ON A GENERALIZATION OF ATOMIC DECOMPOSITIONS

S.K. KAUSHIK AND S.K. SHARMA

ABSTRACT. We generalize atomic decomposition for Banach spaces and called it T-atomic decomposition. A necessary condition for T-atomic decomposition is given. A characterization for a triangular atomic decomposition is also given. Finally, as an application of triangular atomic decompositions, we prove that if a Banach space has a triangular atomic decomposition, then it also has an approximative atomic decomposition, an atomic decomposition and a fusion Banach frame.

1. INTRODUCTION

Coifman and Weiss [3] introduced the notion of atomic decomposition for function spaces. Feichtinger and Gröchenig [5] extended the notion of atomic decomposition to Banach spaces. Frazier and Jawerth [6] had constructed wavelet atomic decompositions for Besov spaces which they called as ϕ -transform. Feichtinger [4] constructed Gabor atomic decompositions for the modulation spaces which are Banach spaces similar in many respects to Besov spaces, defined by smoothness and decay conditions. Atomic decompositions have played a key role in the development of wavelet theory and Gabor theory. Atomic decompositions and Banach frames were further studied in [1, 2, 8].

Motivated by Kozolov [10], we generalize atomic decompositions for Banach spaces. Infact, we introduce the notion of T-atomic decomposition for Banach spaces. Also, a necessary condition for T-atomic decomposition has been obtained. Further, a characterization for triangular atomic decomposition and a characterization for Banach frames have been obtained. Finally, as an application of triangular atomic decompositions, it has been proved that if a Banach space E has a triangular atomic decomposition, then E also has an approximative atomic decomposition, an atomic decomposition and a fusion Banach frame.

2. Preliminaries

Throughout this paper, E will denote a Banach space over the scalar field \mathbb{K} (\mathbb{R} or \mathbb{C}), E^* the dual space of E, L(E) the space of all linear operator on E, $[x_n]$ the closed linear span of $\{x_n\}$ in the norm topology of E, E_d an associated Banach space of scalar-valued sequences, indexed by \mathbb{N} .

A sequence $\{x_n\}$ in E is said to be *complete* if $[x_n] = E$ and a sequence $\{f_n\}$ in E^* is said to be *total* over E if $\{x \in E : f_n(x) = 0, n \in \mathbb{N}\} = \{0\}$. A sequence of projections $\{v_n\}$ on E is *total* on E if $\{x \in E : v_n(x) = 0, n \in \mathbb{N}\} = \{0\}$.

Definition 2.1 ([5]). Let E be a Banach space and E_d be an associated Banach space of scalar-valued sequences, indexed by \mathbb{N} . Let $\{x_n\} \subset E$ and $\{f_n\} \subset E^*$. Then, $(\{f_n\}, \{x_n\})$ is called an *atomic decomposition* for E with respect to E_d , if (i) $\{f_n(x)\} \in E_d, x \in E$

²⁰⁰⁰ Mathematics Subject Classification. 42C15, 42A38.

Key words and phrases. Atomic Decomposition, Banach Frame, Fusion Banach Frame.

(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_{E} \le \|\{f_{n}(x)\}\|_{E_{d}} \le B\|x\|_{E}, \quad x \in E$$

(iii) $x = \sum_{i=1}^{\infty} f_{i}(x)x_{i}, \quad x \in E.$

The constants A and B, respectively, are called lower and upper atomic bounds of the atomic decomposition $(\{f_n\}, \{x_n\})$.

Definition 2.2 ([7]). Let E be a Banach space and E_d be an associated Banach space of scalar-valued sequences, indexed by \mathbb{N} . Let $\{f_n\} \subset E^*$ and $S: E_d \to E$ be given. Then, $(\{f_n\}, S)$ is called a *Banach frame* for E with respect to E_d , if

(i)
$$\{f_n(x)\} \in E_d, x \in E$$

(ii) there exist constants A and B with $0 < A \le B < \infty$ such that

 $(2.1) A \|x\|_E \le \|\{f_n(x)\}\|_{E_d} \le B \|x\|_E, \quad x \in E$

(iii) S is a bounded linear operator such that

$$S(\{f_n(x)\}) = x, \quad x \in E.$$

The constants A and B, respectively, are called lower and upper frame bounds of the Banach frame $(\{f_n\}, S)$. The operator $S : E_d \to E$ is called the reconstruction operator (or, the pre-frame operator). The inequality (2.1) is called the Banach frame inequality.

