ALBANIAN JOURNAL
OF MATHEMATICS
Volume 1, Number 1, Pages 17–30
ISSN 1930-1235: (electronic version)

THE CRITERIA OF RIESZ, HARDY-LITTLEWOOD ET AL. FOR THE RIEMANN HYPOTHESIS REVISITED USING SIMILAR FUNCTIONS

STEFANO BELTRAMINELLI AND DANILO MERLINI

Abstract. The original criteria of Riesz and of Hardy-Littlewood concerning the truth of the Riemann Hypothesis (RH) are revisited and further investigated in light of the recent formulations and results of Maslanka and of Baez-Duarte concerning a representation of the Riemann Zeta function. Then we introduce a general set of similar functions with the emergence of Poisson-like distributions and we present some numerical experiments which indicate that the RH may barely be true.

1. Introduction

It is well known that there are many different criteria for the truth of the Riemann Hypothesis (RH). Some of these are not directly related to the important high level computations and developments concerning the non trivial zeros of the Riemann Zeta function. In fact, at the beginning of the century M. Riesz, and later G.H. Hardy and J.E. Littlewood (among other important results in number theory) found a criterion of "classical type" for the truth of the RH. The above criteria are related to some series involving values of the Zeta function outside the critical strip, i.e. at integers arguments of the Zeta function [8, 10], and in a numerical context, very accurate calculations are needed toward a "possible kind of verification" of the RH.

In the literature important remarks have been given by leading mathematicians (see for example, those cited in [4]). We may think that such criteria may have a limited interest since, with them, one should work outside the critical strip. It is, in fact, true that in dealing with the above criteria one needs the use of arguments of the Zeta function outside the critical strip, and problems of interchange of summations are present. As an example, in the above criteria, if one uses the formula established by the authors, one should give a meaning to an integration over the real line, which exists only for finite intervals. In order to obtain finite numerical results which give "satisfactory" values to the functions supposed to be equal to the reciprocal of the Zeta function outside and inside the critical strip, the integration should be carried out using a special sequence of upper limit of integration extending to infinity [7].

But lately, there have been new developments and rigorous results in connection with this kind of problem: first a "regularization" of the representations of the

Received by the editors August 11, 2006 and, in revised form, November 10, 2006.
2000 *Mathematics Subject Classification.* Primary 11M26.
Key words and phrases. Riemann Zeta function, Riemann Hypothesis, Criteria of Riesz, Hardy-Littlewood and Baez-Duarte.

Zeta function (a pioneering work by Maslanka [9]), followed (in particular) by a new rigorous discrete formulation with theorems concerning the above criteria (the works of Baez-Duarte [1, 2, 3, 4]).

In light of these new approaches, we thought that some of the above criteria deserved still more study, at least in the direction of some numerical experiments. Thus, we introduce additional functions containing two parameters, in order to have further confidence in the numerical results of the experiments.

The content of this work is as follows: in Section 2 we define a general set of functions with two parameters α and β in the spirit of Riesz and of Hardy-Littlewood and then obtain the discrete "representation" of the reciprocal of the Zeta function of our set by means of the two parameter Pochammer's polynomials with their coefficients. For the reader the discussion of the conditions are then given in Appendix A and in Appendix B (they follow strictly the ingenious method of Baez-Duarte for the Riesz case $\alpha = \beta = 2$). In Section 3 we then obtain in some "limit", a Poisson distribution for the coefficients c_k of the Pochammer's polynomials; this is useful in the context of the numerical experiments. These are presented in Section 4 where many various limiting cases are treated. In the case of increasing values of the parameter β, the experiments indicate that the Poisson distribution becomes more and more exact and the sequence c_k becomes a constant which can be evaluated.

We may argue that in the context of the range of validity of the experiments we present the RH may barely be true.

2. The model

We now consider a set of functions with two parameters $(\alpha > 1, \beta > 0)$ to obtain $\frac{1}{\zeta(s)}$. These are simply an extension of these two cases: the first (with $\alpha = \beta = 2$) introduced and studied by Riesz [10], the second one (where $\alpha = 1$ and $\beta = 2$) by Hardy-Littlewood [8].

Let $\mu(n)$ be the Möbius function of argument n, where:

$$\mu(n) = \begin{cases} 1, \text{ if } n = 1 \\ (-1)^k, \text{ if } n \text{ is a product of } k \text{ distinct primes} \\ 0, \text{ if } n \text{ contains a square} \end{cases}$$

Let $s = \sigma + \mathrm{i}t$ be a complex variable. For $\Re(s) > \rho = 1$ one has $\frac{1}{\zeta(s)} = \sum\limits_{n=1}^{\infty} \frac{\mu(n)}{n^s}$.

Following the original idea of Riesz and Hardy-Littlewood, we now introduce the two-parameters family of functions given by:

$$\varphi(s;\alpha,\beta) := \frac{1}{\Gamma(-\frac{s-\alpha}{\beta})} \int_0^\infty \sum_{n=1}^{\infty} \frac{\mu(n)}{n^\alpha} e^{-\frac{x}{n^\beta}} x^{-\left(\frac{s-\alpha}{\beta}+1\right)} dx \tag{1}$$

so that expanding the right-hand side in powers of x, we obtain:

$$\begin{aligned} \varphi(s;\alpha,\beta) &= \frac{1}{\Gamma(-\frac{s-\alpha}{\beta})} \int_0^\infty \sum_{n=1}^{\infty} \frac{\mu(n)}{n^\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k! n^{\beta k}} x^{-\left(\frac{s-\alpha}{\beta}+1\right)} dx \\ &= \frac{1}{\Gamma(-\frac{s-\alpha}{\beta})} \int_0^\infty \psi(x;\alpha,\beta) x^{-\left(\frac{s-\alpha}{\beta}+1\right)} dx \end{aligned}$$

where

$$\psi(x;\alpha,\beta)=\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!}\frac{1}{\zeta(\alpha+\beta k)} \tag{2}$$

The function $\psi(x;\alpha,\beta)$ was introduced by Riesz (case $\psi(x;2,2)$) and by Hardy-Littlewood (case $\psi(x;1,2)$).

