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A NOTE ABOUT INVARIANTS OF ALGEBRAIC CURVES

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Abstract. Let G be the group generated by the transformations $x = \alpha\tilde{x} + b, y = \tilde{y}, \alpha \neq 0, \alpha, b \in \mathbf{k}$, $\operatorname{char}\mathbf{k}$ of the affine plane $\mathbf{k}^2$. For affine algebraic plane curves of the form $y^n = f(x)$ we reduce a calculation of its G-invariants to calculation of the intersection of kernels of some locally nilpotent derivations. We compute a complete set of independent invariants and then reconstruct a curve from given values of these invariants.

1. Introduction

Consider an affine algebraic curve

$$C : F(x, y) = \sum_{i+j\leq d} a_{i,j} x^i y^j = 0, a_{i,j} \in \mathbf{k},$$

defined over field $\mathbf{k}$, $\operatorname{char}\mathbf{k} = 0$. Let $\mathbf{k}[C]$ and $\mathbf{k}(C)$ be the algebras of polynomial and rational functions of coefficients of the curve C. Those affine transformations of plane which preserve the algebraic form of equation $F(x, y)$ generate a group G which is a subgroup of the group of affine plane transformations. A function $\phi(a_{0,0}, a_{1,0}, \ldots, a_{d,0}) \in \mathbf{k}(C)$ is called G-invariant if $\phi(\tilde{a}_{0,0}, \tilde{a}_{1,0}, \ldots, \tilde{a}_{d,0}) = \phi(a_{0,0}, a_{1,0}, \ldots, a_{d,0})$ where $\tilde{a}_{0,0}$, $\tilde{a}_{1,0}, \ldots$, $\tilde{a}_{d,0}$ are defined from the condition

$$F(gx, gy) = \sum_{i+j\leq d} a_{i,j}(gx)^i(gy)^j = \sum_{i+j\leq d} \tilde{a}_{i,j} x^i y^j,$$

for all $g \in G$. The curves C and C' are said to be G-isomorphic if they lies on the same G-orbit.

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The algebras of all G-invariant polynomials and rational functions we denote by $\mathbf{k}[C]^G$ and by $\mathbf{k}(C)^G$, respectively. One way to find elements of the algebra $\mathbf{k}[C]^G$ is the specification of invariants of associated ternary form of order d. In fact, consider a vector space T_d generated by the ternary forms $\sum\limits_{i+j\leq d} b_{i,j}x^{d-(i+j)}y^i z^j$, $b_{i,j} \in \mathbf{k}$ endowed with the natural action of the group $GL_3 := GL_3(\mathbf{k})$. Given GL_3-invariant function f of $\mathbf{k}(T_d)^{GL_3}$, a specification f of the form $b_{i,j} \mapsto a_{i,j}$ or $b_{i,j} \mapsto 0$ in the case when $a_{i,j} \notin \mathbf{k}(C)$, gives us an element of $\mathbf{k}(C)^G$.

But SL_3-invariants(thus and GL_3-invariants) of ternary forms are known only for the cases $d \leq 4$, see [1]. Furthermore, analyzing of the Poincare series of the algebra of invariants of ternary forms, [2], we see that the algebras are very complicated and there is no chance to find theirs minimal generating set.

Since $\mathbf{k}(T_d)^{GL_3}$ coincides with $\mathbf{k}(T_d)^{\mathfrak{gl}_3}$ it implies that the algebra of invariants is the intersection of kernels of some derivations of the algebra $\mathbf{k}(T_d)$. Then in place of the specification of coefficients of the form we may use a "specification" of those derivations.

First, consider a motivating example. Let

$$C_3 : y^2 + a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0,$$

and let G_0 be the group generated by the translations $x \mapsto \alpha\tilde{x} + b$. It is easy to show that j-invariant of the curve C_3 equals ([3], p. 46):

$$j(C_3) = 6912\,\frac{\left(a_0a_2 - {a_1}^2\right)^3}{{a_0}^2\left(4\,{a_1}^3a_3 - 6\,a_3a_0a_1a_2 - 3\,{a_1}^2{a_2}^2 + {a_3}^2{a_0}^2 + 4\,a_0{a_2}^3\right)}.$$

Up to constant factor $j(C_3)$ equal to $\dfrac{S^3}{T}$ where S and T are the specification of two SL_3-invariants of ternary cubic, see [4], p.173.

From another viewpoint a direct calculation yields that the following is true: $\mathcal{D}\left(j(C_3)\right) = 0$ and $\mathcal{H}\left(j(C_3)\right) = 0$ where $\mathcal{D}$, $\mathcal{H}$ denote the following derivations of the algebra of rational functions $\mathbf{k}(C_3) = \mathbf{k}(a_0, a_1, a_2, a_3)$:

$$\mathcal{D}(a_i) = ia_{i-1}, \mathcal{H}(a_i) = (3-i)a_i, i = 0, 1, 2, 3.$$

From the computational point of view, the calculation of $\ker\mathcal{D} \cap \ker\mathcal{H}$ is more effective than the calculating of the algebra of invariants of the ternary cubic. We will derive further that

$$\ker\mathcal{D}_3 \cap \ker H_3 = \mathbf{k}\left(\frac{\left(a_0a_2 - {a_1}^2\right)^3}{a_0^3}, \frac{a_3\,a_0^2 + 2\,{a_1}^3 - 3\,a_1\,a_2\,a_0}{a_0^2}\right).$$

In section 2, we give a full description of the algebras of polynomial and rational invariants for the curve $y^n = f(x)$. We compute a complete set of independent invariants and then reconstruct a curve from given values of these invariants.

2. Invariants of curves $y^n = f(x)$.

Consider the curve

$$C_{n,d} : y^n = a_0x^d + da_1x^{d-1} + \cdots + a_d = \sum_{i=0}^{d} a_d\binom{d}{i}x^{d-i}, n \geq 1,$$

and let G be the group generated by the following transformations

$$x = \alpha\tilde{x} + b, y = \tilde{y}, \alpha \neq 0.$$

It is clear that G is isomorphic to the group of the affine transformations of the complex line $\mathbf{k}^1$.

The algebra $\mathbf{k}(C_{n,d})^G$ consists of functions $\phi(a_0, a_1, \dots, a_d)$ that have the invariance property

$$\phi(\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_d) = \phi(a_0, a_1, \dots, a_d).$$

Here $\tilde{a}_i$ denote the coefficients of the curve $\tilde{C}_{n,d}$:

$$\tilde{C}_{n,d} : \sum_{i=0}^{d} a_d \binom{d}{i} (\alpha\tilde{x} + b)^{d-i} = \sum_{i=0}^{d} \tilde{a}_d \binom{d}{i} \tilde{x}^{d-i}.$$

The coefficients $\tilde{a}_i$ are given by the formulas

$$\tilde{a}_i = \alpha^{n-i} \sum_{k=0}^{i} \binom{i}{k} a_{i-k} b^k. \tag{1}$$

The following statement holds

Theorem 2.1. *We have*

$$\mathbf{k}(C_{n,d})^G = \ker \mathcal{D}_d \cap \ker \mathcal{E}_d,$$

where $\mathcal{D}_d$, $\mathcal{E}_d$ denote the following derivations of the algebra $\mathbf{k}(C_{n,d})$:

$$\mathcal{D}_d(a_i) = i a_{i-1}, \mathcal{E}_d(a_i) = (d-i) a_i. \tag{2}$$

A linear map $D : \mathbf{k}(C_{n,d}) \to \mathbf{k}(C_{n,d})$ is called a derivation of the algebra $\mathbf{k}(C_{n,d})$ if $D(fg) = D(f)g + fD(g)$, for all $f, g \in \mathbf{k}(C_{n,d})$. The subalgebra $\ker D := \{f \in \mathbf{k}(C_{n,d}) \mid D(f) = 0\}$ is called the kernel of the derivation D. The above derivation $\mathcal{D}_d$ is called the basic Weitzenböck derivation.

Proof. Following the arguments of Hilbert [7],page 26, we differentiate with respect to b both sides of the equality

$$\phi(\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_d) = \phi(a_0, a_1, \dots, a_d),$$

and obtain in this way

$$\frac{\partial \phi(\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_d)}{\partial \tilde{a}_0} \frac{\partial \tilde{a}_0}{\partial b} + \frac{\partial \phi(\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_d)}{\partial \tilde{a}_1} \frac{\partial \tilde{a}_1}{\partial b} + \dots + \frac{\partial \phi(\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_d)}{\partial \tilde{a}_d} \frac{\partial \tilde{a}_d}{\partial b} = 0.$$

Substitute $\alpha = 1$, $b = 0$ to $\phi(\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_d)$ and taking into account that $\left.\frac{\partial \tilde{a}_i}{\partial b}\right|_{b=0} = i a_{i-1}$, we get:

$$\tilde{a}_0 \frac{\partial \phi(\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_d)}{\partial \tilde{a}_1} + 2\tilde{a}_1 \frac{\partial \phi(\tilde{a}_0, \dots, \tilde{a}_d)}{\partial \tilde{a}_2} + \cdots d\tilde{a}_{d-1} \frac{\partial \phi(\tilde{a}_0, \dots, \tilde{a}_d)}{\partial \tilde{a}_d} = 0$$

Since the function $\phi(\tilde{a}_0, \dots, \tilde{a}_d)$ depends on the variables $\tilde{a}_i$ in the exact same way as the function $\phi(a_0, a_1, \dots, a_d)$ depends on the a_i then it implies that $\phi(a_0, a_1, \dots, a_d)$ satisfies the differential equation

$$a_0 \frac{\partial \phi(a_0, a_1 \dots, a_d)}{\partial a_1} + 2a_1 \frac{\partial \phi(a_0, a_1 \dots, a_d)}{\partial a_2} + d a_{d-1} \frac{\partial \phi(a_0, a_1 \dots, a_d)}{\partial a_d} = 0.$$

Thus, $\mathcal{D}_d(\phi) = 0$. Now we differentiate with respect to α both sides of the same equality

$$\phi(\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_d) = \phi(a_0, a_1, \dots, a_d).$$

$$\frac{\partial \phi(\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_d)}{\partial \tilde{a}_0} \frac{\partial \tilde{a}_0}{\partial \alpha} + \frac{\partial \phi(\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_d)}{\partial \tilde{a}_1} \frac{\partial \tilde{a}_1}{\partial \alpha} + \dots + \frac{\partial \phi(\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_d)}{\partial \tilde{a}_d} \frac{\partial \tilde{a}_d}{\partial \alpha} = 0.$$

Substitute $\alpha=1, b=0$, to $\phi(\tilde{a}_0,\tilde{a}_1,\ldots,\tilde{a}_d)$ and taking into account

$$\frac{\partial \tilde{a}_i}{\partial \alpha}\Big|_{\alpha=1\, b=0} = (d-i)a_i,$$

we get:

$$\tilde{a}_0\frac{\partial\phi(\tilde{a}_0,\tilde{a}_1,\ldots,\tilde{a}_d)}{\partial\tilde{a}_0}+(d-1)\tilde{a}_1\frac{\partial\phi(\tilde{a}_0,\ldots,\tilde{a}_d)}{\partial\tilde{a}_1}+\cdots+\tilde{a}_{d-1}\frac{\partial\phi(\tilde{a}_0,\ldots,\tilde{a}_d)}{\partial\tilde{a}_{d-1}}=0$$

It implies that $\mathcal{E}_d(\phi(a_0,a_1\ldots,a_d))=0$.

The formulas (1) define a representation of two-parametric Lie group G on the polynomial algebra $\mathbf{k}[a_0,a_1,\ldots,a_d]$. By construction of the operators $\mathcal{D}_d$ and $\ker\mathcal{E}_d$ the formulas (2) define a representation of the corresponding Lie algebra of the group G. It is well-known fact of the representation theory that algebras of invariants of Lie group coincide with the algebra of invariant of its Lie algebra, see [8]. Thus

$$\mathbf{k}(C_{n,d})^G=\ker\mathcal{D}_d\cap\ker\mathcal{E}_d.$$

□

The derivation $\mathcal{E}_d$ sends the monomial $a_0^{m_0}a_1^{m_1}\cdots a_d^{m_d}$ to the term

$$(m_0d+m_1(d-1)+\cdots m_{d-1})a_0^{m_0}a_1^{m_1}\cdots a_d^{m_d}.$$

Let the number $\omega\left(a_0^{m_0}a_1^{m_1}\cdots a_d^{m_d}\right):=m_0d+m_1(d-1)+\cdots m_{d-1}$ be called the weight of the monomial $a_0^{m_0}a_1^{m_1}\cdots a_d^{m_d}$. In particular $\omega(a_i)=d-i$.