A generalization of the concept of Banach frame namely, fusion Banach frame was introduced and studied in [9] and defined as follows:

Definition 2.3. Let E be a Banach space. Let $\{G_n\}$ be a sequence of subspaces of E and $\{v_n\}$ be a sequence of non-zero linear projections such that $v_n(E) = G_n$, $n \in \mathbb{N}$. Let \mathcal{A} be a Banach space associated with E and $S : \mathcal{A} \to E$ be an operator. Then, $(\{G_n, v_n\}, S)$ is called a *frame of subspaces* (or, *fusion Banach frame*) for Ewith respect to \mathcal{A} , if

(i) $\{v_n(x)\} \in \mathcal{A}, x \in E$

(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that

 $A||x||_{E} \le ||\{v_{n}(x)\}||_{\mathcal{A}} \le B||x||_{E}, \quad x \in E$

(iii) S is a bounded linear operator such that

$$S(\{v_n(x)\}) = x, \quad x \in E$$

The constants A and B, respectively, are called lower and upper frame bounds of the frame of subspaces $(\{G_n, v_n\}, S)$.

The following results are referred in this paper and are listed in the form of lemmas:

Lemma 2.4 ([12]). If E is a Banach space and $\{f_n\} \subset E^*$ is total over E, then E is linearly isometric to the associated Banach space $E_d = \{\{f_n(x)\} : x \in E\}$, where the norm is given by $\|\{f_n(x)\}\|_{E_d} = \|x\|_E$, $x \in E$.

Lemma 2.5 ([9]). Let $\{G_n\}$ be a sequence of non-trivial subspaces of E and $\{v_n\}$ be a sequence of non-zero linear projections with $v_n(E) = G_n$, $n \in \mathbb{N}$. If $\{v_n\}$ is total over E, then $\mathcal{A} = \{\{v_n(x)\} : x \in E\}$ is a Banach space with norm $\|\{v_n(x)\}\|_{\mathcal{A}} =$ $\|x\|_E$, $x \in E$.

3. Main Results

We begin with the following definition of T-atomic decomposition

Definition 3.1. Let E be a Banach space, E_d be an associated Banach space of scalar-valued sequences, indexed by \mathbb{N} and $T = (t_{nm})$ be a matrix of scalars such

that

(3.1)
$$\sum_{j=1}^{\infty} |t_{nj}| \le M < \infty, \qquad n = 1, 2, 3, \dots$$

(3.2)
$$\lim_{n \to \infty} t_{nj} = 0, \qquad j = 1, 2, 3, \dots$$

(3.3)
$$\lim_{n \to \infty} \sum_{j=1}^{\infty} t_{nj} = 1.$$

Let $\{x_n\}$ be a sequence in E and $\{f_n\}$ be a sequence in E^* . Then, $(T, \{f_n\}, \{x_n\})$ is called a *T*-atomic decomposition for E with respect to E_d , if

(i) $\{f_n(x)\} \in E_d, x \in E$

(ii) there exist constants A and B with $0 < A \le B < \infty$ such that

$$A\|x\|_{E} \le \|\{f_{n}(x)\}\|_{E_{d}} \le B\|x\|_{E}, \quad x \in E$$

(iii)
$$\lim_{n \to \infty} \sum_{j=1}^{\infty} t_{nj} \left(\sum_{i=1}^{j} f_{i}(x)x_{i}\right) = x, \, x \in E.$$

In case, T is a triangular matrix, then $(T, \{f_n\}, \{x_n\})$ is said to be a *triangular* atomic decomposition for E with respect to E_d .