If $\psi(x;\alpha,\beta)\sim\frac{A}{x^{\frac{\alpha-\rho}{\beta}-\epsilon}}$ for some ϵ and for large x, then

$$|\varphi(s;\alpha,\beta)|\leq\left|\frac{1}{\Gamma(-\frac{s-\alpha}{\beta})}\right|\int_0^\infty\frac{A}{x^{\frac{\alpha-\rho}{\beta}+\frac{\Re(s)-\alpha}{\beta}+1-\epsilon}}dx\leq\left|\frac{1}{\Gamma(-\frac{s-\alpha}{\beta})}\right|\int_0^\infty\frac{A}{x^{1+\frac{\Re(s)-\rho}{\beta}-\epsilon}}dx$$

would exist and would eventually be given by $\frac{1}{\zeta(s)}$ with $\zeta(s)\neq 0$ if we choose $\Re(s)>\rho+\beta\epsilon$.

Let $\rho=\frac{1}{2}$. For $\alpha=\beta=2$ we have:

$$\psi(x;2,2)\sim\frac{A}{x^{3/4-\epsilon}}$$

and for $\alpha=1,\beta=2$:

$$\psi(x;1,2)\sim\frac{A}{x^{1/4-\epsilon}}$$

On the other hand expanding (1) in a similar way, we have that:

$$\begin{aligned}\varphi(s;\alpha,\beta)&=\frac{1}{\Gamma(-\frac{s-\alpha}{\beta})}\int_0^\infty\sum_{n=1}^{\infty}\frac{\mu(n)}{n^\alpha}e^{x(1-\frac{1}{n^\beta})}e^{-x}x^{-(\frac{s-\alpha}{\beta}+1)}dx\\&=\frac{1}{\Gamma(-\frac{s-\alpha}{\beta})}\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{\mu(n)}{n^\alpha}\left(1-\frac{1}{n^\beta}\right)^k\int_0^\infty\frac{1}{k!}x^{k-\frac{s-\alpha}{\beta}-1}e^{-x}dx\\&=\frac{1}{\Gamma(-\frac{s-\alpha}{\beta})}\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{\mu(n)}{n^\alpha}\left(1-\frac{1}{n^\beta}\right)^k\frac{1}{k!}\Gamma(k-\frac{s-\alpha}{\beta})\\&=\frac{1}{\Gamma(-\frac{s-\alpha}{\beta})}\sum_{k=0}^{\infty}c_k\prod_{r=1}^{k}\left(1-\frac{\frac{s-\alpha}{\beta}+1}{r}\right)\Gamma(-\frac{s-\alpha}{\beta})\end{aligned}$$

Thus:

$$\varphi(s;\alpha,\beta)=\sum_{k=0}^{\infty}c_kP_k(\frac{s-\alpha}{\beta}+1) \tag{3}$$

where $P_k(x):=\prod_{r=1}^{k}(1-\frac{x}{r})$ are the Pochhammer polynomials and the sequences:

$$c_k(\alpha,\beta):=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^\alpha}\left(1-\frac{1}{n^\beta}\right)^k \tag{4}$$

were already studied by Baez-Duarte [2, 3] in the special case $\alpha=\beta=2$. For another sequence appearing in an expansion of ζ, still for $\alpha=\beta=2$ see the work of Maslanka [9].

Let $\Re(s)>\rho+\epsilon$ ($\epsilon>0$ and $\rho\in[1,\infty[$). From a theorem of Baez-Duarte [2, 3], which says that $|P_k(s)|\leq A\cdot k^{-\Re(s)}$ where A is a constant depending on $|s|$, for

large values of k we have that:

$$|\varphi(s;\alpha,\beta)| \leq \mathrm{A}\sum_{k=0}^{\infty} |c_k|\, k^{-\left(\frac{\rho+\epsilon-\alpha}{\beta}+1\right)} \tag{5}$$

In Appendix A we show that if $\alpha > 1$ and $\beta > 0$ the following holds unconditionally:

$$q_k \ll \frac{1}{k^{\frac{\alpha-1}{\beta}}}$$

where

$$q_k = \sum_{n=1}^{\infty} \frac{1}{n^\alpha}\left(1-\frac{1}{n^\beta}\right)^k$$

Then we obtain:

$$|\varphi(s;\alpha,\beta)| \leq \sum_{k=0}^{\infty} \frac{1}{k^{\frac{\alpha-1}{\beta}}} \cdot \frac{A}{k^{\frac{\rho+\epsilon-\alpha}{\beta}+1}} \leq \sum_{k=0}^{\infty} \frac{A}{k^{\frac{\rho+\epsilon-1}{\beta}+1}} \leq A\sum_{k=0}^{\infty} \frac{1}{k^{1+\frac{\epsilon}{\beta}}} < \infty$$