A homogeneous polynomial $f\in\mathbf{k}[C_{n,d}]$ be called *isobaric* if all their monomial have equal weights. A weight $\omega(f)$ of an isobaric polynomial f is called a weight of its monomials. Since $\omega(f)>0$, then $\mathbf{k}[C_{n,d}]^{\mathcal{E}_d}=0$. It implies that $\mathbf{k}[C_{n,d}]^G=0$.

If f,g are two isobaric polynomials then

$$\mathcal{E}_d\left(\frac{f}{g}\right)=(\omega(f)-\omega(g))\frac{f}{g}.$$

Therefore the algebra $k(C_{n,d})^{\mathcal{E}_d}$ is generated by rational functions which both denominator and numerator has equal weight.

The kernel of the derivation $\mathcal{D}_d$ also is well-known, see [5], [6]. It is given by

$$\ker\mathcal{D}_d=\mathbf{k}(a_0,z_2,\ldots,z_d),$$

where

$$z_i:=\sum_{k=0}^{i-2}(-1)^k\binom{i}{k}a_{i-k}a_1^ka_0^{i-k-1}+(i-1)(-1)^{i+1}a_1^i,\, i=2,\ldots,d.$$

In particular, for $d=5$ we get

$$\begin{array}{l} z_2=a_2\,a_0-{a_1}^2\\ z_3=a_3\,a_0^2+2\,{a_1}^3-3\,a_1\,a_2\,a_0\\ z_4=a_4\,a_0^3-3\,{a_1}^4+6\,{a_1}^2\,a_2\,a_0-4\,a_1\,a_3\,a_0^2\\ z_5=a_5\,a_0^4+4\,{a_1}^5-10\,{a_1}^3\,a_2\,a_0+10\,{a_1}^2\,a_3\,a_0^2-5\,a_1\,a_4\,a_0^3.\end{array}$$

It is easy to see that $\omega(z_i)=i(n-1)$. The following element $\dfrac{z_i^d}{a_0^{i(d-1)}}$ has the zero weight for any i. Therefore, the statement holds:

Theorem 2.2.

$$\mathbf{k}(C_{n,d})^G = \mathbf{k}\left(\frac{z_2^d}{a_0^{2(d-1)}}, \frac{z_3^d}{a_0^{3(d-1)}}, \cdots, \frac{z_d^d}{a_0^{d(d-1)}}\right).$$

For the curve

$$C_{n,d}^0 : y^n = x^d + da_1x^{d-1} + \cdots + a_d = x^d + \sum_{i=1}^{d} a_d \binom{d}{i} x^{d-i}.$$

and for the group G_0 generated by translations $x = \tilde{x} + b$, the algebra of invariants becomes simpler:

$$\mathbf{k}\left(C_d^0\right)^{G_0} = \mathbf{k}(z_2, z_3, \dots, z_d).$$

Theorem 2.3. (i) *For arbitrary set of $d-1$ numbers j_2,$j_3, \dots, j_d$ there exists a curve C such that $z_i(C) = j_i$.*

(ii) *For two curves C and C' the equalities $z_i(C) = z_i(C')$ hold for $2 \leq i \leq d$, if and only if these curves are G_0-isomorphic.*

Proof. (i). Consider the system of equations

$$\begin{cases} a_2 - {a_1}^2 = j_2 \\ a_3 + 2\,{a_1}^3 - 3\,a_1\,a_2 = j_3 \\ a_4 - 3\,{a_1}^4 + 6\,{a_1}^2\,a_2 - 4\,a_1\,a_3 = j_4 \\ \dots \\ a_d + \displaystyle\sum_{k=1}^{d-2} (-1)^k \binom{d}{k} a_{d-k} a_1^k + (d-1)(-1)^{d+1} a_1^d = j_d \end{cases}$$

Put $a_1 = 0$ we get $a_n = j_n$, i.e., the curve

$$C : y^n = x^d + \binom{d}{2} j_2 x^{d-2} + \cdots + j_d,$$

has the required property $z_i(C) = j_i$.

(ii). We may assume, without loss of generality, that the curve C has the form

$$C : y^2 = x^d + \binom{d}{2} j_2 x^{d-2} + \cdots + j_d.$$

Suppose that for a curve

$$C' : y^2 = x^d + da_1x^{d-1} + \cdots + a_d = x^d + \sum_{i=1}^{d} a_d \binom{d}{i} x^{d-i}.$$

holds $z_i(C') = z_i(C) = j_i$.

By solving the above system we obtain

$$a_i = j_i + a_1^i + \sum_{s=1}^{i-2} \binom{i}{s} a_1^s j_{i-s}, i = 2, 3, \dots, d. \tag{2}$$

Comparing (3) with (1) we deduce that the curve C' is obtained from the curve C by the translation $x + a_1$. □

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ON ISOLATED STRATA OF P-GONAL RIEMANN SURFACES IN THE BRANCH LOCUS OF MODULI SPACES

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Abstract. The moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g has orbifold structure, and the set of singular points of such orbifold is the *branch locus* $\mathcal{B}_g$. Given a prime number $p \geq 7$, $\mathcal{B}_g$ contains isolated strata consisting of p-gonal Riemann surfaces for genera $g \geq \frac{3(p-1)}{2}$, that are multiple of $\frac{p-1}{2}$. This is a generalization of the results obtained in [BCI1] for pentagonal Riemann surfaces, and the results of [K] and [CI3] for zero- and one-dimensional isolated strata in the branch locus.

1. Introduction

In this article we study the topology of moduli spaces of Riemann surfaces. The moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g, being the quotient of the Teichmüller space by the discontinuous action of the mapping class group, has the structure of a complex orbifold, whose set of singular points is called the

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branch locus $\mathcal{B}_g$. The branch locus $\mathcal{B}_g$, $g \geq 3$ consists of the Riemann surfaces with symmetry, i. e. Riemann surfaces with non-trivial automorphism group; see [H] and [B]. $\mathcal{B}_g$ admits an (equisymmetric) stratification where each stratum is given by the symmetry of the surfaces in it, i.e. the conjugacy class in the mapping class group of the automorphism group of the surfaces of the stratum ([B]).

Our goal is to study the topology of $\mathcal{B}_g$ through its connectedness, using this equisymmetric stratification. The connectedness of moduli spaces of hyperelliptic, $p-$gonal and real Riemann surfaces has been widely studied, for instance by [BSS], [K] [CI1], [CI2], [CI3], [BCI2], [G], [S], [BCIP] and [BEMS].

Recently Bartolini, Costa and Izquierdo have shown that $\mathcal{B}_g$ is connected only for genera 3, 4, 7, 13, 17, 19 and 59; see [BCI1] and [BCI2]. The authors found isolated strata in $\mathcal{B}_g$ ($g \neq 3, 4, 7, 13, 17, 19, 59$) given by actions of order five and seven. In [BI] it is shown that the strata induced by actions of order two and three belongs to the same connected component of $\mathcal{B}_g$.

A cyclic p-gonal Riemann surface X is a surface that admits a regular covering of degree p on the Riemann sphere. A 2-gonal Riemann surface is called an hyperelliptic Riemann surface.

The main result in this article is that $\mathcal{B}_g$ contains isolated strata consisting of p-gonal Riemann surfaces ($p \geq 7$) of dimension $d \geq 2$ for genus $g = (d+1)(\frac{p-1}{2})$, according to Riemann-Hurwitz's formula.

Given two Riemann surfaces X_1 and X_2, there is a path of quasiconformal deformations taking X_1 to X_2 since $\mathcal{M}_g$ is connected. The result obtained in this article says that if X_1 belongs to one of the isolated strata and X_2 has another type of symmetry, then the path of quasiconformal deformations must contain surfaces without symmetry.

The main result is a generalization of the results obtained in [BCI1] for isolated strata of cyclic pentagonal Riemann surfaces, and of the results in [K], [CI3] for isolated strata of dimension zero and one. As a consequence we give an infinite family of genera for which $\mathcal{B}_g$ has an increasing number of isolated strata.

2. Riemann surfaces and Fuchsian groups

Let X be a Riemann surface and assume that $Aut(X) \neq \{1\}$. Hence $X/Aut(X)$ is an orbifold and there is a Fuchsian group $\Gamma \leq Aut(\mathcal{D})$, such that $\pi_1(X) \lhd \Gamma$ and

$$\mathcal{D} \to X = \mathcal{D}/\pi_1(X) \to X/Aut(X) = \mathcal{D}/\Gamma$$

where $\mathcal{D} = \{z \in \mathbb{C} : \|z\| < 1\}$.

If the Fuchsian group Γ is isomorphic to an abstract group with canonical presentation

$$\left\langle a_1, b_1, \ldots, a_g, b_g, x_1 \ldots x_k | x_1^{m_1} = \cdots = x_k^{m_k} = \prod_{i=1}^{k} x_i \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle, \tag{1}$$

we say that Γ has *signature*

$$s(\Gamma) = (g; m_1, \ldots, m_k). \tag{2}$$

The generators in presentation (1) will be called *canonical generators*.

Let X be a Riemann surface uniformized by a surface Fuchsian group Γ_g, i.e. a group with signature $(g; -)$. A finite group G is a group of automorphisms of X, i.e. there is a holomorphic action a of G on X, if and only if there is a Fuchsian group Δ and an epimorphism $\theta_a : \Delta \to G$ such that $\ker \theta_a = \Gamma_g$. The epimorphism θ_a is the monodromy of the covering $f_a : X \to X/G = \mathcal{D}/\Delta$.

The relationship between the signatures of a Fuchsian group and subgroups is given in the following theorem of Singerman:

Theorem 1. *(Singerman [Si1]) Let Γ be a Fuchsian group with signature (2) and canonical presentation (1). Then Γ contains a subgroup Γ' of index N with signature*

$$s(\Gamma') = (h; m'_{11}, m'_{12}, ..., m'_{1s_1}, ..., m'_{k1}, ..., m'_{ks_k}).$$

if and only if there exists a transitive permutation representation $\theta : \Gamma \to \Sigma_N$ satisfying the following conditions:

1. The permutation $\theta(x_i)$ has precisely s_i cycles of lengths less than m_i, the lengths of these cycles being $m_i/m'_{i1}, ..., m_i/m'_{is_i}$.

2. The Riemann-Hurwitz formula

$$\mu(\Gamma')/\mu(\Gamma) = N.$$

where $\mu(\Gamma)$, $\mu(\Gamma')$ are the hyperbolic areas of the surfaces $\mathcal{D}/\Gamma$, $\mathcal{D}/\Gamma'$.

For $\mathcal{G}$, an abstract group isomorphic to all the Fuchsian groups of signature $s = (h; m_1, ..., m_k)$, the Teichmüller space of Fuchsian groups of signature s is

$$\{\rho : \mathcal{G} \to PSL(2,\mathbb{R}) : s(\rho(\mathcal{G})) = s\}/ \text{ conjugation in } PSL(2,\mathbb{R}) \ = \ T_s.$$

The Teichmüller space T_s is a simply-connected complex manifold of dimension $3g - 3 + k$. The modular group, $M(\Gamma)$, of Γ, acts on $T(\Gamma)$ as $[\rho] \to [\rho \circ \alpha]$ where $\alpha \in M(\Gamma)$. The moduli space of Γ is the quotient space $\mathcal{M}(\Gamma) = T(\Gamma)/M(\Gamma)$, then $\mathcal{M}(\Gamma)$ is a complex orbifold and its singular locus is $\mathcal{B}(\Gamma)$, called the branch locus of $\mathcal{M}(\Gamma)$. If Γ_g is a surface Fuchsian group, we denote $\mathcal{M}_g = T_g/M_g$ and the branch locus by $\mathcal{B}_g$. The branch locus $\mathcal{B}_g$ consists of surfaces with non-trivial symmetries for $g > 2$.

If $X/Aut(X) = \mathcal{D}/\Gamma$ and genus$(X) = g$, then there is a natural inclusion $i : T_s \to T_g : [\rho] \to [\rho']$, where

$$\rho : \mathcal{G} \to PSL(2,\mathbb{R}),\ \pi_1(X) \subset \mathcal{G},\ \rho' = \rho \mid_{\pi_1(X)}: \pi_1(X) \to PSL(2,\mathbb{R}).$$

If we have $\pi_1(X) \lhd \mathcal{G}$, then there is a topological action of a finite group $G = \mathcal{G}/\pi_1(X)$ on surfaces of genus g given by the inclusion $a : \pi_1(X) \to \mathcal{G}$. This inclusion $a : \pi_1(X) \to \mathcal{G}$ produces $i_a(T_s) \subset T_g$.