Regarding the existence of *T*-atomic decomposition, let *E* be a Banach space, $(\{f_n\}, \{x_n\})(\{f_n\} \subset E^*, \{x_n\} \subset E)$ be an atomic decomposition for *E* with respect to an associated Banach space E_d and $T = (t_{nm})$ be a matrix such that $t_{nn} = 1$, $n \in \mathbb{N}$ and $t_{nm} = 0, m \neq n$. Then $(T, \{f_n\}, \{x_n\})$ is a *T*-atomic decomposition for *E* with respect to E_d .

Also, one may observe that if E is a Banach space and $(T, \{f_n\}, \{x_n\})$ $(T = (t_{nm}), \{f_n\} \subset E^*, \{x_n\} \subset E)$ is a T-atomic decomposition for E with respect to E_d , then $\{x_n\}$ is complete in E and for each $n \in \mathbb{N}, \sigma_n : E \to E$ defined by

$$\sigma_n(x) = \sum_{j=1}^{\infty} t_{nj} \left(\sum_{i=1}^j f_i(x) x_i \right), \quad x \in E$$

is well defined bounded linear operator such that $\sup_{1 \le n < \infty} \|\sigma_n\| < \infty$.

Conversely, we have the following example

Example 3.2. Let $E = c_0$, the space of all sequences convergent to 0 in \mathbb{K} . Let $T = (t_{nm})$ be a matrix such that $t_{nn} = 1$, $n \in \mathbb{N}$ and $t_{nm} = 0$, $n \neq m$. Let $\{e_n\}$ be the sequence of unit vectors in E and $\{f_n\}$ be a sequence in E^* defined by

$$f_n = (0, 0, \dots, (-1)^n, 0, 0, \dots), \quad n \in \mathbb{N}.$$

Then, $\{e_n\}$ is complete in E and for each $n \in \mathbb{N}$, $\sigma_n : E \to E$ defined by

$$\sigma_n(x) = \sum_{j=1}^{\infty} t_{nj} \left(\sum_{i=1}^{j} f_i(x) e_i \right), \quad x \in E$$

is well defined bounded linear operator such that $\sup_{1 \le n < \infty} \|\sigma_n\| < \infty. \text{ But } \lim_{n \to \infty} \sigma_n(x) \neq x,$ for some $x \in E$. Infact, if we take $x = (1, 0, 0, \ldots) \in E$ then $\lim_{n \to \infty} \sigma_n(x) \neq x$. Hence, $(T, \{f_n\}, \{e_n\})$ is not a *T*-atomic decomposition for *E* with respect to any associated Banach space E_d .

In the next result, we prove that for any matrix $T = (t_{nm})$ satisfying (3.1)-(3.3), every atomic decomposition for E is also a T-atomic decomposition for E.

Theorem 3.3. Let $(\{f_n\}, \{x_n\})$ be an atomic decomposition for a Banach space E with respect to E_d . Then, for any matrix $T = (t_{nm})$ satisfying (3.1)-(3.3), $(T, \{f_n\}, \{x_n\})$ is a T-atomic decomposition for E with respect to E_d .

Proof. Let c_E be the Banach space of all convergent sequences of elements of E with the norm $\|\{z_k\}\|_{c_E} = \sup_{1 \le k < \infty} \|z_k\|_E$. For each $n \in \mathbb{N}$, define $u_n : c_E \to E$ by

$$u_n(\{z_k\}) = \sum_{j=1}^{\infty} t_{nj} z_j, \quad \{z_k\} \in c_E.$$

Then, each u_n is well defined on c_E and

$$|u_n\| = \sup_{\{z_k\}\in c_E} \|u_n(\{z_k\})\|$$
$$= \sum_{j=1}^{\infty} |t_{nj}| \le M, \quad n \in \mathbb{N}.$$

Now, for any $\{x_1, x_2, ..., x_m, 0, 0, ...\} \in c_E$, we have

$$\lim_{n \to \infty} u_n(\{x_1, x_2, \dots, x_m, 0, 0, \dots\}) = 0$$

and, for any $\{x, x, x, \ldots\} \in c_E$, we have

$$\lim_{n \to \infty} u_n(x, x, x, \ldots) = x, \quad x \in E.$$

Since, the set of all the elements of the form $\{x_1, x_2, \ldots, x_m, 0, 0, \ldots\}$ and $\{x, x, x, \ldots\}$, where $x_1, x_2, \ldots, x_m \in E$, $1 \le m < \infty$ and $x \in E$ is complete in c_E , we have

$$\lim_{n \to \infty} \sum_{j=1}^{\infty} t_{nj} z_j = \lim_{n \to \infty} u_n(\{z_k\}) = \lim_{k \to \infty} z_k.$$