From this it follows that we can interchange integration and summation in the earlier calculations of φ and thus for $\Re(s) > 1$ we obtain (6) below, i.e. a representation of $[\zeta(s)]^{-1}$:

$$\varphi(s;\alpha,\beta) = \frac{1}{\zeta(s)} = \sum_{k=0}^{\infty} c_k P_k\left(\frac{s-\alpha}{\beta}+1\right), \quad \Re(s) > 1 \tag{6}$$

Now for $\Re(s) > \rho+\epsilon$ ($\epsilon > 0$ and $\rho \in [\frac{1}{2},\infty[$), still from the theorem of Baez-Duarte [2, 3] i.e. that

$$\left|P_k\left(\frac{s-\alpha}{\beta}+1\right)\right| \leq \frac{A}{k^{\frac{\Re(s)-\alpha}{\beta}+1}}$$

and assuming:

$$|c_k| \ll \frac{B}{k^{\frac{1}{\beta}(\alpha-\rho-\epsilon)}}$$

then the above series given by (3) converges uniformely. In fact for $\Re(s) > \rho+\epsilon$ we have:

$$|\varphi(s;\alpha,\beta)| \leq \sum_{k=0}^{\infty} \frac{B}{k^{\frac{1}{\beta}(\alpha-\rho-\epsilon)}} \frac{A}{k^{\frac{\Re(s)-\alpha}{\beta}+1}} = \sum_{k=0}^{\infty} \frac{C}{k^{1+\frac{1}{\beta}(\Re(s)-\rho-\epsilon)}}$$

Following Baez-Duarte the series $\varphi(s;\alpha,\beta)$ extends analytically $\frac{1}{\zeta(s)}$ to the half plane $\Re(s) > \rho = \frac{1}{2}$.

We have thus obtained for our family of sequences with parameters α, β that a necessary and sufficient condition for $\zeta(s) \neq 0$ in the half plane $\Re(s) > \rho$ ($\rho \in [\frac{1}{2},\infty[$) is given by:

$$|c_k(\alpha,\beta)| \leq \frac{\text{const}}{k^{\frac{1}{\beta}(\alpha-\rho-\epsilon)}} \qquad \forall\epsilon > 0, \forall\alpha > 1, \forall\beta > 0 \tag{7}$$

The necessity of the condition (7) is proved in Appendix B.

Remark 1. Instead of using the Möbius function μ in c_k, one may use (for the numerical computations) the formula involving values of the Zeta function:

$$
\begin{aligned}
c_k &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} \left(1 - \frac{1}{n^{\beta}}\right)^k \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{n^{\beta j}} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\zeta(\alpha + \beta j)}
\end{aligned} \tag{8}
$$

Remark 2. From the bound above it follows not only theoretically but also in the context of a numerical analysis that it will be equally difficult to treat the case $\rho \in [\frac{1}{2}, 1]$, for example $\rho = \frac{3}{4}$, as will be the case $\rho = \frac{1}{2} + \epsilon$ with ϵ small. Below in Section 4 we will also treat the case $\alpha = \frac{7}{2}$.

Remark 3. The condition for the truth of the RH using Riesz and Hardy-Littlewood functions $\psi(x)$ is essentially the same as the one using the discrete function c_k with $k \in \mathbb{N}$. In a previous work [7] independent of the present one (which essentially uses the Baez-Duarte idea and theorems) some numerical results were obtained for $\psi(x)$ in the case of the Hardy-Littlewood function ($\alpha = 1, \beta = 2$) by integration in the x-space. The discrete version using the function c_k of the discrete variable k [2, 3, 9] has advantages in the numerical computations which will be presented below. Before this we present another way to control the sequence c_k in a numerical context.

3. Poisson like distribution

We still consider the sequence c_k given by:

$$
c_k = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} \left(1 - \frac{1}{n^{\beta}}\right)^k
$$

Then,

$$
\begin{aligned}
c_k &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} e^{k \ln(1 - \frac{1}{n^{\beta}})} \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} e^{-\frac{k}{n^{\beta}}} e^{k(\ln(1 - \frac{1}{n^{\beta}}) + \frac{1}{n^{\beta}})} \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} e^{-\frac{k}{n^{\beta}}} e^{\Delta(k,n,\beta)}
\end{aligned}
$$

Notice that $\Delta < 0$. For β large we set $\Delta = 0$ to obtain the following approximation:

$$
c_k \cong \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} e^{-\frac{k}{n^{\beta}}}
$$

With this approximation we see that c_k becomes equal to $\psi(x = k)$ of (2) as may easily be checked. Moreover:

$$
c_k \cong \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} e^{k(1 - \frac{1}{n^{\beta}})} e^{-k} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} \sum_{p=0}^{\infty} \frac{k^p}{p!} \left(1 - \frac{1}{n^{\beta}}\right)^p e^{-k}
$$

Thus:

$$c_k \cong \sum_{p=0}^{\infty} c_p \frac{k^p}{p!} e^{-k} \tag{9}$$

We are in the presence of a Poisson distribution for the c_p: in this way, in our numerical computations, we may control in a "more satisfactory" way the values of c_k. The approximation for c_k by means of the Poisson distribution for the c_k we found, will be more satisfactory with increasing values of β and for large values of k. We may also use the approximation given by (9) in which the upper limit of summation will be given by N instead of ∞, i.e. for large k,

$$c_k \cong \sum_{p=0}^{N} c_p \frac{k^p}{p!} e^{-k} \tag{10}$$

4. Numerical experiments

4.1. **The case $\alpha = \frac{7}{2}$ and $\beta = 4$.** This is a case of interest since the behaviour of the c_k at large values of k is expected to be the same as the case $\alpha = \beta = 2$ [2, 3, 10]. In fact from (7) we ask that for $\Re(s) > \frac{1}{2}$:

$$|c_k(7/2, 4)| \leq \frac{C}{k^{\frac{7/2-1/2-\epsilon}{4}}} \sim \frac{k^{\frac{\epsilon}{4}}}{k^{\frac{3}{4}}} \sim |c_k(2,2)| \tag{11}$$