The image of $i_a(T_s)$ by $T_g \to \mathcal{M}_g$ is $\overline{\mathcal{M}}^{G,a}$, where $\overline{\mathcal{M}}^{G,a}$ is the set of Riemann surfaces with automorphisms group containing a subgroup acting in a topologically equivalent way to the action of G on X given by the inclusion a, see [H], the subset $\mathcal{M}^{G,a} \subset \overline{\mathcal{M}}^{G,a}$ is formed by the surfaces whose full group of automorphisms acts in the topological way given by a. The branch locus, $\mathcal{B}_g$, of the covering $T_g \to \mathcal{M}_g$ can be described as the union $\mathcal{B}_g = \bigcup_{G \neq \{1\}} \overline{\mathcal{M}}^{G,a}$, where $\{\mathcal{M}^{G,a}\}$ is the equisymmetric stratification of the branch locus [B]:

Theorem 2. *(Broughton [B]) Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus g, G a finite subgroup of the corresponding modular group M_g. Then:*

(1) $\overline{\mathcal{M}}_g^{G,a}$ is a closed, irreducible algebraic subvariety of $\mathcal{M}_g$.

(2) $\mathcal{M}_g^{G,a}$, if it is non-empty, is a smooth, connected, locally closed algebraic subvariety of $\mathcal{M}_g$, Zariski dense in $\overline{\mathcal{M}}_g^{G,a}$.

There are finitely many strata $\mathcal{M}_g^{G,a}$.

An isolated stratum $\mathcal{M}^{G,a}$ in the above stratification is a stratum that satisfies $\overline{\mathcal{M}}^{G,a} \cap \overline{\mathcal{M}}^{H,b} = \varnothing$, for every group H and action b on surfaces of genus g. Thus $\overline{\mathcal{M}}^{G,a} = \mathcal{M}^{G,a}$

Since each non-trivial group G contains subgroups of prime order, we have the following remark:

Remark 3. *(Cornalba [C])*

$$\mathcal{B}_g = \bigcup_{p\ prime} \overline{\mathcal{M}}^{C_p,a}$$

where $\overline{\mathcal{M}}^{C_p,a}$ is the set of Riemann surfaces of genus g with an automorphism group containing C_p, the cyclic group of order p, acting on surfaces of genus g in the topological way given by a.

3. Isolated strata of p-gonal Riemann surfaces

Definition 4. *A Riemann surface X is said to be p-gonal if it admits a p-sheeted covering $f : X \to \widehat{\mathbb{C}}$ onto the Riemann sphere. If f is a cyclic regular covering then X is called cyclic p-gonal. The covering f will be called the (cyclic) p-gonal morphism.*

A cyclic p-gonal Riemann surface admits an equation of the form $y^p = P(x)$.
By Lemma 2.1 in [A], if the surface X_g has genus $g \geq (p-1)^2 + 1$, then the p-gonal morphism is unique.

We can characterize cyclic p-gonal Riemann surfaces using Fuchsian groups. Let X_g be a Riemann surface, X_g admits a cyclic p-gonal morphism f if and only if there is a Fuchsian group Δ with signature $(0; \overbrace{p, ..., p}^{\frac{2g}{p-1}+2})$ and an index p normal surface subgroup Γ of Δ, such that Γ uniformizes X_g; see [CI4], [CI].

We have the following algorithm to recognize cyclic p-gonal surfaces: A surface X_g admits a cyclic p-gonal morphism f if and only if there is a Fuchsian group Δ with signature $(0; m_1, ..., m_r)$, an order p automorphism $\alpha : X_g \to X_g$, such that $\langle\alpha\rangle \leq G = Aut(X_g)$, and an epimorphism $\theta : \Delta \to G$ with $ker(\theta) = \Gamma$ in such a way that $\theta^{-1}(\langle\alpha\rangle)$ is a Fuchsian group with signature $(0; \overbrace{p, ..., p}^{\frac{2g}{p-1}+2})$. Furthermore the p-gonal morphism f is unique if and only if $\langle\alpha\rangle$ is normal in G (see [G]), and Wootton [W] has proved the following:

Lemma 5. *(Wootton [W]) With the notation above. If $G > C_p$, then $N_G(C_p) > C_p$.*

Isolated strata $\overline{\mathcal{M}}^{C_p,a} = \mathcal{M}^{C_p,a}$ of cyclic p-gonal surfaces correspond to maximal actions of the cyclic group C_p. Isolated strata of dimension 0 where given [K], isolated strata of dimension 1 were studied in [CI3]. We find here isolated strata of any dimension, consisting of p-gonal surfaces also.

Theorem 6. *Let p be a prime number at least seven and let $d \geq 2$. Then there are isolated strata of dimension d consisting of p-gonal surfaces in $\mathcal{B}_g$ if and only if $g = (d+1)(\frac{p-1}{2})$.*

Proof. First of all, an equisymmetric stratum $\mathcal{M}^{C_p,a}$ in $\mathcal{B}_p$ of dimension $d \geq 2$ of p-gonal Riemann surfaces is given by a monodromy $\theta : \Delta(0; \overbrace{p, \ldots, p}^{d+3}) \to C_p$, with Δ a Fuchsian group with maximal signature; see [Si2]. Then, a generic surface X in $\mathcal{M}^{C_p,a}$ will have $C_p = Aut(X)$. The dimension of the stratum is $d = \frac{2g-p+1}{p-1}$ by the Riemann-Hurwitz formula Thus $g = (d+1)(\frac{p-1}{2})$.
If a surface in the stratum has larger automorphism group G, then, by Lemma 5, we can assume that C_p is normal in G by considering $C_p < N_G(C_p)$.
Let X_g, be a p-gonal surface, such that $X_g \in \overline{\mathcal{M}}_g^{C_p,a}$ for some action a, let $\langle\alpha\rangle$ be the group of p-gonal automorphisms of X_g. Consider an automorphism $b \in Aut(X) \setminus \langle\alpha\rangle$, by Lemma 5 and [G], b induces an automorphism $\hat{b}$ of order $t \geq 2$ on the Riemann sphere $X_g/\langle a\rangle = \widehat{\mathbb{C}}$ according to the following diagram

$$\begin{array}{ccc} X_g = \mathcal{D}/\Gamma_g & \overset{b}{\to} & X_g = \mathcal{D}/\Gamma_g \\ f_a \downarrow & & \downarrow f_a \\ X_g/\langle\alpha\rangle = \widehat{\mathbb{C}}(P_1, \ldots, P_k) & \overset{\hat{b}}{\to} & X_g/\langle\alpha\rangle = \widehat{\mathbb{C}}(P_1, \ldots, P_k), \end{array}$$

where Γ_g is a surface Fuchsian group and $f_a : X_g = \mathcal{D}/\Gamma_g \to X_g/\langle\alpha\rangle$ is the p-gonal morphism induced by the group of automorphisms $\langle\alpha\rangle$ with action a. $S = \{P_1, \ldots, P_k\}$ is the branch set in $\widehat{\mathbb{C}}$ of the morphism f_a with monodromy $\theta_a : \Delta(0; p, \overset{d+3}{\ldots}, p) \to C_p$ defined by $\theta_a(x_i) = \alpha^{r_i}$, where $r_i \in \{1, \ldots, p-1\}$ for $1 \leq i \leq d+3$.

Now, $\hat{b}$ induces a permutation on S that either takes singular points with monodromy α^j to points with monodromy $\alpha^{\beta(j)}$, with β an automorphism of C_p, or it acts on each subset formed by points in S with same monodromy α^{r_j}.

We construct monodromies $\theta : \Delta(0; p, \overset{d+3}{\ldots}, p) \to C_p = \langle\alpha\rangle$, where $d = \frac{2g}{p-1} - 2 \geq 2$ by the Riemann-Hurwitz formula. We separate the monodromies in cases according to the congruence of d modulus p.

(1) $d \equiv r \not\equiv 0, 2, p-2, p-1 \, mod(p)$
$\theta : \Delta(0; p, \overset{d+3}{\ldots}, p) \to C_p$ is defined by
$\theta(x_i) = \alpha$, $1 \leq i \leq d$, $\theta(x_{d+1}) = \alpha^2$, $\theta(x_{d+2}) = \alpha^{p-2}$, $\theta(x_{d+3}) = \alpha^{p-r}$.
(2) $d \equiv 0 \, mod(p)$
$\theta : \Delta(0; p, \overset{d+3}{\ldots}, p) \to C_p$ is defined by
$\theta(x_i) = \alpha$, $1 \leq i \leq d$, $\theta(x_{d+1}) = \alpha^3$, $\theta(x_{d+2}) = \alpha^5$, $\theta(x_{d+3}) = \alpha^{p-8}$.
(3) $d \equiv 2 \, mod(p)$
$\theta : \Delta(0; p, \overset{d+3}{\ldots}, p) \to C_p$ is defined by
$\theta(x_i) = \alpha$, $1 \leq i \leq d$, $\theta(x_{d+1}) = \alpha^3$, $\theta(x_{d+2}) = \alpha^{p-3}$, $\theta(x_{d+3}) = \alpha^{p-2}$.
(4) $d \equiv p-2 \, mod(p)$
$\theta : \Delta(0; p, \overset{d+3}{\ldots}, p) \to C_p$ is defined by
$\theta(x_i) = \alpha$, $1 \leq i \leq d$, $\theta(x_{d+1}) = \alpha^3$, $\theta(x_{d+2}) = \alpha^{p-3}$, $\theta(x_{d+3}) = \alpha^2$.
(5) $d \equiv p-1 \, mod(p)$
$\theta : \Delta(0; p, \overset{d+3}{\ldots}, p) \to C_p$ is defined by
$\theta(x_i) = \alpha$, $1 \leq i \leq d$, $\theta(x_{d+1}) = \alpha^4$, $\theta(x_{d+2}) = \alpha^5$, $\theta(x_{d+3}) = \alpha^{p-8}$.

(Notice that $p-8 = 6$ when $p = 7$ in cases 2 and 5)
We see that the given epimorphisms force $\hat{b}$ to be the identity on $\widehat{\mathbb{C}}$. Thus, the surfaces X_g do not admit a larger group of automorphisms than $C_p = \langle\alpha\rangle$ and the equisymmetric strata given by the monodromies above are isolated.

□

Theorem 6 generalizes de results obtained in [BCI1] for isolated strata of pentagonal Riemann surfaces, the results in [CI3] for one-dimensional isolated strata, and the results in [K] for isolated Riemann surfaces. Kulkarni [K] showed that a branch locus $\mathcal{B}_g$ contains isolated Riemann surfaces if and only if $g = 2$ or $g = \frac{p-1}{2}$, with $p \geq 11$ a prime number. The isolated Riemann surfaces are cyclic p-gonal surfaces. Costa and Izquierdo [CI3] showed that $\mathcal{B}_g$ contains one-dimensional isolated strata if and only if $g = p - 1$, with $p \geq 11$ a prime number.

Remark 7. *The isolated strata of heptagonal surfaces with dimension $\frac{g}{3} - 1$ in $\mathcal{B}_g$ obtained here are different of the isolated strata of heptagonal surfaces and dimension $\frac{g}{3} - 1$ obtained in* [BCI2] *since the actions determined by the monodromies are not topologically equivalent, see* [H].

In [BCI1] we showed that $\mathcal{B}_g$ contains isolated strata of cyclic pentagonal surfaces for all even genera greater or equal eighteen. In [BI] (see also [Bo] and [BCIP]) it is shown that the $\mathcal{B}_2$ contains one isolated pentagonal Riemann surface and that $\mathcal{B}_4$, $\mathcal{B}_6$ and $\mathcal{B}_8$ do not contain isolated strata of pentagonal Riemann surfaces. We study the remaining branch loci in the following proposition:

Proposition 8.

(1) $\mathcal{B}_{10}$, $\mathcal{B}_{14}$ and $\mathcal{B}_{16}$ contain isolated strata of cyclic pentagonal Riemann surfaces.
(2) $\mathcal{B}_{12}$ does not contain isolated strata of cyclic pentagonal Riemann surfaces.