Define, $S_n(x) = \sum_{i=1}^n f_i(x)x_i, n \in \mathbb{N}$ and $x \in E$. Then $\lim_{n \to \infty} S_n(x) = x, x \in E$. Therefore, $\lim_{n \to \infty} \sum_{j=1}^{\infty} t_{nj}s_j(x) = x, x \in E$. Hence, $(T, \{f_n\}, \{x_n\})$ is a T-atomic decomposition for E with respect to E_d . \Box

The converse of Theorem 3.3 may not be true as shown by the following example **Example 3.4.** Let $E = \ell^2$. Define $\{x_n\} \subset E$ and $\{f_n\} \subset E^*$ by

$$x_n = e_n - e_{n+1} f_n(x) = \langle e_1 + e_2 + \ldots + e_n, x \rangle, \quad x \in E, \ n = 1, 2, \ldots .$$

Then, $(\{f_n\}, \{x_n\})$ is not an atomic decomposition for E with respect to any associated Banach space E_d . But, by Lemma 2.4, there exist an associated Banach space $E_{d_0} = \{\{f_n(x)\} : x \in E\}$ with the norm $\|\{f_n(x)\}\|_{E_{d_0}} = \|x\|_E, x \in E$ and a matrix $T = (t_{nm})$ given by $t_{nm} = \frac{1}{n}, m = 1, 2, ..., n, t_{nm} = 0$ for m > n (n = 1, 2, ...) such that $(T, \{f_n\}, \{x_n\})$ is a T-atomic decomposition for E with respect to E_{d_0} .

Indeed,

$$\sigma_{n}(x) = \sum_{i=1}^{n} \frac{n-i+1}{n} f_{i}(x) x_{i}$$

$$= \sum_{i=1}^{n} \frac{n-i+1}{n} \left\langle \sum_{j=1}^{i} e_{j}, x \right\rangle (e_{i} - e_{i+1})$$

$$= \langle e_{1}, x \rangle e_{1} + \sum_{i=2}^{n} \left[\frac{n-i+1}{n} \left\langle \sum_{j=1}^{i} e_{j}, x \right\rangle e_{i} - \frac{n-i+2}{n} \left\langle \sum_{j=1}^{i-1} e_{j}, x \right\rangle e_{i} \right]$$

$$- \frac{1}{n} \left\langle \sum_{j=1}^{n} e_{j}, x \right\rangle e_{n+1}$$

$$= \sum_{i=1}^{n} \frac{n-i+1}{n} \langle e_i, x \rangle e_i - \frac{1}{n} \sum_{i=2}^{n+1} \left\langle \sum_{j=1}^{i-1} e_j, x \right\rangle e_i, \quad x \in E, n = 1, 2, 3, \dots$$

Since, $\lim_{n \to \infty} \sum_{i=1}^{n} \langle e_i, x \rangle e_i = x, x \in E$, we have

$$\lim_{n\to\infty}\sum_{i=1}^n\frac{n-i+1}{n}\langle e_i,x\rangle e_i=x,\quad x\in E.$$

For each $n \in \mathbb{N}$, define $v_n : E \to E$ by

$$v_n(x) = \frac{1}{n} \sum_{i=1}^n \left\langle \sum_{j=1}^i e_j, x \right\rangle e_{i+1}, \quad x \in E, n = 1, 2, \dots$$

Then, each v_n is well defined bounded linear operator on E. Also, for each n, k = 1, 2, 3..., we have

$$||v_n(e_k)||^2 = \frac{1}{n^2} \sum_{i=1}^n \left| \left\langle \sum_{j=1}^i e_j, e_k \right\rangle \right|^2 = \frac{n-k+1}{n^2}.$$