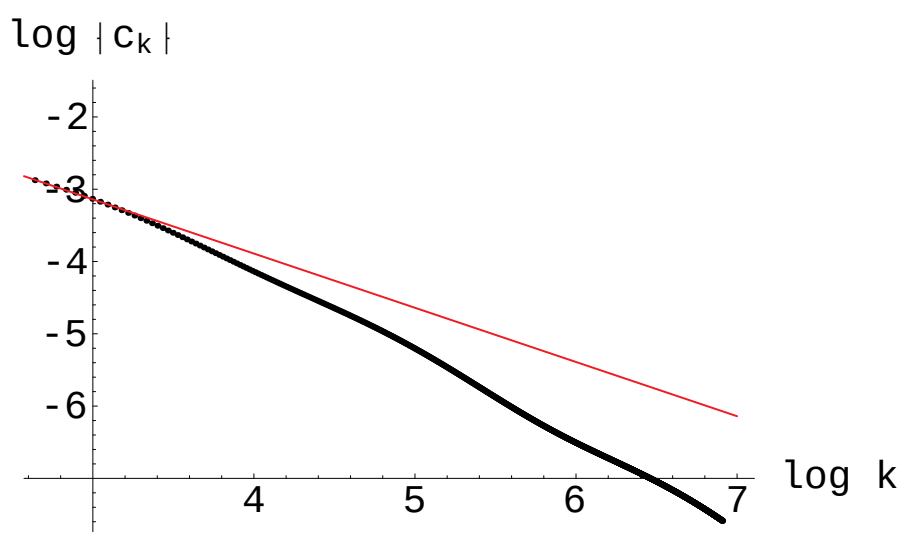

Figure 1. Plot of $\log |c_k| = C - \frac{3}{4} \log k$ together with the straight line of slope$-\frac{3}{4}$.

As a first illustration of the behaviour of c_k (even if k is small) we give in the Figures 1, 2 and 3 the plot respectively of $\log |c_k|, \log(|c_k \log k|)$ and $\log(|c_k (\log k)^2|)$ as a function of $\log k$ for k up to 1000 together with the straight line with slope $-\frac{3}{4}$

which is tangent to the curves at some point. The c_k were computed calculating (4) until $n = 10000$.

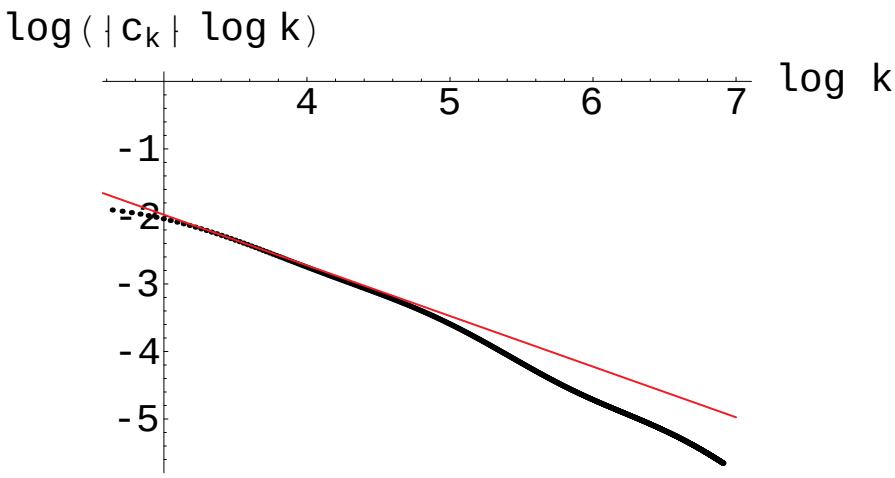

FIGURE 2. Plot of $\log(|c_k| \log k) = C - \frac{3}{4} \log k$ together with the tangent straight line of slope $-\frac{3}{4}$.

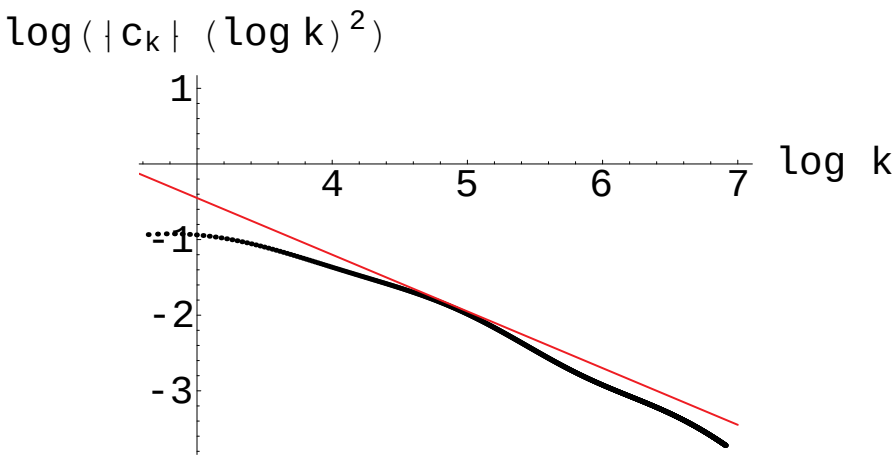

FIGURE 3. Plot of $\log(|c_k|(\log k)^2) = C - \frac{3}{4} \log k$ together with the tangent straight line of slope $-\frac{3}{4}$.