Proof. (1) Consider monodromies:

$\theta_1 : \Delta(0; 5, \overset{7}{\ldots}, 5) \to C_5 = \langle \alpha \rangle$ defined by $\theta_1(x_1) = \theta_1(x_2) = \theta_1(x_3) = \alpha$, $\theta_1(x_4) = \alpha^2$, $\theta_1(x_5) = \theta_1(x_6) = \alpha^3$, $\theta_1(x_7) = \alpha^4$,

$\theta_2 : \Delta(0; 5, \overset{9}{\ldots}, 5) \to C_5 = \langle \alpha \rangle$ defined by $\theta_2(x_i) = \alpha, 1 \leq i \leq 6$, $\theta_2(x_7) = \alpha^2$, $\theta_2(x_8) = \alpha^3$, $\theta_2(x_9) = \alpha^4$,

$\theta_3 : \Delta(0; 5, \overset{10}{\ldots}, 5) \to C_5 = \langle \alpha \rangle$ defined by $\theta_3(x_1) = \alpha$, $\theta_3(x_2) = \alpha^2$, $\theta_3(x_3) = \cdots = \theta_3(x_5)\alpha^3$, $\theta_3(x_6) = \cdots = \theta_3(x_{10}) = \alpha^4$

With the same argument as in Theorem 6 we see that θ_1, θ_2 and θ_3 induce isolated strata in $\mathcal{B}_{10}$, $\mathcal{B}_{14}$ and $\mathcal{B}_{16}$ respectively.

(2) Case $\mathcal{B}_{12}$. The only possible monodromies $\theta : \Delta(0, 5, \overset{8}{\ldots}, 5) \to C_5 = \langle \alpha \rangle$ are, up to an automorphism of C_5 and permuting the order of the generators of Δ:

i) $\theta(x_1) = \cdots = \theta(x_5) = \alpha$, $\theta(x_6) = \alpha^2$, $\theta(x_7) = \theta(x_8) = \alpha^4$;
ii) $\theta(x_1) = \cdots = \theta(x_5) = \alpha$, $\theta(x_6) = \alpha^4$, $\theta(x_7) = \theta(x_8) = \alpha^3$;
iii) $\theta(x_1) = \cdots = \theta(x_4) = \alpha$, $\theta(x_6) = \cdots = \theta(x_8) = \alpha^4$;
iv) $\theta(x_1) = \cdots = \theta(x_4) = \alpha$, $\theta(x_5) = \alpha^2$, $\theta(x_6) = \theta(x_7) = \theta(x_8) = \alpha^4$;
v) $\theta(x_1) = \cdots = \theta(x_4) = \alpha$, $\theta(x_5) = \alpha^2$, $\theta(x_6) = \theta(x_7) = \theta(x_8) = \alpha^3$;
vi) $\theta(x_1) = \cdots = \theta(x_4) = \alpha$, $\theta(x_5) = \theta(x_6) = \alpha^2$, $\theta(x_7) = \alpha^3$, $\theta(x_8) = \alpha^4$;
vii) $\theta(x_1) = \theta(x_2) = \theta(x_3) = \alpha$, $\theta(x_4) = \theta(x_5) = \theta(x_6) = \alpha^2$, $\theta(x_7) = \theta(x_8) = \alpha^3$;
viii) $\theta(x_1) = \theta(x_2) = \theta(x_3) = \alpha$, $\theta(x_4) = \alpha^2$, $\theta(x_5) = \alpha^3$, $\theta(x_6) = \theta(x_7) = \theta(x_8) = \alpha^4$;

ix) $\theta(x_1) = \theta(x_2) = \alpha$, $\theta(x_3) = \theta(x_4) = \alpha^2$, $\theta(x_5) = \theta(x_6) = \alpha^3$, $\theta(x_7) = \theta(x_8) = \alpha^4$.

With the argument in the proof of Theorem 6 the action of C_5 on the pentagonal surfaces $\mathcal{D}/Ker(\theta)$ can be extended to the action of a larger group. For instance the action of C_5 in case **ix)** can be extended to an action of C_{10}, D_5 or $C_5 \rtimes C_4$.

□

Remark 9. *Theorem 6 and Porposition 8 can be interpreted geometrically as follows: Let* (X_g^1, C_p) *and* (X_g^2, G) *be two Riemann surfaces with symmetry, where* X_1 *belongs to one of the isolated strata of cyclic p-gonal surfaces in* $\mathcal{B}_g$ *and* X_g^2 *has another symmetry. Then any path of quasiconformal deformations joining* X_g^1 *and* X_g^2 *must contain surfaces without symmetry.*

We consider the existence of several isolated equisymmetric strata in branch loci. Let $5 \leq p_1 < p_2 < \cdots < p_r$ be prime numbers. We define $\lambda = l.c.m.(\frac{p_i-1}{2})_{i=1}^r$. As a consequence of Theorem 6 , Theorem 3.6 in [K] and Theorem 5 in [CI3] we obtain:

Theorem 10. *Let* $5 \leq p_1 < p_2 < \cdots < p_r$ *be prime numbers. Then, for all* $g = k\,\lambda$, $k \geq 1$ *and* $g > 12$, *the branch locus* $\mathcal{B}_g$ *contains* r *isolated strata formed by cyclic* p_i*-gonal Riemann surfaces,* $1 \leq i \leq r$.

Proof. Observe that the conditions of Theorem 6 are satisfied if $g \geq \frac{3}{2}(p_r - 1)$. The conditions of Theorem 5 in [CI3] and Theorem 6 are satisfied if $g = p_r - 1$. Finally the conditions of Theorem 6, Theorem 5 in [CI3] and Theorem 3.6 in [K] are satisfied if $g = \frac{p_r-1}{2}$. The dimension of the isolated strata of cyclic p_i-gonal surfaces is $d_i = \frac{2g}{p_i-1} - 1$ by the Riemann-Hurwitz formula.

$\mathcal{B}_{12}$ does not contain isolated strata of cyclic pentagonal Riemann surfaces, it contains isolated strata of cyclic heptagonal Riemann surfaces.

□

As a consequence we have:

Corollary 11. *Given a number* $r \in \mathbb{N}$, *there is an infinite number of genera* g *such that* $\mathcal{B}_g$ *contains at least* r *isolated equisymmetric strata.*

We finish with some examples for small genera.

3.1. **Examples.**

(1) By Theorem 5 in [CI3] and Proposition 8, $\mathcal{B}_{10}$ contains one isolated stratum of cyclic pentagonal surfaces of dimension four, and one 1-dimensional startum of cyclic 11-gonal surfaces.
(2) By Theorem 6 and Theorem 5 in [CI3], the smallest genus for which the branch locus contains isolated strata of cyclic heptagonal and 13-gonal Riemann surfaces is twelve.The dimensions of the isolated strata are 3 and 1 respectively.
(3) By Theorem 10, $\mathcal{B}_{20}$ contains both isolated strata of cyclic pentagonal and 11-gonal Riemann surfaces. The dimensions of the isolated strata are 9 and 3 respectively. By [K], $\mathcal{B}_{20}$ contains isolated Riemann surfaces that are cyclic 41-gonal.

(4) The smallest genus for which the branch locus contains both isolated strata of cyclic heptagonal and 11-gonal Riemann surfaces is fifteen. The dimensions of the isolated strata are 3 and 2 respectively. By [K], $\mathcal{B}_{15}$ contains isolated Riemann surfaces that are cyclic 31-gonal.

(5) The smallest genus g for which the branch locus $\mathcal{B}_g$ contains both isolated strata of cyclic pentagonal and heptagonal Riemann surfaces is eighteen. The dimensions of the strata are 8 and 5 respectively. It contains also isolated strata of cyclic 13-gonal Riemann surfaces of dimension 3. By [CI3] and [K], $\mathcal{B}_{18}$ contains one-dimensional isolated strata of cyclic 19-gonal surfaces and isolated cyclic 37-gonal Riemann surfaces.

(6) By Theorem 10, $\mathcal{B}_{24}$ contains isolated strata of cyclic pentagonal, heptagonal, 13-gonal and 17-gonal Riemann surfaces. The dimensions of the isolated strata are 11, 7, 3 and 2 respectively.

(7) The smallest genus for which the branch locus contains isolated strata of cyclic pentagonal, heptagonal and 11-gonal Riemann surfaces is thirty, the dimensions of these strata are 14, 9 and 5 respectively. By Theorem 10, $\mathcal{B}_{30}$ contains also isolated strata of cyclic 13-gonal Riemann, surfaces with dimension 4. By [CI3] and [K], $\mathcal{B}_{30}$ contains one-dimensional isolated strata of cyclic 31-gonal surfaces and isolated cyclic 61-gonal Riemann surfaces.

(8) $\mathcal{B}_{60}$ contains isolated strata of cyclic pentagonal, heptagonal, 11-gonal, 13-gonal, 31-gonal, 41-gonal, 61-gonal surfaces, with dimensions 29, 19, 11, 9, 3, 2 and 1 respectively.

(9) $\mathcal{B}_{1000}$ contains isolated strata of cyclic pentagonal, 11-gonal, 17-gonal, 41-gonal, 101-gonal, 251-gonal and 401-gonal surfaces, with dimensions 499, 199, 124, 49, 19, 7 and 4 respectively.

(10) $\mathcal{B}_{2012}$ contains 1005-dimensional isolated strata of cyclic pentagonal surfaces.

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PISOT DUAL TILINGS OF LOW DEGREE AND THEIR DISCONNECTEDNESS

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Abstract. We study the connectedness of the graph-directed self-affine tiles associated to $\beta-$expansions, called Pisot dual tilings. These tiles are examples of Rauzy fractals and play an important role in the study of β-expansion, substitution and symbolic dynamical system. Using the complete classification of the β-expansion of 1 for quartic Pisot units and the classification of the connected tilings given in [4] and [5], here we continue studying connectedness of Pisot dual tilings generated by a Pisot unit with integral minimal equation $x^4-ax^3-bx^2-cx-1=0$ in the special case when $a+c-2\lfloor\beta\rfloor=1$. It is shown that every tile is disconnected having infinitely many connected components.

1. Introduction

Let $\beta > 1$ be a real number which is not an integer. A *greedy expansion* of a positive real x in base β is an expansion of the form:

$$x=\sum_{i=N_0}^{\infty} a_{-i}\beta^{-i}=a_{-N_0},a_{-N_0-1},\cdots$$

with $a_i \in [0,\beta)\cap\mathbb{Z}$ and a *greedy condition*

$$0\le x-\sum_{N_0}^{N} a_{-i}\beta^{-i}<\beta^{-N} \qquad \forall N\ge N_0$$

Let $1=d_{-1}\beta^{-1}+d_{-2}\beta^{-2}+\cdots$ be an expansion of 1 defined by the algorithm

$$c_{-i}=\beta c_{-i+1}-\lfloor\beta c_{-i+1}\rfloor,\quad d_{-i}=\lfloor\beta c_{-i+1}\rfloor \tag{1}$$

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with $c_0 = 1$, where $\lfloor x \rfloor$ denotes the maximal integer not exceeding x. The sequence $d_\beta(1) = .d_{-1}, d_{-2}, \cdots$ is called $\beta-$*expansion of* 1.

Parry [12] has shown that a sequence $x = x_1, x_2, \cdots$ of nonnegative integers is realized as a $\beta-$expansion of some positive real number if and only if it satisfies the following lexicographical condition:

$$\forall p \geq 0, \quad \sigma^p(x) <_{\text{lex}} d^*(1)$$

with

$$d^*(1) = \begin{cases} d_\beta(1), & \text{if } d_\beta(1) \text{ is infinite;} \\ (d_{-1}, d_{-2}, \cdots, d_{-n+1}, (d_{-n} - 1),)^\omega, & \text{if } d_\beta(1) = d_{-1}, \cdots, d_{-n}, \end{cases}$$

where for a string of symbols u, u^ω represents the periodic expansion $u, u, \cdots$ and σ is the shift defined by $\sigma((x_i)_{i \leq M}) = (x_{i-1})_{i \leq M}$. In this case this sequence $x = x_1, x_2, \cdots$ is called *admissible sequence*.

2. Tiling Construction

Let β be a *Pisot number* which is an algebraic integer greater than 1 whose Galois conjugates other than itself have modulus smaller than 1. Let $\mathbb{Q}(\beta)_{\geq 0}$ be the nonnegative elements of the minimum field containing the rational numbers $\mathbb{Q}$ and β. We call a *Pisot unit* a Pisot number which is also a unit of the integer ring of $\mathbb{Q}(\beta)$.

The symbolic dynamical system attached to the β-expansion is sofic if and only if the $\beta-$expansion of 1 is eventually periodic. Especially when β is a Pisot number it gives a sofic system. Thurston [15] introduced an idea to construct a self-affine tiling generated by a Pisot unit β in connection to this sofic system. Akiyama [2] and Praggastis [13] studied in detail such self-affine tilings. G. Rauzy [14] already constructed this kind of tiling in a different approach closely related to substitutions. This tiling has a strong connection to the explicit construction of Markov partitions of dynamical systems, hopefully toral automorphisms; see also P. Arnoux-Sh. Ito [6].