Therefore, $\lim_{n\to\infty} ||v_n(e_k)||^2 = 0$. Hence, $\lim_{n\to\infty} v_n(x) = 0$, $x \in \operatorname{span}\{x_i\}_{i=1}^{\infty}$. Also, since

$$||v_n(x)||^2 = \frac{1}{n^2} \sum_{i=1}^n \left| \left\langle \sum_{j=1}^i e_j, x \right\rangle \right|^2$$

$$\leq \frac{1}{n^2} \sum_{i=1}^n \left\| \sum_{j=1}^i e_j \right\|^2 ||x||^2$$

$$= \frac{n(n+1)}{2n^2} ||x||^2$$

$$\leq ||x||^2, \quad x \in E, \ n = 1, 2, 3, \dots,$$

we have, $\sup_{1 \le n < \infty} ||v_n|| < \infty$. Hence, $\lim_{n \to \infty} \sigma_n(x) = x, x \in E$.

Next, we give a necessary condition for a $T\mbox{-}{\rm atomic}$ decomposition in a Banach space.

Theorem 3.5. Let E be a Banach space and $T = (t_{nm})$ be a matrix satisfying (3.1)-(3.3). If $(T, \{f_n\}, \{x_n\})(\{f_n\} \subset E^*, \{x_n\} \subset E)$ is a T-atomic decomposition for E with respect to E_d . Then for each $n, m \in \mathbb{N}$, there exists a linear operator $v_{nm} \in L(E)$ such that

 $\lim_{n \to \infty} \lim_{m \to \infty} v_{nm}(x) = x, \quad x \in E.$

Proof. For each n, m = 1, 2, 3..., define

$$v_{nm}(x) = \sum_{j=1}^{m} t_{nj} \left(\sum_{i=1}^{j} f_i(x) x_i \right), \quad x \in E.$$

Then, $v_{nm} \in L(E)$. Also

$$\lim_{m \to \infty} v_{nm}(x) = \sum_{j=1}^{\infty} t_{nj} \left(\sum_{i=1}^{j} f_i(x) x_i \right), \quad x \in E.$$

Since, $(T, \{f_n\}, \{x_n\})$ is a T-atomic decomposition for E, therefore

$$\lim_{n \to \infty} \sum_{j=1}^{\infty} t_{nj} \left(\sum_{i=1}^{j} f_i(x) x_i \right) = x, \quad x \in E.$$

Let E be a Banach space and $T = (t_{nm})$ be a triangular matrix satisfying (3.1)-(3.3). Let $\{x_n\}$ be any sequence in E and $\{f_n\}$ be any sequence in E^* . For each $n \in \mathbb{N}$, define

$$\sigma_n(x) = \sum_{j=1}^n t_{nj} \sum_{i=1}^j f_i(x) x_i, \quad x \in E, \ n = 1, 2, 3, \dots,$$

$$E_0^{(T)} = \{x \in E : \lim_{n \to \infty} \sigma_n(x) = x\} \text{ and }$$

$$E_1^{(T)} = \{x \in E : \lim_{n \to \infty} \sigma_n(x) \text{ exists}\}.$$

The following result characterizes triangular atomic decompositions in terms of $\{\sigma_n\}$ and $E_0^{(T)}$

Theorem 3.6. Let E be a Banach space and $T = (t_{nm})$ be a triangular matrix satisfying (3.1)-(3.3). Let $\{f_n\} \subset E^*$ and $\{x_n\} \subset E$. Then there exists an associated Banach space E_{d_0} such that $(T, \{f_n\}, \{x_n\})$ is a triangular atomic decomposition for E with respect to E_{d_0} if and only if $\{\sigma_n\}$ is total on E and $E_0^{(T)} = E$.

Proof. Assume that $E_0^{(T)} = E$ and $\{\sigma_n\}$ is total on E. Let $x \in E$ such that $f_n(x) = 0$ for all $n \in \mathbb{N}$. Then $\sigma_n(x) = 0$, $n \in \mathbb{N}$. So totality of $\{\sigma_n\}$ yields x = 0. Therefore, by Lemma 2.4, there exists an associated Banach space $E_{d_0} = \{\{f_n(x)\}: x \in E\}$ with the norm $\|\{f_n(x)\}\|_{E_{d_0}} = \|x\|_E$, $x \in E$. Also, by hypothesis, we have $\lim_{n \to \infty} \sum_{j=1}^{\infty} t_{nj} \left(\sum_{i=1}^{j} f_i(x)x_i\right) = x, x \in E$. Hence, $(T, \{f_n\}, \{x_n\})$ is a triangular atomic decomposition for E with respect to E_{d_0} .