This experiment indicates that c_k, for k up to 1000, may decay more fast than $\frac{C}{(\log k)^2 k^{\frac{3}{4}}}$ as announced by Baez-Duarte in [2] for the case $\alpha = \beta = 2$, i.e. more fast then the bound (11) if the RH is true (see the necessary condition in Appendix B), this of course in the above range of k. For bigger values of k see the Footnote and [3].

4.2. **The case $\alpha = \frac{7}{2}$ with $\beta \to \infty$.** Let $\alpha = \frac{7}{2}$ be fixed, from (8) as β increases we get:

$$\begin{aligned}\lim_{\beta\to\infty} c_k &= \lim_{\beta\to\infty} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \tfrac{1}{\zeta(7/2+\beta j)} = \binom{k}{0}\tfrac{1}{\zeta(7/2)} - 1 + 1 + \sum_{j=1}^{k} \binom{k}{j}(-1)^j \\ &= \tfrac{1}{\zeta(7/2)} - 1 + \sum_{j=0}^{k} \binom{k}{j}(-1)^j\end{aligned}$$

Thus:

$$\lim_{\beta\to\infty} c_k = \frac{1}{\zeta(7/2)} - 1 \cong -0.112479 \qquad \forall k \in \mathbb{N} \tag{12}$$

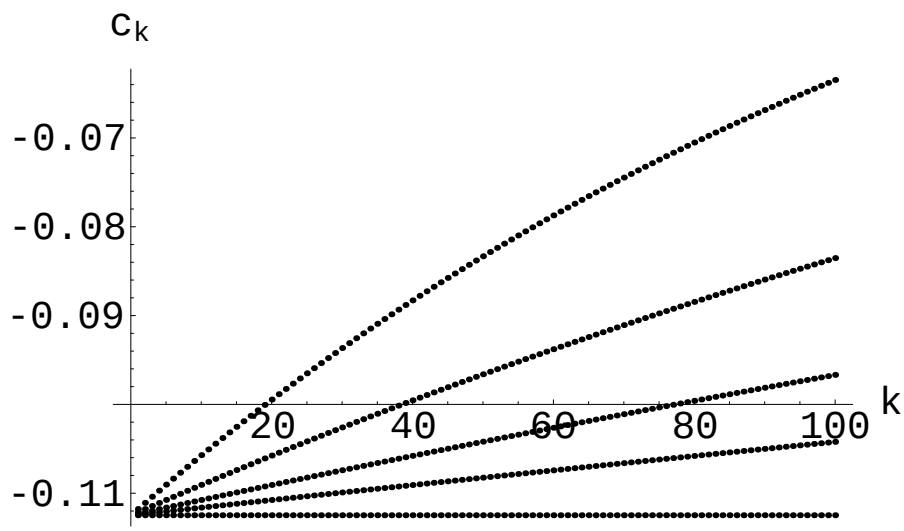

FIGURE 4. Plot of c_k for $\alpha = 7/2$ and $\beta = 4, 5, 6, 7, 20$ (from top to bottom).

Our numerical experiments convalidate these results. We calculated the first 100 c_k for $\beta = 4, 5, 6, 7, 20$. For $\beta = 20$ we get already a convergence to the theoretical limit (12), see Figure 4. So, this infinite β limit obtained by the numerical calculations for low values of k (up to 100) indicates that RH may barely be true (see (7) as $\beta \to \infty$).

4.3. **The Poisson distribution.** To demonstrate the goodness of the approximation's formula (10) we computed the c_k until $k = 1000$ for the case $\alpha = \frac{7}{2}, \beta = 4$ (using (4)). Then using these already computed c_k we calculated also the first 500 c_k of (10). We plotted these two curves together. In Figure 5 we see that from $k \cong 40$ the Poisson approximation is essentially the same as the real sequence.

