2}, \cdots$. Here we consider two cases:

- $\lfloor\beta\rfloor \oplus \omega$ is admissible which is equivalent to $0 \leq .\omega < .d_{-2}, d_{-3}, \cdots$. We have that
$$T_{.\omega} = \bigcup_{i=0}^{\lfloor\beta\rfloor} G_{-1}(T_{.i\oplus\omega})$$
Since $\lfloor\beta\rfloor > d_{-2}$ then $.\lfloor\beta\rfloor \oplus \omega \geq .d_{-2}, d_{-3}, \cdots$. Using Lemma 3.1 we get that $\left(\bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega})\right) \bigcap G_{-1}(T_{.\lfloor\beta\rfloor\oplus\omega}) = \emptyset$, which shows that $T_{.\omega}$ is a disconnected tile.

- $\lfloor\beta\rfloor \oplus \omega$ is not admissible which is equivalent to $.\,d_{-2}, d_{-3}, \cdots \leq .\,\omega < .\,d_{-1}, d_{-2}, \cdots$. We have that

$$T_{.\,\omega} = \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}),$$

Let us show that $.\lfloor\beta\rfloor - 1 \oplus \omega \geq .d_{-2}, d_{-3}, \cdots$. If we suppose the contrary then $\lfloor\beta\rfloor - 1 \leq d_{-2}$, which implies that $d_{-2} = \lfloor\beta\rfloor - 1$ and $.\,\omega < .\,d_{-3}, d_{-4}, \cdots$. Since $\lfloor\beta\rfloor \oplus \omega$ is not admissible we get that

$$(5) \qquad .\,d_{-2}, d_{-3}, \cdots < .\,d_{-3}, d_{-4}, \cdots$$

Since $a + c - 2\lfloor\beta\rfloor = 1$, let us consider the 4 possibilities such that $d_{-2} = \lfloor\beta\rfloor - 1$.

(1) If $c = a - 3$, then
$d_\beta(1) = .a-2, 2a+b-2, (3a+2b-4, 3a+2b-5, 2a+b-3, 0, 1-a-b, 2-a-b, 0, 2a+b-3)^\omega$
So $d_{-2} = \lfloor\beta\rfloor - 1 = a - 3$ implies that $d_{-3} = \lfloor\beta\rfloor - 4$, which contradicts (5).

(2) If $c = a - 1$ then
$$d_\beta(1) = .a-1, a+b, (a+b, 0, -b, 0, a+b-1)^\omega$$
So $d_{-2} = \lfloor\beta\rfloor - 1 = a - 2$ implies that $.d_{-2}, d_{-3}, \cdots = .a-2, (a-2, 0, 2, 0, a-3)^\omega$ and $.d_{-3}, d_{-4}, \cdots = .(a-2, 0, 2, 0, a-3)^\omega$, which contradicts (5).

(3) If $c = a + 1$ then
$$d_\beta(1) = .\,a, b+1, (0, a-b, b, b, a-b+1, 0, b)^\omega$$
So $d_{-2} = \lfloor\beta\rfloor - 1 = a - 1 \geq 2$ contradicts (5).

(4) If $c = a + 3$ then $a \geq 3$ and
$$d_\beta(1) = .\,a+1, b-a-1, (2a-b+3, b-a-1, 0, 2a-b+3, 2b-3a-5, 4a-2b+6, 2b-3a-4, 2a-b+3, 0, b-a-2)^\omega$$
So $d_{-2} = \lfloor\beta\rfloor - 1 = a \geq 3$ implies that $d_{-3} = 2$, which contradicts (5).

So we proved that $.\lfloor\beta\rfloor - 1 \oplus \omega \geq .d_{-2}, d_{-3}, \cdots$. Using Lemma 3.1 we get that $\left(\bigcup_{i=0}^{\lfloor\beta\rfloor-2} G_{-1}(T_{.i\oplus\omega})\right) \bigcap G_{-1}(T_{.\lfloor\beta\rfloor-1\oplus\omega}) = \emptyset$, which shows that $T_{.\omega}$ is a disconnected tile.