The converse part is straight forward.

We conclude this section with the following characterization of Banach frames in terms of $E_0^{(T)}$ and $E_1^{(T)}$

Theorem 3.7. Let E be a Banach space and $T = (t_{nm})$ be a triangular matrix satisfying (3.1)-(3.3). Let $\{x_n\} \subset E$ and $\{f_n\} \subset E^*$ such that $f_i(x_j) = \delta_{ij}, i, j \in \mathbb{N}$. Then there exist an associated Banach space E_d and a bounded linear operator $S: E_d \to E$ such that $(\{f_n\}, S)$ is Banach frame for E with respect to E_d if and only if $E_0^{(T)} = E_1^{(T)}$.

Proof. Let $(\{f_n\}, S)$ be a Banach frame for E. Then

$$f_m(\sigma_n(x)) = f_m\left(\sum_{j=1}^{\infty} t_{nj}\left(\sum_{i=1}^{j} f_i(x)x_i\right)\right)$$
$$= \left(\sum_{j=m}^{\infty} t_{nj}\right)f_m(x), \quad n, m = 1, 2, 3... \text{ and } x \in E$$

Let $x \in E_1^{(T)}$. Then

$$f_m(x - \lim_{n \to \infty} \sigma_n(x)) = f_m(x) - \lim_{n \to \infty} \left(\sum_{j=m}^{\infty} t_{nj}\right) f_m(x)$$
$$= f_m(x) \left[1 - \lim_{n \to \infty} \sum_{j=1}^{\infty} t_{nj} + \lim_{n \to \infty} \sum_{j=1}^{m-1} t_{nj}\right] = 0$$

Therefore, by the frame inequality for the Banach frame $(\{f_n\}, S)$, we have $x \in E_0^{(T)}$.

Conversely, let $x \in E$ be such that $f_n(x) = 0$, n = 1, 2, 3... Then $\sigma_n(x) = 0$ for all $n \in \mathbb{N}$. Since, $E_0^{(T)} = E_1^{(T)}$, we have x = 0. Therefore, by Lemma 2.4, there exist associated Banach space $E_{d_0} = \{\{f_n(x)\} : x \in E\}$ with the norm $\|\{f_n(x)\}\|_{E_{d_0}} = \|x\|_E$, $x \in E$ and a bounded linear operator $S : E_{d_0} \to E$ defined by $S(\{f_n(x)\}) = x, x \in E$ such that $(\{f_n\}, S)$ is a Banach frame for E with respect to E_{d_0} .

4. Applications

In this section, we give some applications of triangular atomic decompositions. First, we give the definition of approximative atomic decomposition introduced in [11].

Let *E* be a Banach space and let E_d be an associated Banach space of scalarvalued sequences, indexed by N. Let $\{x_n\} \subset E$ and $\{h_{n,i}\}_{i=1,2,\ldots,m_n} \subset E^*$, where $\{m_n\}$ is an increasing sequence of positive integers. Then, $(\{h_{n,i}\}_{i=1,2,\ldots,m_n}, \{x_n\})$ is called an approximative atomic decomposition for *E* with respect to E_d , if

(i) $\{h_{n,i}(x)\}_{i=1,2,...,m_n} \in E_d, x \in E$

(ii) there exist constants A and B with $0 < A \le B < \infty$ such that $A \| x \|_{T} \le \| \| h_{T} \|_{T} \le B \| \| x \|_{T} = x \in F$

$$A\|x\|_{E} \le \|\{h_{n,i}(x)\}_{i=1,2,\dots,m_{n}}\|_{E_{d}} \le B\|x\|_{E}, \quad x \in E$$

(iii) $x = \lim_{n \to \infty} \sum_{i=1}^{m_n} h_{n,i}(x) x_i, x \in E.$

In the following result, we prove that if a Banach space has a triangular atomic decomposition, then it also has an approximative atomic decomposition

Theorem 4.1. If a Banach space has a triangular atomic decomposition then it also has an approximative atomic decomposition.

