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ABSOLUTELY SUMMING OPERATORS IN $m_1(l_1)$

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ABSTRACT. A scalar sequence (a_i) is said to be a *p*-multiplier of a Banach space X, if it satisfies the following condition:

$$m_p(X) = \left\{ a = (a_i) : \sum_i ||a_i x_i||^p < \infty, \forall (x_i) \in l_w^p(X) \right\}.$$

In this paper we will prove the following: every bounded linear operator from Banach space l_1 into $m_1(l_1)$, is an absolutely summing operator.

1. INTRODUCTION

The theory of absolutely summing operators has as a starting point from the famous resume of Grothendick [5], in which it was proved that every bounded linear operator from Banach space l_1 into l_2 is an absolutely summing operator. In later works by Pietsch necessary and sufficient conditions are given under which an operator is an absolutely summing operator; see [4], [6]. In this context the absolutely summing operators were studied in the sequence spaces and function spaces by several authors; see [4] for further references. The sequence space $m_1(l_1)$ was defined by the authors S. Aywa and J. H. Fourie in [2]. In this paper we prove that every bounded linear operator from Banach space l_1 into $m_1(X)$ is an absolutely summing operator, after taking in consideration the definition of the 1-colacunary sequences given in [1] and their properties. Also we give a result which characterizes the absolutely summing operators from space $m_1(X)$ into l_2 , in case where X contains a basis which satisfies the 1-colacunarity.

2. Preliminaries

In the first part of the paper we will prove that every bounded linear operator from Banach space l_1 into $m_1(l_1)$, is an absolutely summing operator. In the second part we will prove the following: If X contains a basic and 1-colacunary vector sequence (x_i) , then every bounded linear operator from l_1 into $m_1(X)$ is an absolutely summing operator. In the sequel we will briefly describe the notation and definitions which are used throughout the paper.

Let Λ denote the vector space of scalar sequences (a_i) , where (a_i) are from \mathbb{R} or \mathbb{C} , i.e.,

$$\Lambda = \{ a = (a_i) : a_i \in \mathbb{R} \quad \text{or} \quad a_i \in \mathbb{C} \}.$$

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The space $m_p(X)$, is defined as follows

(1)
$$m_p(X) = \left\{ a = (a_i) \in \Lambda : \sum_i ||a_i x_i||^p < \infty, \forall (x_i) \in l_w^p(X) \right\},$$

and is a Banach space under the norm

$$||(a_i)||_{p,p} = \sup_{\epsilon_p((x_i)) \le 1} \left(\sum_{n \in \mathbb{N}} |a_n|^p ||x_n||^p \right)^{\frac{1}{p}},$$

where $\epsilon_p((x_i)) = \sup_{||a|| \le 1} ||a(x_i)||_p$, $a \in X^*$ (see [2]). By $l_w^p(X)$ we will denote the Banach space

$$l_w^{\ p}(X) = \left\{ x = (x_i) \in X : \left(\sum_i |x^*(x_i)|^p \right)^{\frac{1}{p}} < \infty, x^* \in X^* \right\}.$$

For the class of the scalar sequences $m_p(X)$, the following inclusion holds

$$l_p \subseteq m_p(X) \subset l_{\infty},$$

for any $1 \leq p \leq \infty$.

Definition 1. Let X be a Banach space. A sequence $(x_n)_{n \in \mathbb{N}}$ in X is p-colacunary if there is a $\delta > 0$ such that

$$\left\|\sum_{i\leq n} a_i x_i\right\| \geq \delta\left(\sum_{i\leq n} |a_i|^p\right)^{\frac{1}{p}},$$

for any sequence of scalars a_0, a_1, \cdots, a_n .

The following theorem is proved in [5], [6].

Theorem 1. Every bounded linear operator defined from Banach space l_1 into space l_2 , is an absolutely summing operator.

The next result is known as the "Ideal property of p-summing operators"; see [4] for details.

Theorem 2. Let $1 \le p < \infty$ and $v \in \prod_p(X, Y)$. Then the composition of v with any bounded linear operator is p-summing.