□

**Lemma 3.3.** *Let $d_\beta(1) = .\lfloor\beta\rfloor, \lfloor\beta\rfloor, d_{-3} \cdots$ be the $\beta-$expansion of 1. If $d_{-3} < \lfloor\beta\rfloor$ then each tile is disconnected.*

*Proof.* Let $T_{.\,\omega}$ be a tile, for $\omega = \omega_1, \omega_2, \cdots$, which means that $0 \leq .\,\omega < .d_{-1}, d_{-2}, \cdots$. Since

$$.d_{-3}, d_{-4}, \cdots < .\lfloor\beta\rfloor, d_{-3}, d_{-4}, \cdots < \lfloor\beta\rfloor, \lfloor\beta\rfloor, d_{-3}, d_{-4}, \cdots$$

here we consider three cases:

- $\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \omega$ is admissible which is equivalent to $0 \le .\,\omega < .\,d_{-3}, d_{-4}, \cdots$. Here we have that

$$\begin{aligned} T_{.\,\omega} &= \bigcup_{i=0}^{\lfloor\beta\rfloor} G_{-1}(T_{.i\oplus\omega}) = \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \cup G_{-1}(T_{.\lfloor\beta\rfloor\oplus\omega}) \\ &= \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \cup \bigcup_{i=0}^{\lfloor\beta\rfloor} (G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor\oplus\omega}) \\ &= \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \cup \bigcup_{i=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor\oplus\omega}) \cup (G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor\oplus\omega}) \end{aligned}$$

Using Lemma 3.1, since $.\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \omega \ge .d_{-2}, d_{-3}, \cdots$, we get that

$$\bigcup_{i=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor\oplus\omega}) \cap (G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor\oplus\omega}) = \emptyset \tag{6}$$

Also, from the second part of Lemma 3.1, we get that

$$\bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i}) \cap \bigcup_{i=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.i,\lfloor\beta\rfloor}) \tag{7}$$

Since $G_{-1}(T_{.i\oplus\omega}) \subset G_{-1}(T_{.i}) + \phi(\omega_1\beta^{-1} + \omega_2\beta^{-2} + \cdots)$ and $(G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor\oplus\omega}) \subset (G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor}) + \phi(\omega_1\beta^{-1} + \omega_2\beta^{-2} + \cdots)$, using (6) and (7), we get that

$$\bigcup_{i=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor\oplus\omega}) \cap \left( (G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor\oplus\omega}) \cup \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \right) = \emptyset$$

which shows that $T_{.\omega}$ is a disconnected tile.

- $\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \omega$ is not admissible but $\lfloor\beta\rfloor \oplus \omega$ is admissible, which is equivalent to $.\,d_{-3}, d_{-4}, \cdots \le .\,\omega < .\,\lfloor\beta\rfloor, d_{-3}, \cdots$. Here we have that

$$T_{.\,\omega} = \bigcup_{i=0}^{\lfloor\beta\rfloor} G_{-1}(T_{.i\oplus\omega}) = \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \cup G_{-1}(T_{.\lfloor\beta\rfloor\oplus\omega})$$

Since $.\lfloor\beta\rfloor \oplus \omega \ge .d_{-2}, d_{-3}, \cdots$, using Lemma 3.1, we get that

$$\bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \cap G_{-1}(T_{.\lfloor\beta\rfloor\oplus\omega}) = \emptyset$$

which shows that $T_{.\omega}$ is a disconnected tile.

- $\lfloor\beta\rfloor \oplus \omega$ is not admissible which is equivalent to $.\,\lfloor\beta\rfloor, d_{-3}, \cdots \le .\,\omega < .\,\lfloor\beta\rfloor, \lfloor\beta\rfloor, d_{-3}, \cdots$.