Proof. Let E be a Banach space having a triangular atomic decomposition $(T, \{f_n\}, \{x_n\})(T = (t_{nm}), \{f_n\} \subset E^*, \{x_n\} \subset E)$ with respect to E_d . Since, T is a triangular matrix, for each $n, m \in \mathbb{N}, m \ge n$,

$$\sum_{j=1}^{m} t_{nj} \left(\sum_{i=1}^{j} f_i(x) x_i \right) = \sum_{j=1}^{n} t_{nj} \left(\sum_{i=1}^{j} f_i(x) x_i \right), \quad x \in E.$$

For each $n \in \mathbb{N}$, define $\sigma_n : E \to E$ by

$$\sigma_n(x) = \sum_{j=1}^n t_{nj} \left(\sum_{i=1}^j f_i(x) x_i \right), \quad x \in E.$$

Then, each σ_n is well defined finite rank linear operator on E. Since, for each $n \in \mathbb{N}, \sigma_n(E)$ is finite dimensional. So, there exist a sequence $\{y_{n,i}\}_{i=m_{n-1}+1}^{m_n}$ in E and a total sequence $\{g_{n,i}\}_{i=m_{n-1}+1}^{m_n}$ in E^* such that

$$\sigma_n(x) = \sum_{i=m_{n-1}+1}^{m_n} g_{n,i}(x) y_{n,i}, \quad x \in E, \ n \in \mathbb{N},$$

where $\{m_n\}$ is an increasing sequence of positive integers with $m_0 = 0$. Define, $\{z_n\} \subset E$ and $\{h_{n,i}\}_{i=1,2,\dots,m_n} \subset E^*$ by

$$z_i = y_{n,i}, \quad i = m_{n-1} + 1, \dots, \quad m_n,$$

$$h_{n,i} = \begin{cases} 0, & \text{if } i = 1, 2, \dots, m_{n-1} \\ g_{n,i}, & \text{if } i = m_{n-1} + 1, \dots, m_n, \qquad n \in \mathbb{N}. \end{cases}$$

Then, for each $x \in E$,

$$\lim_{n \to \infty} \sum_{i=1}^{m_n} h_{n,i}(x) z_i = \lim_{n \to \infty} \sigma_n(x) = x \,.$$

Let $x \in E$ be such that $h_{n,i}(x) = 0$, for all $i = 1, 2, ..., m_n$, $n \in \mathbb{N}$. Then x = 0. Therefore, by Lemma 2.4, there exists an associated Banach space $E_{d_0} = \{\{h_{n,i}(x)\}_{i=1,2,...,m_n} : x \in E\}$ with the norm given by $\|\{h_{n,i}(x)\}_{i=1,2,...,m_n}\|_{E_{d_0}} = \|x\|_E$, $x \in E$ such that, $(\{h_{n,i}\}_{i=1,2,...,m_n}, \{z_n\})$ is an approximative atomic decomposition for E with respect to E_{d_0} .

Corollary 4.2. If a Banach space E has a triangular atomic decomposition, then it also has an atomic decomposition.

Proof. Follows in view of Theorem 4.1.

Finally, we prove that, if for a suitably chosen triangular matrix T satisfying (3.1)-(3.3), E has a triangular atomic decomposition, then it also has a fusion Banach frame.

Theorem 4.3. Let E be a Banach space and $T = (t_{nm})$ be a triangular matrix such that $t_{nm} \neq 0$, $n \geq m$. If $(T, \{f_n\}, \{x_n\})(\{f_n\} \subset E^*, \{x_n\} \subset E)$ is a triangular atomic decomposition for E, then E has a fusion Banach frame.