3. Results

Theorem 3. Every bounded linear operator from Banach space l_1 into Banach space $m_1(l_1)$, is an absolutely summing operator.

Proof. From the facts mentioned above, in order to prove the Theorem, it is enough to prove the fact that the Banach space $m_1(l_1)$, is a subspace of the space l_2 and the norm in $m_1(l_1)$ is equivalent with the standard norm given in l_2 . Let us consider that $a = (a_i)$, is any scalar sequence from space $m_1(l_1)$,

(2)
$$m_1(l_1) = \left\{ a = (a_i) \in \Lambda : \sum_i ||a_i x_i|| < \infty, \forall (x_i) \in l_w^{-1}(l_1) \right\}.$$

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Let us denote by (e_i) the standard unit vector basis in l_1 , and let us define the operator A from l_1 into l_1 , by the following relation:

$$A: x = \sum_{i} b_i e_i \to \sum_{i} a_i b_i e_i,$$

for any sequence $(a_i) \in m_1(l_1)$. The above operator is well defined, because $(a_i b_i) \in l_1$. Indeed, from the above it was shown that $m_1(l_1) \subset l_\infty$ and from this follows that the following relation is true,

$$\sum_{i} |a_i \cdot b_i| \le \sup_{i} |a_i| \sum_{i} |b_i| < \infty.$$

We have

$$||Ax|| = \left\| A\left(\sum_{i} b_{i} \cdot e_{i}\right) \right\| = \left\| \left(\sum_{i} b_{i} \cdot a_{i} \cdot e_{i}\right) \right\|$$
$$= \sum_{i} |b_{i} \cdot a_{i}| \le \sup_{i} |a_{i}| \cdot \sum_{i} |b_{i}|$$
$$= \sup_{i} |a_{i}| \cdot \left\| \sum_{i} b_{i} \cdot e_{i} \right\| = \sup_{i} |a_{i}| \cdot ||x||.$$

Thus, it follows that the operator ${\cal A}$ is a bounded linear operator. Hence, we have

(3)
$$\sum_{i \in \mathbb{N}} ||A(a_i x_i)|| \le S \sum_{i \in \mathbb{N}} ||a_i x_i|| < \infty,$$

where $S = \sup_i |a_i|$ for any $(a_i) \in m_1(l_1)$ and $(x_i) \in l_1$. Without loss of generality we can assume that the sequence of vectors (x_i) is normalized. Taking in consideration the relation (3) we have:

$$\sum_{i \in \mathbb{N}} ||A(a_i x_i)|| = \sum_{i \in \mathbb{N}} |a_i| \cdot ||A(x_i)|| = \sum_{i \in \mathbb{N}} |a_i| \cdot ||a_i x_i|| = \sum_{i \in \mathbb{N}} |a_i|^2 < \infty.$$

The last relation proves that $(a_i) \in l_2$. Next, we aim to prove that the norm $||(a_i)||_{1,1}$ is equivalent with $||(a_i)||_{l_2}$. Let $(a_i) \in m_1(l_1)$. Then

(4)
$$||(a_i)||_{1,1} = \sup_{\epsilon_1((x_i)) \le 1} \sum_{n \in \mathbb{N}} |a_n| \cdot ||x_n|| \le \sup_n |a_n| \sup_{\epsilon_1((x_i)) \le 1} \sum_{n \in \mathbb{N}} ||x_n|| \le \left(\sum_{n \in \mathbb{N}} |a_n|^2\right)^{\frac{1}{2}} \cdot N = N \cdot ||(a_n)||_{l_2},$$