Let us recall this tiling, which is called *dual tiling*, in the notion of [2]. Let

$$\beta = \beta^{(1)}, \beta^{(2)}, \cdots, \beta^{(r_1)} \text{ and } \beta^{(r_1+1)}, \overline{\beta^{(r_1+1)}}, \cdots, \beta^{(r_1+r_2)}, \overline{\beta^{(r_1+r_2)}}$$

be the real and the complex conjugates of β, respectively. We also denote by $x^{(j)}$ $(j = 1, 2, \cdots, r_1 + 2r_2)$ the corresponding conjugates of $x \in \mathbb{Q}(\beta)$.

Define a map

$$\Phi \colon \mathbb{Q}(\beta) \to \mathbb{R}^{\, r_1+2\, r_2-1},$$

by

$$\Phi(x) = \left(x^{(2)}, \cdots, x^{(r_1)}, \Re(x^{(r_1+1)}), \Im(x^{(r_1+1)}), \cdots, \Re(x^{(r_1+r_2)}), \Im(x^{(r_1+r_2)})\right).$$

Let $A = .a_{-1}, a_{-2}, \cdots$ be a greedy expansion in base β. Define S_A to be the set of elements of $\mathbb{Z}[\beta]_{\geq 0}$ whose greedy expansion has the tail A. In other words we just classify all elements of $\mathbb{Z}[\beta]_{\geq 0}$ by their fractional part and map via Φ to have a protile $T_A = \overline{\Phi(S_A)}$. It is not so easy to show that these T_A will give a non overlapping tiling of the space $\mathbb{R}^{r_1+2r_2-1}$. The finiteness condition described in [9] or its weaker version, namely *weakly finiteness* condition, described in [3] implies that these T_A will give a non overlapping tiling of the space $\mathbb{R}^{r_1+2r_2-1}$; see also [2].

One of important aspects of the self-affine tiles is connectedness. Note that if a tile is connected then it must be arcwise connected. This is seen by Hata in [11].

The aim of this paper is to explore the disconnectedness problem of Pisot dual tilings of degree 4 with minimal equation $x^4 - ax^3 - bx^2 - cx - 1 = 0$ where $a, b, c \in \mathbb{Z}$. A general arcwise connectedness criterion for Pisot dual tilings is established in [4]. It is proved that each tile corresponding to a Pisot unit β is arcwise connected if $d_\beta(1)$ terminates with 1.

To treat all Pisot units the above result is not enough since $\beta-$expansion of 1 is not finite in general. If $p(0) = 1$ then $\beta-$expansion of 1 can not be finite (see Proposition 1 of [1]). Even when $p(0) = -1$ there are many such cases. The main result of [4] that we want to extend here is:

Let β be a Pisot unit of degree 4 with minimal polynomial $p(x) = x^4 - ax^3 - bx^2 - cx - 1$. Then each tile is arcwise connected except for the following cases:

$$\begin{cases} a \geq 5 \\ c = a - 3 \\ \frac{5-3a}{2} \leq b \leq -a \end{cases} \quad \begin{cases} a \geq 3 \\ c = a - 1 \\ \frac{1-a}{2} \leq b \leq -1 \end{cases} \quad \begin{cases} a \geq 3 \\ c = a + 1 \\ \frac{1+a}{2} \leq b \leq a-1 \end{cases} \quad \begin{cases} a \geq 1 \\ c = a + 3 \\ \frac{5+3a}{2} \leq b \leq 2a+2 \end{cases}$$

The above result was proved in [4] and [5] to be equivalent to:

Let β be a Pisot unit of degree 4 with minimal polynomial $p(x) = x^4 - ax^3 - bx^2 - cx - 1$. Then

- $a + c - 2\lfloor \beta \rfloor \leq 1$,
- each tile is arcwise connected if and only if $a + c - 2\lfloor \beta \rfloor \leq 0$.

In fact, here we prove that if $\deg \beta = 4$, $p(0) = -1$ and $a + c - 2[\beta] = 1$, each tile is disconnected having infinitely many connected components. As far as we know, no example of such type of disconnected Pisot dual tiles was known before. As the Pisot dual tiles are generated by consecutive integers, it was expected that they are always connected. Thus this result gives an unfortunate surprise that there exists a concrete family of Pisot units each tile of whose dual tiling is disconnected.

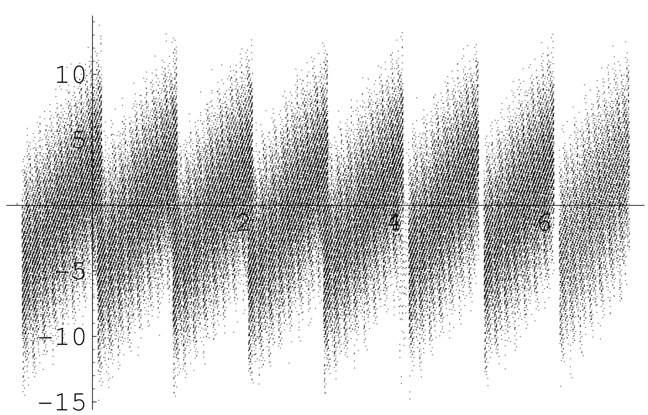

FIGURE 1. The projection of the central tile (disconnected) generated by the Pisot unit β with minimal equation $x^4 - 7x^3 - 4x^2 - 8x - 1 = 0$

3. Disconnectedness

Let β be a quartic Pisot unit of degree 4 with integral minimal polynomial

$$p(x) = x^4 - ax^3 - bx^2 - cx - 1.$$

From now on, we will assume that

$$a + c - 2\lfloor\beta\rfloor = 1. \tag{2}$$

Let $d_\beta(1) = .d_{-1}, d_{-2}, \cdots$ be the $\beta-$expansion of 1.

Lemma 3.1. *Suppose that $\xi = \xi_{-1}, \xi_{-2}, \cdots$ is an admissible expansion with $.\xi \geq .d_{-2}, d_{-3}, \cdots$. Then for every $m \in \{1, 2, \cdots, \xi_{-1}\}$ the following holds*

$$T_{.\xi} \cap (T. + \phi(\xi - m\beta^{-1}) = \emptyset$$

or written in an equivalent way we can say that under Condition (2), if $T_{.e_1,e_2} \cap T_{.f_1,f_2} \neq \emptyset$ and $e_2 - f_2 \geq 1$ then $e_1 = \lfloor\beta\rfloor$ and $f_1 = 0$.

Proof. First, recall that the condition $a+c-2\lfloor\beta\rfloor = 1$ is equivalent to the following 4 cases:

$$i) \begin{cases} a \geq 5 \\ c = a - 3 \\ \dfrac{5-3a}{2} \leq b \leq -a \end{cases}$$

$$ii) \begin{cases} a \geq 3 \\ c = a - 1 \\ \dfrac{1-a}{2} \leq b \leq -1 \end{cases}$$

$$iii) \begin{cases} a \geq 3 \\ c = a + 1 \\ \dfrac{1+a}{2} \leq b \leq a-1 \end{cases}$$

$$iv) \begin{cases} a \geq 1 \\ c = a + 3 \\ \dfrac{5+3a}{2} \leq b \leq 2a+2 \end{cases}$$

Let γ be the negative root of the equation $x^2 - \lfloor\beta\rfloor x - 1$. Let us show that in 4 possible cases we have that $p(\gamma) > 0$.

i) If $c=a-3$ then $\lfloor\beta\rfloor=a-2$ and $b\leq-a$. So

$$p(\gamma)\geq\gamma^4-a\gamma^3+a\gamma^2-(a-3)\gamma-1=-\gamma^3+2\gamma^2>0$$

ii) If $c=a-1$ then $\lfloor\beta\rfloor=a-1$ and $b\leq-1$. So

$$p(\gamma)\geq\gamma^4-a\gamma^3+\gamma^2-(a-1)\gamma-1=\gamma^4-a\gamma^3>0$$

iii) If $c=a+1$ then $\lfloor\beta\rfloor=a$ and $b\leq a-1$. So

$$p(\gamma)\geq\gamma^4-a\gamma^3-(a-1)\gamma^2-(a+1)\gamma-1=-\gamma^3+\gamma^2>0$$

iv) If $c=a+3$ then $\lfloor\beta\rfloor=a+1$ and $b\leq2a+2$. So

$$p(\gamma)\geq\gamma^4-a\gamma^3-(2a+2)\gamma^2-(a+3)\gamma-1=-\gamma^3+2\gamma^2>0$$

Let θ be the biggest among the negative roots of the polynomial $p(x) = x^4 - ax^3 - bx^2 - cx - 1$. The existence of such root is implied from the fact that $p(-1) > 0$ and $p(0) = 1$. Since $x \in (\theta, 0) \Rightarrow p(x) < 0$ then $\gamma < \theta < 0$. So $\theta^2 - \lfloor\beta\rfloor\theta - 1 < 0$.

If we suppose that $\exists m \in \{1, 2, \cdots, \xi_{-1}\}$ such that

$$T_{.\xi} \cap (T_{.} + \phi(\xi - m\beta^{-1}) \neq \emptyset$$

then exists an expansion of 0 in base θ

$$m\theta^{-1} + c_0 + \sum_{i=1}^{\infty} c_i\theta^i$$

such that $1 \leq m \leq \xi_{-1}$, $\forall i \in \{0, 1, 2, \cdots\}$ we have that $c_i \in \mathbb{Z} \cap [-\beta, \beta]$ and $c_0 \leq \beta - 1$. So we have that

$$0 = m\theta^{-1} + c_0 + \sum_{i=1}^{\infty} c_i\theta^i \leq \theta^{-1} + \lfloor\beta\rfloor - 1 - \frac{\lfloor\beta\rfloor\theta}{1+\theta} = \frac{\theta^2 - \lfloor\beta\rfloor\theta - 1}{-\theta(1+\theta)} < 0.$$

This contradiction ends the proof of the current lemma. $\square$

For a Pisot number of degree d let G_{-1} be the natural map defined by the following commutative diagram:

(3)
$$\begin{array}{ccc} \mathbb{Q}(\beta) & \xrightarrow{\times\beta} & \mathbb{Q}(\beta) \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \mathbb{R}^{d-1} & \xrightarrow[G_{-1}]{} & \mathbb{R}^{d-1}. \end{array}$$

where, we denote by $a \oplus b$ the concatenation of words a, b. Then G_{-1} is contractive since β is a Pisot number. The set equations are given in this form:

(4)
$$T_{.A} = \bigcup_{.i\oplus A} G_{-1}(T_{.i\oplus A}),$$

where the summation is taken over all possible $i \in [0, \beta) \cap \mathbb{Z}$ such that $i \oplus A$ is admissible (see [3]). Note that we identify $.i \oplus A$ with the corresponding $\beta-$expansion to realize it as a non negative real number. For the $\beta-$expansion of 1 that appear in the following lemmas see [10].

Lemma 3.2. *Let $d_\beta(1) = .\lfloor\beta\rfloor, d_{-2}, \cdots$ be the $\beta-$expansion of 1. If $d_{-2} < \lfloor\beta\rfloor$ then each tile is disconnected.*

Proof. Let $T_{.\omega}$ be a tile, which means that $0 \leq .\omega < .d_{-1}, d_{-2}, \cdots$. Here we consider two cases:

- $\lfloor\beta\rfloor \oplus \omega$ is admissible which is equivalent to $0 \leq .\omega < .d_{-2}, d_{-3}, \cdots$. We have that
$$T_{.\omega} = \bigcup_{i=0}^{\lfloor\beta\rfloor} G_{-1}(T_{.i\oplus\omega})$$
Since $\lfloor\beta\rfloor > d_{-2}$ then $.\lfloor\beta\rfloor \oplus \omega \geq .d_{-2}, d_{-3}, \cdots$. Using Lemma 3.1 we get that $\left(\bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega})\right) \bigcap G_{-1}(T_{.\lfloor\beta\rfloor\oplus\omega}) = \emptyset$, which shows that $T_{.\omega}$ is a disconnected tile.