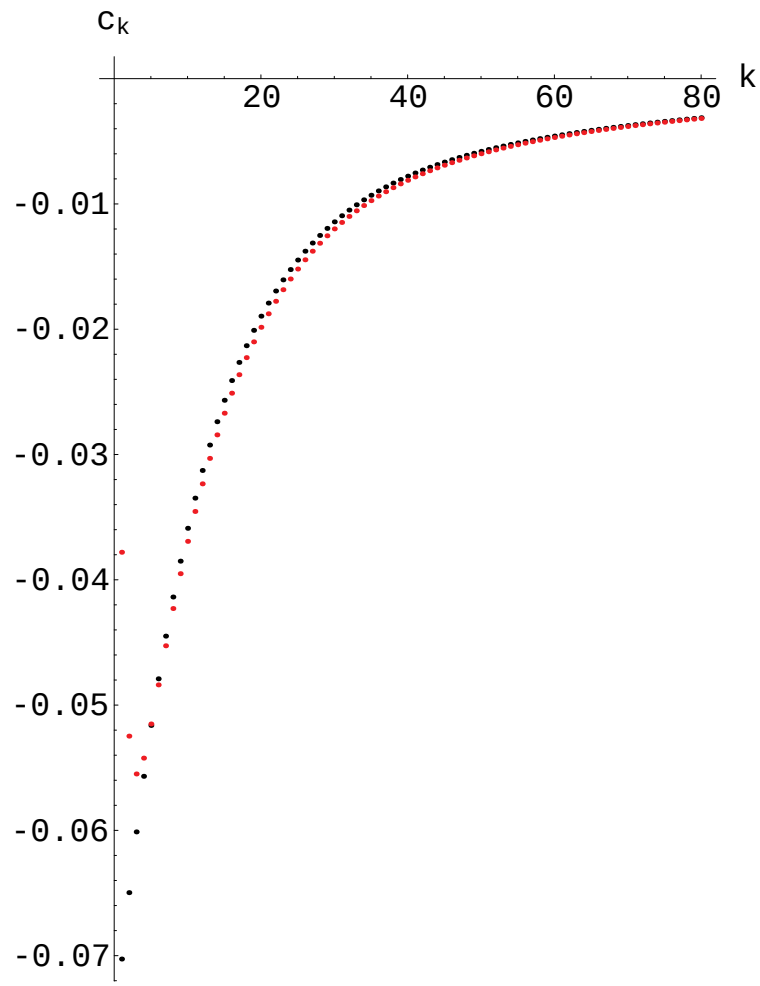

FIGURE 5. Plot of c_k for $\alpha = \frac{7}{2}, \beta = 4$ (black) vs. the Poisson approximation (red).

4.4. **The case $\alpha = \frac{1}{2}$.** In this case ($\alpha < 1$!) we cannot employ the argument of Appendix A, but we have for $\Re(s) - \epsilon \geq \rho = \frac{1}{2}$ and assuming $|c_k|$ increases with β:

$$\left|\varphi(s; \frac{1}{2}, \beta)\right| < \sum_{k=0}^{\infty} k^{-\left(\frac{\epsilon}{\beta}+1\right)} |c_k(1/2, \beta)| \leq \sum_{k=0}^{\infty} k^{-\left(\frac{\epsilon}{\beta}+1\right)} |c_k(1/2, \infty)|$$

From Subsection 4.2 we know that $|c_k(1/2,\infty)| = |\frac{1}{\zeta(1/2)} - 1| \cong 1.68477$, thus for any finite β, $\varphi(s;1/2,\beta)$ is also finite under the assumption that $\sup_\beta |c_k(1/2,\beta)|$ is bounded by $|c_k(1/2,\infty| \cong 1.68477$.

We remember that the great mathematician of the beginning of the century, J.F. Littlewood, has shown on the RH that the series

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\frac{1}{2}+\epsilon}} \qquad \epsilon > 0$$

converges, even if with the Pochhammer's approach is not possible to have absolute convergence. We can verify the numerical bound of this series which should be smaller (or equal) than our predicted bound $A \cong 1.68477$ (Figure 6).

In the strong coupling limit we observe, with the help of our numerical results, the phenomena of a kind of "annihilation of the wave" in a macroscopic region of increasing width with β (α should be understood as $\alpha = \frac{1}{2}+$).

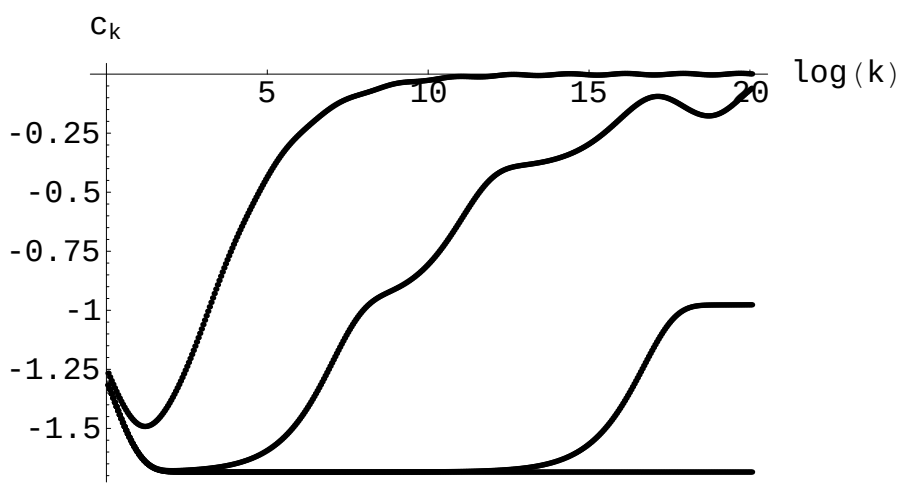

FIGURE 6. Plot of c_k for $\alpha = \frac{1}{2}, \beta = 4, 10, 24, 50$ from top to bottom (data obtained with $n = 10^6$ in (4)).

4.5. **Some cases with the same decay as the Hardy-Littlewood one.** We present some cases which should give the same behaviour as the original Hardy-Littlewood case ($\alpha = 1, \beta = 2$), then:

$$\alpha = \frac{\beta}{4} + \frac{1}{2}$$

For all the cases we present ($\beta = 6, 8, 10$), the c_k seem to decay as $k^{-\frac{1}{4}}$ up to $k = 500$ million (see Figure 7).