 $\forall (x_n) \in l_w^{-1}(l_1)$. From Schur's l_1 -theorem (see [4] for details) it follows that

$$N = \sup_{\epsilon_1((x_i)) \le 1} \sum_{n \in \mathbb{N}} ||x_n|| < \infty.$$

Take $(a_i) \in l_2$ and consider that $\sum_{i=1}^n a_i^2 = 1$. Then, from Dvoretzky-Rogers theorem it follows that there exists an unconditional sequence $(y_n) \in l_1$, such that $||y_n||_{l_1} = |a_n|$; see [6] for details. From unconditionality of (y_n) , we have that $\sum_{i \in \mathbb{N}} \theta_i y_i < \infty$, for any sequence of signs (θ_i) (see [6, Prop.1, c.1]), respectively $\sum_{i \in \mathbb{N}} |y_i| < \infty$, from which follows that it converges $\sum_{i \in \mathbb{N}} |y^*(y_i)| < \infty$ and

 $\sum_{i\in\mathbb{N}} |a_i| \cdot ||y_i|| = \sum_{i\in\mathbb{N}} |a_i|^2 < \infty$. Which means that the relation

$$\sup_{i_1(y_i) \le 1} \sum_{n \in \mathbb{N}} |a_n| \cdot ||y_n||$$

defines a norm $||(a_i)||_{1,1}$ on $m_1(l_1)$. Now we have the estimation,

(5)
$$||(a_n)||_{l_2} = \left(\sum_{n \in \mathbb{N}} |a_n|^2\right)^{\frac{1}{2}} = \left(\sum_{n \in \mathbb{N}} |a_n| \cdot ||y_n||\right)^{\frac{1}{2}} \le \left(\sup_{\epsilon_1((y_i)) \le 1} \sum_{n \in \mathbb{N}} |a_n|||y_n||\right)^{\frac{1}{2}} \le ||(a_n)||_{1,1}.$$

From relations (4) and (5) it follows that the norms $||(a_i)||_{l_2}$ and $||(a_i)||_{1,1}$ are equivalent. This completes the proof.

Proposition 4. Let $(x_n)_{n \in \mathbb{N}}$ be a basic and 1-colacunary sequence of vectors in a Banach space X. Then every bounded linear operator T from l_1 , into $m_1(X)$, is an absolutely summing operator.

Proof. Let (x_n) , be a 1-colacunary sequence of vectors in Banach space X, then there follows the following relation

$$\delta \cdot \sum_{i \le n} |a_i| \le \left\| \sum_{i \le n} a_i x_i \right\| \le \sum_{i \le n} |a_i|,$$

which means that sequence of vectors (x_n) is equivalent with standard unit vector basis of l_1 . The rest of the proof is similar to that of Theorem 3.

Theorem 5. Let $(x_n)_{n \in \mathbb{N}}$ be a basic and 1-colacunary sequence of vectors in a Banach space X. Then every bounded linear operator T from $m_1(X)$, into l_2 , is an absolutely summing operator.

Proof. Let us denote by (f_i) the basic sequence in Banach space $m_1(X)$, and let (x_i) be a basic and 1-colacunary sequence of vectors in X. Then,

$$\left\|\sum_{i\leq n}a_ix_i\right\|\geq \delta\sum_{i\leq n}|a_i|,$$

for any finite sequence of scalars a_0, a_1, \dots, a_n . From this relation it follows that

(6)
$$\delta \cdot \sum_{i \le n} |a_i| \le \left\| \sum_{i \le n} a_i x_i \right\| \le \sum_{i \le n} |a_i|,$$

...

for any sequence of scalars a_0, a_1, \dots, a_n . Let A be an operator defined from l_1 into $m_1(X)$, by the relation

$$A: x = \sum_i a_i e_i \to \sum_i a_i f_i,$$

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where (e_i) is the standard unit vector basis in l_1 . The operator A is well defined. Next, we prove that A is bounded from the upper side, lower side, and bijective, from which it follows that it has bounded inverse A^{-1} ; see [8]. We have

$$\begin{split} ||Ax|| &= \left\| A\left(\sum_{i} a_{i}e_{i}\right) \right\| = \left\| \sum_{i} a_{i}f_{i} \right\| = \left\| (a_{i}) \right\|_{m_{1}(X)} \\ &= \sup_{\epsilon_{1}((y_{i})) \leq 1} \left(\sum_{n} |a_{n}|^{1} \cdot ||y_{n}||^{1} \right) \leq \\ &\sup_{\epsilon_{1}((y_{i})) \leq 1} \sup_{n} ||y_{n}|| \cdot \sum_{n} |a_{n}| = M \cdot \left\| \sum_{i} a_{i}e_{i} \right\|_{l_{1}} = M \cdot ||x||, \end{split}$$