Here we have that

$$\begin{aligned}
T_{.\omega} &= \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) = \bigcup_{i=0}^{\lfloor\beta\rfloor-2} G_{-1}(T_{.i\oplus\omega}) \cup G_{-1}(T_{.\lfloor\beta\rfloor-1\oplus\omega}) \\
&= \bigcup_{i=0}^{\lfloor\beta\rfloor-2} G_{-1}(T_{.i\oplus\omega}) \cup \bigcup_{i=0}^{\lfloor\beta\rfloor} (G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor-1\oplus\omega}) \\
&= \bigcup_{i=0}^{\lfloor\beta\rfloor-2} G_{-1}(T_{.i\oplus\omega}) \cup \bigcup_{i=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor-1\oplus\omega}) \cup (G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor-1\oplus\omega})
\end{aligned}$$

This case happens when
△ $c = a-3$, $\lfloor\beta\rfloor = a-2$ and $d_{-3} = a-4 = \lfloor\beta\rfloor - 2$,
△ $c = a+1$, $\lfloor\beta\rfloor = a \geq 3$, $b = a-1$ and $d_{-3} = 0$
△ $c = a+3$, $\lfloor\beta\rfloor = a+1 \geq 2$ and $d_{-3} = 1$.
So $d_{-3} < \lfloor\beta\rfloor - 1$, which implies that $\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor - 1 \oplus \omega \geq .d_{-2}, d_{-3}, \cdots$. ($d_{-3} = \lfloor\beta\rfloor - 1$ happens only when $a = 1$ and $c = a+3$. In this case also the previous inequality is true.) Using Lemma 3.1 we get that $\bigcup_{i=0}^{\lfloor\beta\rfloor-1} T_{.i\oplus\lfloor\beta\rfloor-1\oplus\omega} \cap T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor-1\oplus\omega} = \emptyset$ which implies that

$$\bigcup_{i=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor-1\oplus\omega}) \cap (G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor-1\oplus\omega}) = \emptyset \tag{8}$$

Also, from the second part of Lemma 3.1, we get that

$$\bigcup_{i=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor-1}) \cap \bigcup_{i=0}^{\lfloor\beta\rfloor-2} G_{-1}(T_{.i}) = \emptyset \tag{9}$$

Since $G_{-1}(T_{.i}) + \phi(\omega_1\beta^{-1} + \cdots) \supset G_{-1}(T_{.i\oplus\omega})$ and $(G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor-1}) + \phi(\omega_1\beta^{-1} + \cdots) \supset (G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor-1\oplus\omega})$, using (8) and (9), we get that

$$\bigcup_{i=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor-1\oplus\omega}) \cap \left( (G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor-1\oplus\omega}) \cup \bigcup_{i=0}^{\lfloor\beta\rfloor-2} G_{-1}(T_{.i\oplus\omega}) \right) = \emptyset$$

which shows that $T_{.\omega}$ is a disconnected tile.

□

**Lemma 3.4.** *Let $d_\beta(1) = .\lfloor\beta\rfloor, \lfloor\beta\rfloor, \lfloor\beta\rfloor, d_{-4}\cdots$ be the $\beta-$expansion of 1. Then each tile is disconnected.*

*Proof.* The supposition of lemma is equivalent to $a \geq 3$, $b = -1$ and $c = a-3$. Here $\lfloor\beta\rfloor = a-1$ and the $\beta-$ expansion of 1 is:

$$d_\beta(1) = .\lfloor\beta\rfloor, \lfloor\beta\rfloor, (\lfloor\beta\rfloor,\ 0\ ,\ 1\ ,\ 0\ , \lfloor\beta\rfloor - 1)^\omega$$

For $\omega = \omega_1, \omega_2, \cdots$, let $T_{.\omega}$ be a tile which means that $0 \leq .\omega < d_\beta(1)$. Since

$$.d_{-4}, d_{-5}, \cdots < .\lfloor\beta\rfloor, d_{-4}, \cdots < .\lfloor\beta\rfloor, \lfloor\beta\rfloor, d_{-4}, \cdots < .\lfloor\beta\rfloor, \lfloor\beta\rfloor, \lfloor\beta\rfloor, d_{-4}, \cdots$$

here we consider four possible cases:

**Case i)** $.\lfloor\beta\rfloor, \lfloor\beta\rfloor, d_{-4}\cdots \leq .\omega < d_\beta(1)$ which is equivalent to $\lfloor\beta\rfloor \oplus \omega$ is not admissible. Here we have that

$$\begin{aligned}
T_{.\omega} =& G_{-1}(T_{.0\oplus\omega}) \cup \bigcup_{i=1}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \\
=& \bigcup_{j=1}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.j\oplus0\oplus\omega}) \cup (G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus0\oplus\omega}) \cup \bigcup_{i=1}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \\
=& \bigcup_{j=1}^{\lfloor\beta\rfloor-1}(G_{-1})^2(T_{.j\oplus0\oplus\omega}) \cup \bigcup_{k=1}^{\lfloor\beta\rfloor-1}(G_{-1})^3(T_{.k\oplus\lfloor\beta\rfloor\oplus0\oplus\omega}) \cup (G_{-1})^3(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor\oplus0\oplus\omega}) \\
& \cup \bigcup_{i=1}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega})
\end{aligned}$$

Using Lemma 3.1, since $.\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus 0 \oplus \omega \geq .d_{-2}, d_{-3}, \cdots$, we get that

$$\bigcup_{k=0}^{\lfloor\beta\rfloor-1} (G_{-1})^3(T_{.k\oplus\lfloor\beta\rfloor\oplus0\oplus\omega}) \cap (G_{-1})^3(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor\oplus0\oplus\omega}) = \emptyset \tag{10}$$

Also, from the second part of Lemma 3.1, we get that

$$\begin{aligned}
&\bigcup_{k=1}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.k,\lfloor\beta\rfloor}) \cap \bigcup_{j=1}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.j}) = \emptyset \\
&\bigcup_{k=1}^{\lfloor\beta\rfloor-1} (G_{-1})^3(T_{.k,\lfloor\beta\rfloor}) \cap \bigcup_{i=1}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i}) = \emptyset
\end{aligned} \tag{11}$$

Since

$$(G_{-1})^2(T_{.j}) + \phi(\omega_1\beta^{-1} + \omega_2\beta^{-2} + \cdots) \supset (G_{-1})^2(T_{.j\oplus0\oplus\omega}),$$

then

$$\bigcup_{k=1}^{\lfloor\beta\rfloor-1} (G_{-1})^3(T_{.k,\lfloor\beta\rfloor}) + \phi(\omega_1\beta^{-1} + \cdots) \supset (G_{-1})^3(T_{.k\oplus\lfloor\beta\rfloor\oplus0\oplus\omega})$$

and

$$G_{-1}(T_{.i}) + \phi(\omega_1\beta^{-1} + \cdots) \supset G_{-1}(T_{.i\oplus\omega}).$$

Using (10) and (11), we get that

$$\begin{aligned}
&\bigcup_{k=1}^{\lfloor\beta\rfloor-1} (G_{-1})^3(T_{.k\oplus\lfloor\beta\rfloor\oplus0\oplus\omega}) \cap \\
&\left(\bigcup_{j=1}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.j\oplus0\oplus\omega}) \cup (G_{-1})^3(T_{.\lfloor\beta\rfloor,\lfloor\beta\rfloor,0,\omega}) \cup \bigcup_{i=1}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i,\omega})\right) = \emptyset
\end{aligned}$$

which shows that $T_{.\omega}$ is a disconnected tile.

**Case ii)** $.\lfloor\beta\rfloor, d_{-4}\cdots \leq .\omega < .\lfloor\beta\rfloor, \lfloor\beta\rfloor, d_{-4}\cdots$ which is equivalent to $\lfloor\beta\rfloor \oplus \omega$ is admissible but $\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \omega$ is not admissible. In the previous Lemma, for $.d_{-3}, d_{-4}, \cdots \leq .\omega < .d_{-2}, d_{-3}, \cdots$, we did not use the supposition that $d_{-3} < \lfloor\beta\rfloor$, so the proof shows that $T_{.\omega}$ is disconnected even if $d_{-3} = \lfloor\beta\rfloor$.