- $\lfloor\beta\rfloor \oplus \omega$ is not admissible which is equivalent to $.\,d_{-2}, d_{-3}, \cdots \leq .\,\omega < .\,d_{-1}, d_{-2}, \cdots$. We have that

$$T_{.\,\omega} = \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}),$$

Let us show that $.\lfloor\beta\rfloor - 1 \oplus \omega \geq .d_{-2}, d_{-3}, \cdots$. If we suppose the contrary then $\lfloor\beta\rfloor - 1 \leq d_{-2}$, which implies that $d_{-2} = \lfloor\beta\rfloor - 1$ and $.\,\omega < .\,d_{-3}, d_{-4}, \cdots$. Since $\lfloor\beta\rfloor \oplus \omega$ is not admissible we get that

$$.\,d_{-2}, d_{-3}, \cdots < .\,d_{-3}, d_{-4}, \cdots \tag{5}$$

Since $a + c - 2\lfloor\beta\rfloor = 1$, let us consider the 4 possibilities such that $d_{-2} = \lfloor\beta\rfloor - 1$.

(1) If $c = a - 3$, then
$d_\beta(1) = .a{-}2, 2a{+}b{-}2, (3a{+}2b{-}4, 3a{+}2b{-}5, 2a{+}b{-}3, 0, 1{-}a{-}b, 2{-}a{-}b, 0, 2a{+}b{-}3)^\omega$
So $d_{-2} = \lfloor\beta\rfloor - 1 = a - 3$ implies that $d_{-3} = \lfloor\beta\rfloor - 4$, which contradicts (5).

(2) If $c = a - 1$ then
$$d_\beta(1) = .a{-}1, a{+}b, (a{+}b, 0, {-}b, 0, a{+}b{-}1)^\omega$$
So $d_{-2} = \lfloor\beta\rfloor - 1 = a - 2$ implies that $.d_{-2}, d_{-3}, \cdots = .a - 2, (a - 2, 0, 2, 0, a - 3)^\omega$ and $.d_{-3}, d_{-4}, \cdots = .(a - 2, 0, 2, 0, a - 3)^\omega$, which contradicts (5).

(3) If $c = a + 1$ then
$$d_\beta(1) = .\,a, b{+}1, (0, a{-}b, b, b, a{-}b{+}1, 0, b)^\omega$$
So $d_{-2} = \lfloor\beta\rfloor - 1 = a - 1 \geq 2$ contradicts (5).

(4) If $c = a + 3$ then $a \geq 3$ and
$$d_\beta(1) = .\,a{+}1, b{-}a{-}1, (2a{-}b{+}3, b{-}a{-}1, 0, 2a{-}b{+}3, 2b{-}3a{-}5, 4a{-}2b{+}6, 2b{-}3a{-}4, 2a{-}b{+}3, 0, b{-}a{-}2)^\omega$$
So $d_{-2} = \lfloor\beta\rfloor - 1 = a \geq 3$ implies that $d_{-3} = 2$, which contradicts (5).

So we proved that $.\lfloor\beta\rfloor - 1 \oplus \omega \geq .d_{-2}, d_{-3}, \cdots$. Using Lemma 3.1 we get that $\left(\bigcup_{i=0}^{\lfloor\beta\rfloor-2} G_{-1}(T_{.i\oplus\omega})\right) \bigcap G_{-1}(T_{.\lfloor\beta\rfloor-1\oplus\omega}) = \emptyset$, which shows that $T_{.\omega}$ is a disconnected tile.

□

Lemma 3.3. *Let $d_\beta(1) = .\lfloor\beta\rfloor, \lfloor\beta\rfloor, d_{-3} \cdots$ be the $\beta-$expansion of 1. If $d_{-3} < \lfloor\beta\rfloor$ then each tile is disconnected.*

Proof. Let $T_{.\,\omega}$ be a tile, for $\omega = \omega_1, \omega_2, \cdots$, which means that $0 \leq .\,\omega < .d_{-1}, d_{-2}, \cdots$. Since

$$.d_{-3}, d_{-4}, \cdots < .\lfloor\beta\rfloor, d_{-3}, d_{-4}, \cdots < \lfloor\beta\rfloor, \lfloor\beta\rfloor, d_{-3}, d_{-4}, \cdots$$

here we consider three cases:

- $\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \omega$ is admissible which is equivalent to $0 \leq .\,\omega < .\,d_{-3}, d_{-4}, \cdots$. Here we have that

$$\begin{aligned} T_{.\,\omega} = \bigcup_{i=0}^{\lfloor\beta\rfloor} G_{-1}(T_{.i\oplus\omega}) &= \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \cup G_{-1}(T_{.\lfloor\beta\rfloor\oplus\omega}) \\ &= \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \cup \bigcup_{i=0}^{\lfloor\beta\rfloor}(G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor\oplus\omega}) \\ &= \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \cup \bigcup_{i=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor\oplus\omega}) \cup (G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor\oplus\omega}) \end{aligned}$$

Using Lemma 3.1, since $.\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \omega \geq .d_{-2}, d_{-3}, \cdots$, we get that

$$\bigcup_{i=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor\oplus\omega}) \cap (G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor\oplus\omega}) = \emptyset \tag{6}$$

Also, from the second part of Lemma 3.1, we get that

$$\bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i}) \cap \bigcup_{i=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.i,\lfloor\beta\rfloor}) \tag{7}$$

Since $G_{-1}(T_{.i\oplus\omega}) \subset G_{-1}(T_{.i}) + \phi(\omega_1\beta^{-1} + \omega_2\beta^{-2} + \cdots)$ and $(G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor\oplus\omega}) \subset (G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor}) + \phi(\omega_1\beta^{-1} + \omega_2\beta^{-2} + \cdots)$, using (6) and (7), we get that

$$\bigcup_{i=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor\oplus\omega}) \cap \left((G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor\oplus\omega}) \cup \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \right) = \emptyset$$

which shows that $T_{.\omega}$ is a disconnected tile.

- $\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \omega$ is not admissible but $\lfloor\beta\rfloor \oplus \omega$ is admissible, which is equivalent to $.\,d_{-3}, d_{-4}, \cdots \leq .\,\omega < .\,\lfloor\beta\rfloor, d_{-3}, \cdots$. Here we have that

$$T_{.\,\omega} = \bigcup_{i=0}^{\lfloor\beta\rfloor} G_{-1}(T_{.i\oplus\omega}) = \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \cup G_{-1}(T_{.\lfloor\beta\rfloor\oplus\omega})$$

Since $.\lfloor\beta\rfloor \oplus \omega \geq .d_{-2}, d_{-3}, \cdots$, using Lemma 3.1, we get that

$$\bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \cap G_{-1}(T_{.\lfloor\beta\rfloor\oplus\omega}) = \emptyset$$

which shows that $T_{.\omega}$ is a disconnected tile.

- $\lfloor\beta\rfloor \oplus \omega$ is not admissible which is equivalent to $.\,\lfloor\beta\rfloor, d_{-3}, \cdots \leq .\,\omega < .\,\lfloor\beta\rfloor, \lfloor\beta\rfloor, d_{-3}, \cdots$.

Here we have that

$$\begin{aligned}
T_{.\omega} &= \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) = \bigcup_{i=0}^{\lfloor\beta\rfloor-2} G_{-1}(T_{.i\oplus\omega}) \cup G_{-1}(T_{.\lfloor\beta\rfloor-1\oplus\omega}) \\
&= \bigcup_{i=0}^{\lfloor\beta\rfloor-2} G_{-1}(T_{.i\oplus\omega}) \cup \bigcup_{i=0}^{\lfloor\beta\rfloor}(G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor-1\oplus\omega}) \\
&= \bigcup_{i=0}^{\lfloor\beta\rfloor-2} G_{-1}(T_{.i\oplus\omega}) \cup \bigcup_{i=0}^{\lfloor\beta\rfloor-1}(G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor-1\oplus\omega}) \cup (G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor-1\oplus\omega})
\end{aligned}$$

This case happens when
△ $c = a-3$, $\lfloor\beta\rfloor = a-2$ and $d_{-3} = a-4 = \lfloor\beta\rfloor - 2$,
△ $c = a+1$, $\lfloor\beta\rfloor = a \geq 3$, $b = a-1$ and $d_{-3} = 0$
△ $c = a+3$, $\lfloor\beta\rfloor = a+1 \geq 2$ and $d_{-3} = 1$.
So $d_{-3} < \lfloor\beta\rfloor - 1$, which implies that $\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor - 1 \oplus \omega \geq .d_{-2}, d_{-3}, \cdots$. ($d_{-3} = \lfloor\beta\rfloor - 1$ happens only when $a = 1$ and $c = a+3$. In this case also the previous inequality is true.) Using Lemma 3.1 we get that $\bigcup_{i=0}^{\lfloor\beta\rfloor-1} T_{.i\oplus\lfloor\beta\rfloor-1\oplus\omega} \cap T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor-1\oplus\omega} = \emptyset$ which implies that

$$\bigcup_{i=0}^{\lfloor\beta\rfloor-1}(G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor-1\oplus\omega}) \cap (G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor-1\oplus\omega}) = \emptyset \tag{8}$$

Also, from the second part of Lemma 3.1, we get that

$$\bigcup_{i=0}^{\lfloor\beta\rfloor-1}(G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor-1}) \cap \bigcup_{i=0}^{\lfloor\beta\rfloor-2} G_{-1}(T_{.i}) = \emptyset \tag{9}$$

Since $G_{-1}(T_{.i}) + \phi(\omega_1\beta^{-1} + \cdots) \supset G_{-1}(T_{.i\oplus\omega})$ and $(G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor-1}) + \phi(\omega_1\beta^{-1} + \cdots) \supset (G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor-1\oplus\omega})$, using (8) and (9), we get that

$$\bigcup_{i=0}^{\lfloor\beta\rfloor-1}(G_{-1})^2(T_{.i\oplus\lfloor\beta\rfloor-1\oplus\omega}) \cap \left((G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor-1\oplus\omega}) \cup \bigcup_{i=0}^{\lfloor\beta\rfloor-2} G_{-1}(T_{.i\oplus\omega}) \right) = \emptyset$$

which shows that $T_{.\omega}$ is a disconnected tile.

□

Lemma 3.4. *Let $d_\beta(1) = .\lfloor\beta\rfloor, \lfloor\beta\rfloor, \lfloor\beta\rfloor, d_{-4}\cdots$ be the $\beta-$expansion of 1. Then each tile is disconnected.*

Proof. The supposition of lemma is equivalent to $a \geq 3,\ b = -1$ and $c = a-3$. Here $\lfloor\beta\rfloor = a-1$ and the $\beta-$ expansion of 1 is:

$$d_\beta(1) = .\lfloor\beta\rfloor, \lfloor\beta\rfloor, (\lfloor\beta\rfloor,\ 0\ ,\ 1\ ,\ 0\ , \lfloor\beta\rfloor-1)^\omega$$

For $\omega = \omega_1, \omega_2, \cdots$, let $T_{.\omega}$ be a tile which means that $0 \leq .\omega < d_\beta(1)$. Since

$$.d_{-4}, d_{-5}, \cdots < .\lfloor\beta\rfloor, d_{-4}, \cdots < .\lfloor\beta\rfloor, \lfloor\beta\rfloor, d_{-4}, \cdots < .\lfloor\beta\rfloor, \lfloor\beta\rfloor, \lfloor\beta\rfloor, d_{-4}, \cdots$$

here we consider four possible cases:

Case i) $.\lfloor\beta\rfloor,\lfloor\beta\rfloor,d_{-4}\cdots\le .\omega<d_\beta(1)$ which is equivalent to $\lfloor\beta\rfloor\oplus\omega$ is not admissible. Here we have that

$$\begin{aligned}
T_{.\omega}=&G_{-1}(T_{.0\oplus\omega})\cup\bigcup_{i=1}^{\lfloor\beta\rfloor-1}G_{-1}(T_{.i\oplus\omega})\\
&=\bigcup_{j=1}^{\lfloor\beta\rfloor-1}(G_{-1})^2(T_{.j\oplus0\oplus\omega})\cup(G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus0\oplus\omega})\cup\bigcup_{i=1}^{\lfloor\beta\rfloor-1}G_{-1}(T_{.i\oplus\omega})\\
&=\bigcup_{j=1}^{\lfloor\beta\rfloor-1}(G_{-1})^2(T_{.j\oplus0\oplus\omega})\cup\bigcup_{k=1}^{\lfloor\beta\rfloor-1}(G_{-1})^3(T_{.k\oplus\lfloor\beta\rfloor\oplus0\oplus\omega})\cup(G_{-1})^3(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor\oplus0\oplus\omega})\\
&\cup\bigcup_{i=1}^{\lfloor\beta\rfloor-1}G_{-1}(T_{.i\oplus\omega})
\end{aligned}$$