Proof. By Theorem 4.1, E has approximative atomic decomposition. Let $\{x_n\} \subset E$ and $\{h_{n,i}\}_{i=1,2,\ldots,m_n} \subset E^*$ be sequences such that $(\{h_{n,i}\}_{i=1,2,\ldots,m_n}, \{x_n\})$ is an approximative atomic decomposition for E with respect to E_d , where $\{m_n\}$ is an increasing sequence of positive integers. For each $n \in \mathbb{N}$, define $u_n : E \to E$ by

$$u_n(x) = \sum_{i=1}^{m_n} h_{n,i}(x) x_i, \quad x \in E.$$

Then, each u_n is a well defined continuous linear operator on E with dim $u_n(E) < \infty$ and $\lim_{n \to \infty} u_n(x) = x, x \in E$. Define $G_n = u_n(E), n \in \mathbb{N}$. Then, each G_n is finite dimensional. Therefore, there exist a sequence $\{y_{n,i}\}_{i=1}^{m_n}$ in E and a total sequence $\{g_{n,i}\}_{i=1}^{m_n}$ in E^* such that

$$u_n(x) = \sum_{i=1}^{m_n} g_{n,i}(x) y_{n,i}, \quad x \in E \text{ and } n \in \mathbb{N}.$$

Now, for each $n \in \mathbb{N}$, define $v_n : E \to E$ by

$$v_n(x) = \sum_{i=1}^{m_n} g_{n,i}(x) y_{n,i}, \quad x \in E$$

Then, each v_n is a projection on G_n such that $\{x \in E : v_n(x) = 0, n \in \mathbb{N}\} = \{0\}$. Therefore, by Lemma 2.5, there exist an associated Banach space $\mathcal{A} = \{v_n(x) : x \in E\}$ with the norm given by $\|\{v_n(x)\}\|_{\mathcal{A}} = \|x\|_E, x \in E$ and a bounded linear operator $S : \mathcal{A} \to E$ given by $S(\{v_n(x)\}) = x, x \in E$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to \mathcal{A} .

Acknowledgement

The research of first author is supported by the UGC vide letter No.F. No. 8-1(5)/2010(MRP/NRCB) dated 23.03.2010.

References

- [1] O. Christensen, An introduction to Frames and Riesz Bases, Birkhäuser, 2003.
- [2] O. Christensen and C. Heil, Perturbation of Banach frames and atomic decompositions, *Math. Nach.*, 185 (1997) 33-47.
- [3] R.R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc., 83 (1977) 569-645.
- [4] H.G. Feichtinger, Atomic characterizations of Modulation spaces through Gabor-Type Representation, *Rocky Mountain J. Math.*, 19(1989) 113-126.
- [5] H.G. Feichtinger and K. Gröchenig, A unified approach to atomic decompositions via integrable group representations, In: *Proc. Conf. "Function Spaces and Applications"*, Lecture Notes Math. 1302, Berlin-Heidelberg-New York: Springer (1988) 52-73.
- [6] M. Frazier and B. Jawerth, Decompositions of Besov spaces, Indiana Univ. Math. J., 34 (1985) 777-799.
- [7] K. Gröchenig, Describing functions: Atomic decompositions versus frames, Monatsh. Math., 112(1991) 1-41.
- [8] P.K. Jain, S.K. Kaushik and Nisha Gupta, On near exact Banach frames in Banach spaces, Bull. Aust. Math. Soc., 78 (2008) 335-342.
- [9] S.K. Kaushik and Varinder Kumar, Frames of subspaces for Banach spaces, International Jour. of Wavelets, Multiresolution and Information Processing, 8(2) (2010) 243–252.
- [10] V. Ya. Kozolov, On a generalization of the notion of basis, Doklady Akad. Nauk. SSSR, 73 (1950) 643-646.
- [11] S.K. Kaushik and S.K. Sharma, On approximative atomic decompositions in Banach spaces, General Mathematics, (to appear).
- [12] I. Singer, Bases in Banach Spaces II, Springer-Verlag, Berlin, Heidelberg, New York, 1981.

Department of Mathematics, Kirori Mal College, University of Delhi, Delhi 110 007, INDIA.

 $E\text{-}mail\ address:\ \texttt{shikk2003@yahoo.co.in}$

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI, DELHI 110 007, INDIA. *E-mail address*: sumitkumarsharma@gmail.com