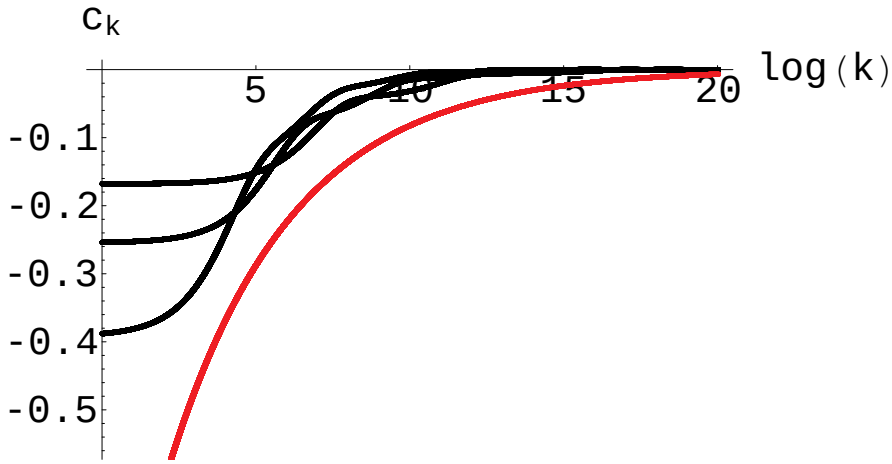

FIGURE 7. Plot of c_k if $\alpha = \frac{\beta}{4} + \frac{1}{2}$ for $\beta = 6, 8, 10$ [black curves from the bottom to the top] and the reference function $-k^{-\frac{1}{4}}$ [red curve] (data obtained with $n = 10^6$ in (4)).

5. Conclusions

In this work we have revisited the original criteria of Riesz and of Hardy-Littlewood for the Riemann Hypothesis in light of recent pioneering works concerning the possible representations of the Riemann Zeta function by means of the Pochhammer's polynomials. The discrete representation in the case $\alpha = \beta = 2$ is due to Baez-Duarte. In order to carry out our numerical experiments related to the criteria, we have first extended the analytical formulation to a more general class of sequences containing two parameters α and β; using a theorem of Baez-Duarte we have specified a sufficient and necessary condition for the truth of the RH for our general class of sequences i.e. for the decay of the coefficients c_k as a power law of k. Moreover in doing this we have found the emergence of a Poisson-like distribution for the c_k which should be exact in the large β limit. Numerical experiments have been carried out for various cases for low values of k.

(1) For $\alpha = \frac{7}{2}$ and $\beta = 4$ we have presented intensive calculation using the Möbius function up to $n = 10000$ and for k up to some hundreds. For this case, the power law decay $k^{-\frac{3}{4}}$ is the same as that appearing in the original work of Riesz ($\alpha = \beta = 2$) and also investigated numerically by Baez-Duarte. The experiments confirm the correctness of the power law within the range of the values of n and of k we were able to treat here. For large values of k see Footnote.
(2) For α and β such that the c_k should all give the power law decay $k^{-\frac{1}{4}}$ at large values of k to ensure the truth of the RH, i.e those where $\alpha = \frac{\beta}{4} + \frac{1}{2}$, we have presented experiments for some values of β which indicates this

power law decay. All sequences c_k have plots lying above a fixed curve of equation $y = Ak^{-\frac{1}{4}}$ for some fixed constant A independent of β, in the range of k we have considered.

(3) Finally we have considered some experiments in the large β limit which indicate that the plots of c_k become more and more flat, well approximated by the mean value of the Poisson-type distribution we have found. As β becomes large and large the c_k approaches in absolut value a constant, for all k, indicating that in this sense the RH may barely be true.

This work will be expanded with numerical experiments for bigger values of k [5] and in the search of other new representations of the Riemann Zeta function, different of the one considered here [6]. Moreover there is the aim that the new criteria will be useful in the context of additional numerical experiments. These works will be presented in a near future.

Appendix A

We follow strictly the lines of calculations of Baez-Duarte [2, 3] to show that the representation (6) for $[\zeta(s)]^{-1}$ is unconditionally valid for $\Re(s) > \rho = 1, \alpha > 1$ and $\beta > 0$. We consider the quantity:

$$q_k = \sum_{n=1}^{\infty} \frac{1}{n^\alpha}\left(1 - \frac{1}{n^\beta}\right)^k$$

Using the Euler-MacLaurin series (restricting ourselves to the main contribution), we have that:

$$q_k \cong \int_1^\infty \frac{1}{x^\alpha}\left(1 - \frac{1}{x^\beta}\right)^k dx$$

Then with the variable change $y = \frac{1}{x^\beta}$ we obtain:

$$\begin{aligned} q_k &\cong \frac{1}{\beta}\int_0^1 y^{\frac{\alpha-1}{\beta}-1}(1-y)^{k+1-1} dy \\ &= B(\frac{\alpha-1}{\beta}, k+1) \end{aligned}$$

where

$$B(\lambda, \mu) = \int_0^1 x^{\lambda-1}(1-x)^{\mu-1} dx = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}$$

is the Beta function.

Thus for k large, we have

$$q_k \cong \frac{1}{\beta}\Gamma(\frac{\alpha-1}{\beta})C\frac{k}{k^{\frac{\alpha-1}{\beta}+1}} \cong \frac{1}{k^{\frac{\alpha-1}{\beta}}}$$

Appendix B

Still following Baez-Duarte [2, 3] and here for the family of sequences with parameters α and β, we now show the necessity of the condition (7), assuming the RH to be true in the seminfinite strip $\Re(s) > \rho = \frac{1}{2}$.