where $(y_i) \in l_w^{-1}(X)$ and $M = \sup_{\epsilon_1((y_i)) \leq 1} \sup_n ||y_n||$. In the similar way we can prove the lower bound of ||Ax||. In the following, it is enough to prove that A is onto (because injectivity follows from the definition). Let $y = (c_i) \in m_1(X)$ be any element from that space, then it is enough to prove that there follows

$$\sum_{i} |c_i| < \infty.$$

Relation (6) is true for any scalar sequence (a_i) , so it remains true if we are using the scalar sequence (c_i) , instead (a_i) i.e., the following relation is valid

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$$\sum_{i} |c_{i}| \leq \frac{1}{\delta} \cdot \left\| \sum_{i} c_{i} x_{i} \right\| \leq \frac{1}{\delta} \cdot \sum_{i} ||c_{i} x_{i}|| < \infty.$$

This proved that A is a bijective operator with bounded inverse. The following diagram is commutative



Let C denote the operator which is the composition of the operators A and T, i.e.,

(7)
$$C = T \cdot A$$

The operator C is a bounded linear operator from Banach space l_1 into space l_2 , so it is absolutely summing operator between them (Theorem 1). Then from relation (7) we will have that $C \cdot A^{-1} = T$. From Theorem 2, follows that the operator Tis also an absolutely summing operator.

The proof of the following Proposition is similar to that of Proposition 10 in [3].

Proposition 6. Let $(x_n)_{n \in \mathbb{N}}$ be a basic and 1-colacunary sequence of vectors in X. Then every infinite dimensional subspace Y of $m_1(X)$ is isomorphic to $m_1(X)$ and complemented in $m_1(X)$. Hence, the Banach space $m_1(X)$ is a Prime space.

Proof. Let H be an operator defined from the Banach space $m_1(X)$ into the space l_1 by the relation

$$H: x = \sum_{i} a_i f_i \to \sum_{i} a_i e_i,$$

where (f_i) and (e_i) , are basic sequences in $m_1(X)$, l_1 respectively. This operator is invertible (exactly as operator A in Theorem 4). Let Y be any infinite dimensional subspace of $m_1(X)$. Let us denote by $Y_1 = H(Y)$, the subspace of l_1 . From the decomposition method of Pelczynski it follows that

$$l_1 = Y_1 \oplus B$$

for some Banach space B; see [7]. Let $x \in m_1(X)$. Then $H(x) = y \in l_1$ and y has unique representation

$$(8) y = a + b$$

for suitable $a \in Y_1$ and $b \in B$. From this there is a $a_1 \in Y, H(a_1) = a$, $y = H(a_1) + b \Rightarrow H^{-1}(y) = H^{-1}(H(a_1)) + H^{-1}(b) \Rightarrow$

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(9)
$$x = a_1 + H^{-1}(b)$$

and the last representation of x is unique. If we use another representation of x we will have $x = a_1^{'} + H^{-1}(b^{'})$, then $H(x) = H(a_1^{'}) + b^{'} \Rightarrow$

(10)
$$y = H(a_1) + b'.$$

But relation (10) is in contradiction with relation (8). So every $x \in m_1(X)$ has unique representation through space Y, and we can use the notation

$$m_1(X) = Y \oplus C$$

for some Banach space C, with Y isomorphic to $m_1(X)$. Thus, $H(Y) = Y_1$ is isomorphic to l_1 . Let us denote by B that isomorphism between them. Then $B(l_1) =$ $BH(m_1(X)) = Y_1 \Rightarrow BH(m_1(X)) = H(Y)$ and from this follows that $H^{-1} \cdot B \cdot H$ is isomorphism between spaces $m_1(X)$ and Y. This completes the proof.

Corollary 7. The space $m_1(l_1)$ is a Prime space.

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