Using Lemma 3.1, since $.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor\oplus0\oplus\omega\ge .d_{-2},d_{-3},\cdots$, we get that

$$\bigcup_{k=0}^{\lfloor\beta\rfloor-1}(G_{-1})^3(T_{.k\oplus\lfloor\beta\rfloor\oplus0\oplus\omega})\cap(G_{-1})^3(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor\oplus0\oplus\omega})=\emptyset \tag{10}$$

Also, from the second part of Lemma 3.1, we get that

$$\begin{aligned}
&\bigcup_{k=1}^{\lfloor\beta\rfloor-1}(G_{-1})^2(T_{.k,\lfloor\beta\rfloor})\cap\bigcup_{j=1}^{\lfloor\beta\rfloor-1}G_{-1}(T_{.j})=\emptyset\\
&\bigcup_{k=1}^{\lfloor\beta\rfloor-1}(G_{-1})^3(T_{.k,\lfloor\beta\rfloor})\cap\bigcup_{i=1}^{\lfloor\beta\rfloor-1}G_{-1}(T_{.i})=\emptyset
\end{aligned}\tag{11}$$

Since

$$(G_{-1})^2(T_{.j})+\phi(\omega_1\beta^{-1}+\omega_2\beta^{-2}+\cdots)\supset(G_{-1})^2(T_{.j\oplus0\oplus\omega}),$$

then

$$\bigcup_{k=1}^{\lfloor\beta\rfloor-1}(G_{-1})^3(T_{.k,\lfloor\beta\rfloor})+\phi(\omega_1\beta^{-1}+\cdots)\supset(G_{-1})^3(T_{.k\oplus\lfloor\beta\rfloor\oplus0\oplus\omega})$$

and

$$G_{-1}(T_{.i})+\phi(\omega_1\beta^{-1}+\cdots)\supset G_{-1}(T_{.i\oplus\omega}).$$

Using (10) and (11), we get that

$$\begin{aligned}
&\bigcup_{k=1}^{\lfloor\beta\rfloor-1}(G_{-1})^3(T_{.k\oplus\lfloor\beta\rfloor\oplus0\oplus\omega})\cap\\
&\left(\bigcup_{j=1}^{\lfloor\beta\rfloor-1}(G_{-1})^2(T_{.j\oplus0\oplus\omega})\cup(G_{-1})^3(T_{.\lfloor\beta\rfloor,\lfloor\beta\rfloor,0,\omega})\cup\bigcup_{i=1}^{\lfloor\beta\rfloor-1}G_{-1}(T_{.i,\omega})\right)=\emptyset
\end{aligned}$$

which shows that $T_{.\omega}$ is a disconnected tile.

Case ii) $.\lfloor\beta\rfloor,d_{-4}\cdots\le .\omega<.\lfloor\beta\rfloor,\lfloor\beta\rfloor,d_{-4}\cdots$ which is equivalent to $\lfloor\beta\rfloor\oplus\omega$ is admissible but $\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor\oplus\omega$ is not admissible. In the previous Lemma, for $.d_{-3},d_{-4},\cdots\le .\omega<.d_{-2},d_{-3},\cdots$, we did not use the supposition that $d_{-3}<\lfloor\beta\rfloor$, so the proof shows that $T_{.\omega}$ is disconnected even if $d_{-3} = \lfloor\beta\rfloor$.

Case iii) $.d_{-4}, d_{-5}, \cdots \leq .\omega < .\lfloor\beta\rfloor, d_{-4}\cdots$ which is equivalent to $\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \omega$ is admissible ($\lfloor\beta\rfloor \oplus \omega$ is admissible also) but $\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \omega$ is not admissible.

$$\begin{aligned} T_{.\omega} &= \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \cup G_{-1}(T_{.\lfloor\beta\rfloor\oplus\omega}) \\ &= \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \cup \bigcup_{j=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.j\oplus\lfloor\beta\rfloor\oplus\omega}) \cup (G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor\oplus\omega}) \end{aligned}$$

Since $.\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \omega \geq .d_{-2}, d_{-3}, \cdots$, using Lemma 3.1 we have that

$$\bigcup_{j=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.j\oplus\lfloor\beta\rfloor\oplus\omega}) \cap (G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor\oplus\omega}) = \emptyset \tag{12}$$

Also from the second part of Lemma 3.1 we get that

$$\bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i}) \cap \bigcup_{j=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.j,\lfloor\beta\rfloor}) = \emptyset \tag{13}$$

Since

$$G_{-1}(T_{.i}) + \phi(\omega_1\beta^{-1} + \omega_2\beta^{-2} + \cdots) \supset G_{-1}(T_{.i\oplus\omega}),$$

and

$$(G_{-1})^2(T_{.j\oplus\lfloor\beta\rfloor}) + \phi(\omega_1\beta^{-1} + \cdots) \supset (G_{-1})^2(T_{.j\oplus\lfloor\beta\rfloor\oplus\omega}),$$

using (12) and (13) we get that

$$\bigcup_{j=0}^{\lfloor\beta\rfloor-1} (G_{-1})^2(T_{.j\oplus\lfloor\beta\rfloor\oplus\omega}) \cap \left(\bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i\oplus\omega}) \cup (G_{-1})^2(T_{.\lfloor\beta\rfloor\oplus\lfloor\beta\rfloor\oplus\omega}) \right) = \emptyset$$

which shows that $T_{.\omega}$ is a disconnected tile.

Case iv) $0 \leq .\omega < .d_{-4}, d_{-5}, \cdots$ which is equivalent to $\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \omega$ is admissible ($\lfloor\beta\rfloor \oplus \lfloor\beta\rfloor \oplus \omega$ and $\lfloor\beta\rfloor \oplus \omega$ are admissible also). Since $T_{.\omega}$ is a translation of the central tile $T.$, it is enough to show that $T.$ is disconnected.

$$\begin{aligned} T. &= \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i}) \cup G_{-1}(T_{.\lfloor\beta\rfloor}) \\ &= \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i}) \cup (G_{-1})^2(T_{.0,\lfloor\beta\rfloor}) \cup \bigcup_{j=1}^{\lfloor\beta\rfloor} (G_{-1})^2(T_{.j,\lfloor\beta\rfloor}) \\ &= \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i}) \cup \bigcup_{k=0}^{\lfloor\beta\rfloor-1} (G_{-1})^3(T_{.k,0,\lfloor\beta\rfloor}) \cup (G_{-1})^3(T_{.\lfloor\beta\rfloor,0,\lfloor\beta\rfloor}) \cup \bigcup_{j=1}^{\lfloor\beta\rfloor} (G_{-1})^3(T_{.j,\lfloor\beta\rfloor}) \\ &= \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i}) \cup \bigcup_{k=0}^{\lfloor\beta\rfloor-1} (G_{-1})^3(T_{.k,0,\lfloor\beta\rfloor}) \cup \bigcup_{l=0}^{\lfloor\beta\rfloor-1} (G_{-1})^4(T_{.l,\lfloor\beta\rfloor,0,\lfloor\beta\rfloor}) \\ &\qquad \cup (G_{-1})^4(T_{.\lfloor\beta\rfloor,\lfloor\beta\rfloor,0,\lfloor\beta\rfloor}) \cup \bigcup_{j=1}^{\lfloor\beta\rfloor} (G_{-1})^2(T_{.j,\lfloor\beta\rfloor}) \end{aligned}$$

Since $.\lfloor\beta\rfloor,\lfloor\beta\rfloor,0,\lfloor\beta\rfloor \geq .d_{-2},d_{-3},\cdots$, using Lemma 3.1, we get that

$$\bigcup_{l=0}^{\lfloor\beta\rfloor-1}(G_{-1})^4(T_{.l,\lfloor\beta\rfloor,0,\lfloor\beta\rfloor}) \cap (G_{-1})^4(T_{.\lfloor\beta\rfloor,\lfloor\beta\rfloor,0,\lfloor\beta\rfloor}) = \emptyset \tag{14}$$

Also, using the second part of Lemma 3.1, we get that

$$(G_{-1})^2(T_{.0,\lfloor\beta\rfloor}) \cap \bigcup_{i=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.i}) = \emptyset, \quad \bigcup_{l=0}^{\lfloor\beta\rfloor-1}(G_{-1})^2(T_{.l,\lfloor\beta\rfloor}) \cap \bigcup_{k=0}^{\lfloor\beta\rfloor-1} G_{-1}(T_{.k}) = \emptyset$$

$$(G_{-1})^2(T_{.\lfloor\beta\rfloor}) \cap \bigcup_{j=1}^{\lfloor\beta\rfloor} G_{-1}(T_{.j}) = \emptyset$$

Since

$$(G_{-1})^2(T_{.0,\lfloor\beta\rfloor}) \supset \bigcup_{l=0}^{\lfloor\beta\rfloor-1}(G_{-1})^4(T_{.l,\lfloor\beta\rfloor,0,\lfloor\beta\rfloor}),$$

then

$$\bigcup_{l=0}^{\lfloor\beta\rfloor-1}(G_{-1})^4(T_{.l,\lfloor\beta\rfloor}) + \phi(\lfloor\beta\rfloor) \supset \bigcup_{l=0}^{\lfloor\beta\rfloor-1}(G_{-1})^4(T_{.l,\lfloor\beta\rfloor,0,\lfloor\beta\rfloor})$$

and

$$\bigcup_{k=0}^{\lfloor\beta\rfloor-1}(G_{-1})^3(T_{.k}) + \phi(\lfloor\beta\rfloor) \supset \bigcup_{k=0}^{\lfloor\beta\rfloor-1}(G_{-1})^3(T_{.k,0,\lfloor\beta\rfloor}),$$

$$(G_{-1})^3(T_{.\lfloor\beta\rfloor}) + \phi(\lfloor\beta\rfloor) \supset \bigcup_{l=0}^{\lfloor\beta\rfloor-1}(G_{-1})^4(T_{.l,\lfloor\beta\rfloor,0,\lfloor\beta\rfloor})$$

$$\bigcup_{j=1}^{\lfloor\beta\rfloor}(G_{-1})^2(T_{.j}) + \phi(\lfloor\beta\rfloor) \supset \bigcup_{j=1}^{\lfloor\beta\rfloor}(G_{-1})^2(T_{.j,\lfloor\beta\rfloor})$$

we get that

$$\begin{aligned}&\bigcup_{l=0}^{\lfloor\beta\rfloor-1}(G_{-1})^4(T_{.l,\lfloor\beta\rfloor,0,\lfloor\beta\rfloor}) \cap \\ &\cap\left(\bigcup_{i=0}^{\lfloor\beta\rfloor-1}G_{-1}(T_{.i})\cup\bigcup_{k=0}^{\lfloor\beta\rfloor-1}(G_{-1})^3(T_{.k,0,\lfloor\beta\rfloor})\cup\bigcup_{j=1}^{\lfloor\beta\rfloor}(G_{-1})^2(T_{.j,\lfloor\beta\rfloor})\right)=\emptyset\end{aligned} \tag{15}$$

Formulas (14) and (15) show that the central tile is disconnected.

□

Combining the results of Lemmas 3.2, 3.3 and 3.4 we get the following theorem:

Theorem 3.1. *Let β be a Pisot unit with integral minimal equation $x^4 - ax^3 - bx^2 - cx - 1 = 0$ such that $a + c - 2\lfloor\beta\rfloor = 1$. Then each tile is disconnected having infinitely many connected components.*

4. Conclussions, Comments and Open Problems

In the previous works of Akiyama and Gjini in [4] and [5], it was proved that at least one of such tiles is disconnected. This result was generalized here. We proved that every dual tile is disconnected and furthermore each of them has infinitely many connected components which is a surprise because the digits of quartic $\beta-$expansions are consecutive integers which leads one to expect connected tiles. As a result we have a complete classification of connectedness of Pisot dual tiles with respect to quartic Pisot units. It remains to be found the characterization of β-expansion of 1 for Pisot numbers of lower degree as well as to study the connectedness, Hausdorff dimension of the boundary for the tiles generated by Pisot numbers of higher degree.

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NUMERICAL EXPERIMENTS RELATED TO THE INFORMATION–THEORETIC SCHOTTKY AND TORELLI PROBLEMS

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Abstract. The Schottky Problem is to characterize the Jacobians in the space of principally polarized abelian varieties (PPAVs). Besides its intrinsic interest, it is related to deep questions in PDE and Physics. The perspective of Information Theory leads to the numerical investigation of properties of the distribution of the periods, especially in cases of relatively large genus. In turn, numerical results lead to conjectures that the periods of hyperelliptic curves are band-limited, and that the squared moduli of the periods of a general surface are Zipfian.