We set $M(x) = \sum_{n \le x} \mu(n)$, then we obtain $\forall \epsilon > 0$:

$$M(x) \le x^{\rho+\epsilon}$$

Summation by parts gives for the main contribution:

$$|c_k| = \left| \int_1^\infty M(x) \frac{d}{dx} \left(\frac{1}{x^\alpha} \left(1 - \frac{1}{x^\beta} \right)^k \right) dx \right|$$

With the variable change $y = \frac{1}{x}$, using $M(\frac{1}{y}) \ll y^{-\rho-\epsilon}$ for $y \downarrow 0$ (RH) we have:

$$|c_k| \ll \alpha \int_0^1 y^{\alpha-\rho-\epsilon-1} (1-y^\beta)^k dy + \beta k \int_0^1 y^{\alpha+\beta-\rho-\epsilon-1} (1-y^\beta)^{k-1} dy$$

and finally with $y^\beta = z$ we obtain

$$|c_k| \ll \frac{\alpha}{\beta} \int_0^1 z^{\frac{\alpha-\rho-\epsilon}{\beta}-1} (1-z)^{k+1-1} dz + k \int_0^1 z^{\frac{\alpha-\rho-\epsilon+\beta}{\beta}-1} (1-z)^{k-1} dz$$

which for large k is given by:

$$|c_k| \ll \frac{\alpha}{\beta} \frac{\Gamma(\frac{\alpha-\rho-\epsilon}{\beta})}{k^{\frac{\alpha-\rho-\epsilon}{\beta}}} + \frac{\Gamma(\frac{\alpha-\rho-\epsilon+\beta}{\beta})}{k^{\frac{\alpha-\rho-\epsilon+\beta}{\beta}+1}} < \frac{C}{k^{\frac{\alpha-\rho-\epsilon}{\beta}}}$$

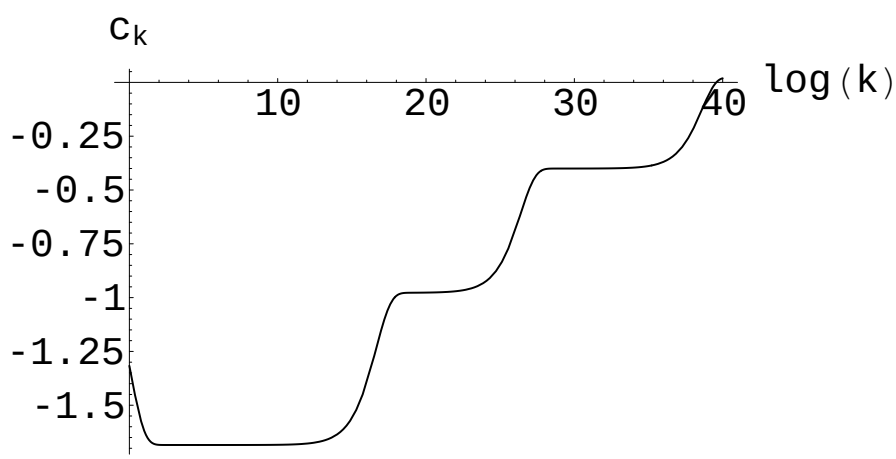

FIGURE 8. Plot of c_k for $\alpha = \frac{1}{2}, \beta = 24$ (data obtained with $n = 10^6$ in (4)).

FOOTNOTE

Our calculations have been carried out only to values of k not exceeding $k = 1000$ except for the Subsections 4.4 and 4.5. It is our pleasure to thank Prof. Luis Baez-Duarte for sending us, after the first draft of this paper, a copy of two recent published works by the author (in particular [3]), now added to our references. The paper contains the plot of the results of advanced numerical experiments up to $k = 100000$ by Krzystof Maslanka for the Riesz case, which clearly indicate that c_k becomes of oscillatory type with a wavelength related in first approximation to the first zero of the Riemann Zeta function.

We also thank Prof. Luis Baez-Duarte for sending us a picture of more numerical results, also concerning the Riesz case, by Marek Wolf, where values of k extend up to $k = 200000$ and confirming the oscillatory character of the sequence c_k, as well. Now a refinement of these results is published in arXiv [11].

We are currently performing an extension of our calculations, using the Poisson-Möbius formula considered in this work, for various cases, with the aim of obtaining satisfactory numerical results up to some billions for k and these will be presented in a forthcoming note. For the case $\alpha = \frac{1}{2}+, \beta = 24$, the plot of c_k up to $k = e^{40} \cong 2.3 \cdot 10^{17}$ is already given in Fig. 8 without comments, see only the emergence of plateau up to $\log(k) = 40$.

References

[1] Baez-Duarte L 2003 *arXiv:math.NT/0307214v1* 16 July 2003
[2] Baez-Duarte L 2003 *arXiv:math.NT/0307215v1* 16 July 2003
[3] Baez-Duarte L 2005 *International Journal of Mathematics and Mathematical Sciences* 2005:21 3527-3537
[4] Baez-Duarte L 2005 *arXiv:math.NT/0504402v1* 20 April 2005
[5] Beltraminelli S and Merlini D 2007 *in preparation, to be posted*
[6] Beltraminelli S and Merlini D 2007 *in preparation, to be posted*
[7] D'Errico M 2004 (unpublished) *presented at the International Workshop on Complex Systems (Cerfim-Issi)* held in Locarno (Switzerland), 16-18 september 2004
[8] Hardy GH and Littlewood JE 1918 *Acta Mathematica* **41** 119
[9] Maslanka K 2001 *arXiv:math-ph/0105007v1* 4 May 2001
[10] Riesz F 1916 *Acta Mathematica* **40** 185
[11] Wolf M 2006 *arXiv:math.NT/06054857v1* 17 May 2006

CERFIM, Research Center for Mathematics and Physics, PO Box 1132, 6600 Locarno, Switzerland

E-mail address: `stefano.beltraminelli@ti.ch`

E-mail address: `merlini@cerfim.ch`