**Case iii)** $.d_{-4}, d_{-5}, \cdots \leq .\omega < .\lfloor\beta\rfloor, d_{-4}\cdots$ which is equivalent to $\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \omega$ is admissible ($\lfloor\beta\rfloor \oplus \omega$ is admissible also) but $\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \omega$ is not admissible.

$$\begin{aligned} T_{.\omega} &= \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \cup G_{-1}(T_{.\lfloor\beta\rfloor\oplus\omega}) \\ &= \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \cup \bigcup_{j=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.j\oplus\lfloor\beta\rfloor\oplus\omega}) \cup (G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor\oplus\omega}) \end{aligned}$$

Since $.\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \omega \geq .d_{-2}, d_{-3}, \cdots$, using Lemma 3.1 we have that

$$\bigcup_{j=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.j\oplus\lfloor\beta\rfloor\oplus\omega}) \cap (G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor\oplus\omega}) = \emptyset \tag{12}$$

Also from the second part of Lemma 3.1 we get that

$$\bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i}) \cap \bigcup_{j=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.j,\lfloor\beta\rfloor}) = \emptyset \tag{13}$$

Since

$$G_{-1}(T_{.i}) + \phi(\omega_1\beta^{-1} + \omega_2\beta^{-2} + \cdots) \supset G_{-1}(T_{.i\oplus\omega}),$$

and

$$(G_{-1})^2(T_{.j\oplus\lfloor\beta\rfloor}) + \phi(\omega_1\beta^{-1} + \cdots) \supset (G_{-1})^2(T_{.j\oplus\lfloor\beta\rfloor\oplus\omega}),$$

using (12) and (13) we get that

$$\bigcup_{j=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.j\oplus\lfloor\beta\rfloor\oplus\omega}) \cap \left( \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \cup (G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor\oplus\omega}) \right) = \emptyset$$

which shows that $T_{.\omega}$ is a disconnected tile.

**Case iv)** $0 \leq .\omega < .d_{-4}, d_{-5}, \cdots$ which is equivalent to $\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \omega$ is admissible ($\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \omega$ and $\lfloor\beta\rfloor \oplus \omega$ are admissible also). Since $T_{.\omega}$ is a translation of the central tile $T.$, it is enough to show that $T.$ is disconnected.

$$\begin{aligned} T. &= \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i}) \cup G_{-1}(T_{.\lfloor\beta\rfloor}) \\ &= \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i}) \cup (G_{-1})^2(T_{.0,\lfloor\beta\rfloor}) \cup \bigcup_{j=1}^{\lfloor\beta\rfloor} (G_{-1})^2(T_{.j,\lfloor\beta\rfloor}) \\ &= \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i}) \cup \bigcup_{k=0}^{\lfloor\beta\rfloor-1} (G_{-1})^3(T_{.k,0,\lfloor\beta\rfloor}) \cup (G_{-1})^3(T_{.\lfloor\beta\rfloor,0,\lfloor\beta\rfloor}) \cup \bigcup_{j=1}^{\lfloor\beta\rfloor} (G_{-1})^3(T_{.j,\lfloor\beta\rfloor}) \\ &= \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i}) \cup \bigcup_{k=0}^{\lfloor\beta\rfloor-1} (G_{-1})^3(T_{.k,0,\lfloor\beta\rfloor}) \cup \bigcup_{l=0}^{\lfloor\beta\rfloor-1} (G_{-1})^4(T_{.l,\lfloor\beta\rfloor,0,\lfloor\beta\rfloor}) \\ &\qquad \cup (G_{-1})^4(T_{.\lfloor\beta\rfloor,\lfloor\beta\rfloor,0,\lfloor\beta\rfloor}) \cup \bigcup_{j=1}^{\lfloor\beta\rfloor} (G_{-1})^2(T_{.j,\lfloor\beta\rfloor}) \end{aligned}$$

 

Since $.\lfloor\beta\rfloor, \lfloor\beta\rfloor, 0, \lfloor\beta\rfloor \geq .d_{-2}, d_{-3}, \cdots$, using Lemma 3.1, we get that

$$\bigcup_{l=0}^{\lfloor\beta\rfloor-1}(G_{-1})^4(T_{.l,\lfloor\beta\rfloor,0,\lfloor\beta\rfloor}) \cap (G_{-1})^4(T_{.\lfloor\beta\rfloor,\lfloor\beta\rfloor,0,\lfloor\beta\rfloor}) = \emptyset \tag{14}$$

Also, using the second part of Lemma 3.1, we get that

$$(G_{-1})^2(T_{.0,\lfloor\beta\rfloor}) \cap \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i}) = \emptyset, \quad \bigcup_{l=0}^{\lfloor\beta\rfloor-1}(G_{-1})^2(T_{.l,\lfloor\beta\rfloor}) \cap \bigcup_{k=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.k}) = \emptyset$$

$$(G_{-1})^2(T_{.\lfloor\beta\rfloor}) \cap \bigcup_{j=1}^{\lfloor\beta\rfloor} G_{-1}(T_{.j}) = \emptyset$$

Since

$$(G_{-1})^2(T_{.0,\lfloor\beta\rfloor}) \supset \bigcup_{l=0}^{\lfloor\beta\rfloor-1} (G_{-1})^4(T_{.l,\lfloor\beta\rfloor,0,\lfloor\beta\rfloor}),$$

then

$$\bigcup_{l=0}^{\lfloor\beta\rfloor-1} (G_{-1})^4(T_{.l,\lfloor\beta\rfloor}) + \phi(\lfloor\beta\rfloor) \supset \bigcup_{l=0}^{\lfloor\beta\rfloor-1} (G_{-1})^4(T_{.l,\lfloor\beta\rfloor,0,\lfloor\beta\rfloor})$$

and

$$\bigcup_{k=0}^{\lfloor\beta\rfloor-1} (G_{-1})^3(T_{.k}) + \phi(\lfloor\beta\rfloor) \supset \bigcup_{k=0}^{\lfloor\beta\rfloor-1} (G_{-1})^3(T_{.k,0,\lfloor\beta\rfloor}),$$

$$(G_{-1})^3(T_{.\lfloor\beta\rfloor}) + \phi(\lfloor\beta\rfloor) \supset \bigcup_{l=0}^{\lfloor\beta\rfloor-1} (G_{-1})^4(T_{.l,\lfloor\beta\rfloor,0,\lfloor\beta\rfloor})$$

$$\bigcup_{j=1}^{\lfloor\beta\rfloor}(G_{-1})^2(T_{.j}) + \phi(\lfloor\beta\rfloor) \supset \bigcup_{j=1}^{\lfloor\beta\rfloor}(G_{-1})^2(T_{.j,\lfloor\beta\rfloor})$$

we get that

$$\begin{aligned} &\bigcup_{l=0}^{\lfloor\beta\rfloor-1} (G_{-1})^4(T_{.l,\lfloor\beta\rfloor,0,\lfloor\beta\rfloor}) \cap \\ &\cap \left( \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i}) \cup \bigcup_{k=0}^{\lfloor\beta\rfloor-1}(G_{-1})^3(T_{.k,0,\lfloor\beta\rfloor}) \cup \bigcup_{j=1}^{\lfloor\beta\rfloor}(G_{-1})^2(T_{.j,\lfloor\beta\rfloor}) \right) = \emptyset \end{aligned} \tag{15}$$

Formulas (14) and (15) show that the central tile is disconnected.

□

Combining the results of Lemmas 3.2, 3.3 and 3.4 we get the following theorem:

**Theorem 3.1.** *Let $\beta$ be a Pisot unit with integral minimal equation $x^4 - ax^3 - bx^2 - cx - 1 = 0$ such that $a + c - 2\lfloor\beta\rfloor = 1$. Then each tile is disconnected having infinitely many connected components.*

## 4. Conclussions, Comments and Open Problems

In the previous works of Akiyama and Gjini in [4] and [5], it was proved that at least one of such tiles is disconnected. This result was generalized here. We proved that every dual tile is disconnected and furthermore each of them has infinitely many connected components which is a surprise because the digits of quartic $\beta-$expansions are consecutive integers which leads one to expect connected tiles. As a result we have a complete classification of connectedness of Pisot dual tiles with respect to quartic Pisot units. It remains to be found the characterization of $\beta$-expansion of 1 for Pisot numbers of lower degree as well as to study the connectedness, Hausdorff dimension of the boundary for the tiles generated by Pisot numbers of higher degree.

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