1. Introduction

1.1. **Introduction.** Interest in two classical problems from the study of compact Riemann Surfaces of genus at least two continues to grow as new interpretations and new computational tools arise. The first of these, the Schottky Problem, is to characterize the Jacobians in the space of principally polarized abelian varieties (PPAVs). Besides its intrinsic interest, its solutions involve such diverse areas as classical algebraic geometry and partial differential equations, especially the Korteweig–deVries (KdV) and the Kadomtsev–Petviashvili (KP) equations [9], which relate it to problems of physics. Additionally, one of the earliest modern solutions to the Schottky problem, that of Andreootti and Mayer [2], used the heat equation in an essential way.

The second of the two problems, the Torelli problem, seeks methods to determine properties of a Riemann Surface from its period matrix, which, in principle, is possible due to Torelli's Theorem [10], but which, in practice, has proven to be rather difficult, involving detailed properties of Riemann's theta function.

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More recently, *Information Theory* has become a prominent technique in many areas of mathematics and applications. For present purposes, the basic idea of information theory is to understand the minimum number of bits required for Alice to send a message to Bob. A simple but telling example comes from the idea of a Huffman code, which optimally determines the length of symbols used with respect to their frequency of use. In a Huffman code for ordinary English, the letters `T` and `E` are assigned the smallest number of bits, because they are used most often, while seldom–used letters like `Q` and `Z` are encoded with more bits.

This work describes numerical exploration of the Schottky and Torelli problems from the perspective of information theory. Alice wants to tell Bob about a Riemann surface, and, by Torelli's theorem, she can do so by giving him its period matrix. These experiments lead to conjectures about the information-theoretic nature of the period matrix of a general Riemann surface (a Schottkly problem), about distinguishing the period matrix of a hyperelliptic Riemann surface from that of a general Riemann surface (a Torelli problem), and about the effects of truncated arithmetic.

1.2. **Computational Perspective.** New computational tools now allow us to examine the Schottky and Torelli problems from a computational perspective. When Alice tells Bob about a Riemann surface using its period matrix,this message's length is $g(g+1)/2$ complex numbers, although in practice the information might be $g(g+1)b/2$ bits, where each complex number is approximated by a binary number with b or fewer bits.

However, since the moduli space of compact Riemann Surfaces of genus g has dimension $3g-3$ when $g>1$, the message is intrinsically compressible, that is, it contains much less information than its naive length would indicate. The nature of the *Information-Theoretic Schottky Problem* is to exploit the compressibility to try to characterize period matrices in $\mathcal{H}_g$, and that of the *Information-Theoretic Torelli Problem* is to exploit properties of the signal to determine properties of the curve.

In a sense, the Information Theoretic Schottky Problem was anticipated by Rauch over 50 years ago [11]. He proved that if there is a set of g periods $\pi_{ij} = \int_{b_j} d\zeta_i$ on a non-hyperelliptic compact Riemann Surface W of genus g such that the products $d\zeta_{i_1} d\zeta_{i_2}$ formed a basis for the quadratic differentials (there being $3g-3$ such pairs), then any Riemann surface with periods π'_{ij}that agreed with π_{ij} at the associated indices is conformally equivalent to W. In other words, in some cases one can choose $3g-3$ elements of the period matrix as (local) moduli, or, put differently, Alice need only send these $3g-3$ periods to Bob.

Communication complexity depends on the distribution of the messages to be sent, so one is led to consider the distribution of the periods. When the genus is small, the periods have no statistical properties to speak of, but when the genus is large there is enough "data" to look for statistical properties. In practice this means to take the set of all periods, create a corresponding set of real numbers (by, *e.g.*, taking the modulus-squared, the argument, or imaginary part), and sorting that set. One thus obtains a distribution (in the sense of statistics). One can view this as a generalization of the approach of Buser and Sarnak [BS], who showed that the smallest period of a Riemann surface is smaller than expected.

2. General Conjectures

Numerical experiments, described below, lead to three conjectures. The first two are specific to the case of hyperelliptic surfaces.

If the arguments of the periods of a compact hyperelliptic Riemann surface were distributed uniformly over the unit circle, the expected value of the distance between arguments would be $R_g = \pi/g(g+1)$.

Conjecture 1. *The periods of a hyperelliptic Riemann surface are* band-limited*, that is, there exists an argument α and a radius $\varepsilon < R_g$ such that no arguments fall outside of the intervals $(\alpha - \varepsilon, \alpha + \varepsilon)$ and $(\alpha + \pi - \varepsilon, \alpha + \pi + \varepsilon)$. In other words, there is a large interval containing no argument of periods.*

Conjecture 2. *The distribution of the magnitudes of the periods of a hyperelliptic Riemann surface are characterized by large gaps.*

See below for more specifics about the nature of the gaps.

The third conjecture is more general.

Conjecture 3. *The distribution of the squared modulus of the periods of a non-hyperelliptic Riemann Surface is* Zipfian*, that is, there is a power $p < 1$ such that the n^{th} modulus squared grows like n^{-p}.*

3. Remarks on Truncated Arithmetic

While Alice can send Bob an exact representation of an irrational or transcendental period matrix entry (*e.g.*, $\frac{1}{2}+i\frac{\sqrt{5}}{2}$), it is important to be aware of the effect of truncation to something like $(.5+1.12i)$. Since the various loci of compact Riemann Surfaces with interesting properties are small, it seems likely that an approximation to the period matrix of a compact Riemann Surface with some interesting property will fail to reveal the property. For example, if a Riemann Surface has a non-trivial conformal automorphism, period matrices of Riemann Surfaces without non-trivial automorphisms are arbitrarily close.

This section presents some reassuring results in this regard. Consider *Bring's curve*, which is the unique compact Riemann surface of genus four with the full symmetric group S_5 of automorphisms. Its period matrix was determined by Riera and Rodriguez [12]. Their matrix depends on a parameter, that is, there is a 1−parameter family of such matrices in $\mathcal{H}_4$, one of which is actually the period matrix for Bring's curve. This value is transcendental, so any numerical investigation of the matrix will necessarily involve an approximation.

Using the techniques of Accola [1], it is possible to determine a large number of vanishings of Riemann's theta function at quarter periods of the Jacobian of Bring's curve from the various involutions in the automorphism group. Remarkably, an exhaustive search for vanishings of θ at the quarter periods of the principally polarized abelian variety constructed from an *approximation* to the period matrix of Riera and Rodríguez found the vanishings predicted by Accola'a method. In other words, while one must be cautious about using approximations, it may still be possible to determine interesting properties of a compact Riemann surface from an approximation to its period matrix

(The data files from this investigation are too large to appear in print; the author will gladly make them available on request.)

4. Notation

All Riemann surfaces to be considered are compact and and of genus $g > 1$. Choose a *symplectic homology basis*, that is, a basis $\{A_1, \ldots, A_g,\ B_1, \ldots, B_g\}$ for the singular homology group $H_1(W, \mathbb{Z})$. Let $A \cdot B$ denote the intersection product of the cycles A and B. In a symplectic basis, these are, by definition, as follows: for all i and j, $A_i \cdot A_j = B_i \cdot B_j = 0$, and $A_i \cdot B_j = \delta_{ij}$.

One can also choose a normalized basis ω_i for $H^{(1,0)}(W)$, the vector space of holomorphic 1-forms; normalization means that $\int_{A_i} \omega_j = \delta_{ij}$. The matrix

$$\Omega = \left[\int_{B_i} \omega_j\right]$$

is called the *period matrix*, and the columns of $[I, \Omega]$, where I is the $g \times g$ identity, define a lattice L_Ω in $\mathbb{C}^g$. The complex torus

$$\mathrm{Jac}W = \mathbb{C}^g / L_\Omega$$

is the *Jacobian* of W. Torelli's Theorem asserts that either Ω or, equivalently, $\mathrm{Jac}W$ completely determines the conformal type of a Riemann surface W.

Now, let $\mathcal{H}_g$ denote the *Siegel upper half space* of symmetric $g \times g$ complex matrices with positive-definite imaginary part; every period matrix lies in the Siegel upper half space. The symplectic group $SP(2g, \mathbb{Z})$ acts on $\mathcal{H}_g$, and the quotient $\mathcal{A}_g = \mathcal{H}_g / SP(2g, \mathbb{Z})$ is a Hausdorff analytic space, providing a moduli space for PPAVs. The geometric meaning of this action is change-of-basis in the homology and cohomology groups.

Let $z \in \mathbb{C}^g$ and $\Omega \in \mathcal{H}_g$, and define *Riemann's theta function* by

$$\theta(z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp \pi i({}^t n \Omega n + 2{}^t n z);$$

here, ${}^t A$ denotes the transpose of the matrix A. The series converges absolutely and uniformly on compact subsets when $\Omega \in \mathcal{H}_g$.

It is important to keep in mind that most of the results presented here are *experimental*.

5. Hyperelliptic Period Matrices

Of particular interest are the periods of hyperelliptic Riemann Surfaces. In the communications scenario, the question becomes “If Alice sends Bob a period matrix, can an eavesdropper Eve determine whether it comes from a hyperelliptic surface?” In principle, Eve could apply the by–now classical theorems of Farkas [8] showing that certain vanishings of Riemann's Theta function at half-periods determine whether a surface is hyperelliptic. However, this calculation is daunting when the genus g is large, since there are 2^{2g} half periods to check.

Rauch's Theorem, mentioned in the previous section, does not apply in the hyperelliptic case.

Numerical experiments (using Maple) suggest that Eve might have an easier way to reject the hypothesis that the period matrix comes from a hyperelliptic curve. Note that if the arguments of the periods were distributed uniformly about the circle, the expected distance between an argument and its nearest neighbor would be $R_g = 4\pi/g(g+1)$. Deviations from uniformity indicate some special property of the associated Riemann Surface.

These conjectures only make sense when the genus is large.s

Previous results about the periods, such as the results of Bujalence, Costa, Gamboa, and Riera [3] on the periods of Accola–MacLachlan and Kulkarni Surfaces, depend on special properties of the surface. They did not examine the distribution of the arguments of the periods. These results apply to a specific family of hyperelliptic surfaces as well.

The rest of this paper will present some of the numerical evidence in support of the conjecture as well as some arguments supporting the idea that the distribution of hyperelliptic periods is somewhat intrinsic.

5.1. **Numerical Experiments.** Maple includes a package `algcurves` for computing period matrices, described in [De]. There is an intrinsic limitation that the coefficients must be Gaussian rationals, and there are extrinsic limitations in computing power. These experiments used Version 14 of Maple running on a 12 core 3.47GHz server, as well as on a variety of smaller machines.

5.2. **Procedures.** To generate a "random" hyperelliptic curve, choose a degree d, number of terms r, and degrees $d_1, \ldots, d_{r-1}$. The coefficient c_j of x^{d_j} is formed by choosing four "random" integers a_{j_1}, b_{j_1}, a_{j_2}, b_{j_2} and setting

$$c_j = \frac{a_{j_1}}{b_{j_1}} + i\frac{a_{j_2}}{b_{j_2}}.$$

Maple then computes the period matrix of the curve $y^2 - \prod(x - \hat{c_j})$, where $\hat{c_j}$ denotes Maple's internal representation of c_j (the `algcurves` routine uses decimal representations of the coefficients of the curve).

Once the period matrix is computed, the complex argument and complex modulus functions are mapped onto the matrix. The moduli and arguments are extracted as a vector, whose entries are sorted in decreasing order of argument, and finally plotted.

The computational limitation appears to be on the size of d and on the number of terms r. Maple is unable to handle curves when either is large. For many of the cases here the clock time for generating the period matrix was approximately 1 minute, but for some the computation did not terminate in a reasonable amount of time.

5.3. **Results.** Figure 1 shows a scatterplot of the periods of a "random" hyperellliptic of genus 39. The horizontal axis is the argument; the vertical axis is the magnitude.

Notice that the two clusters differ by about π, *i.e.*, the periods are clustered around a line in $\mathbb{C}$. This occurred for all of the hyperelliptic curves tested.

To show that this pattern is not general, consider in Figure 2 the scatterplot of a "random" trigonal curve of similar genus.

While there is evidence of clustering, there are no large gaps between the arguments as there were in the hyperelliptic case.

For further contrast, Figure 3 is a scatterplot of the periods of the Fermat curve whose projective equation is $x^{10}+y^{10}+z^{10} = 0$. While there seems to be a preferred argument, the remaining arguments are well distributed around the circle.

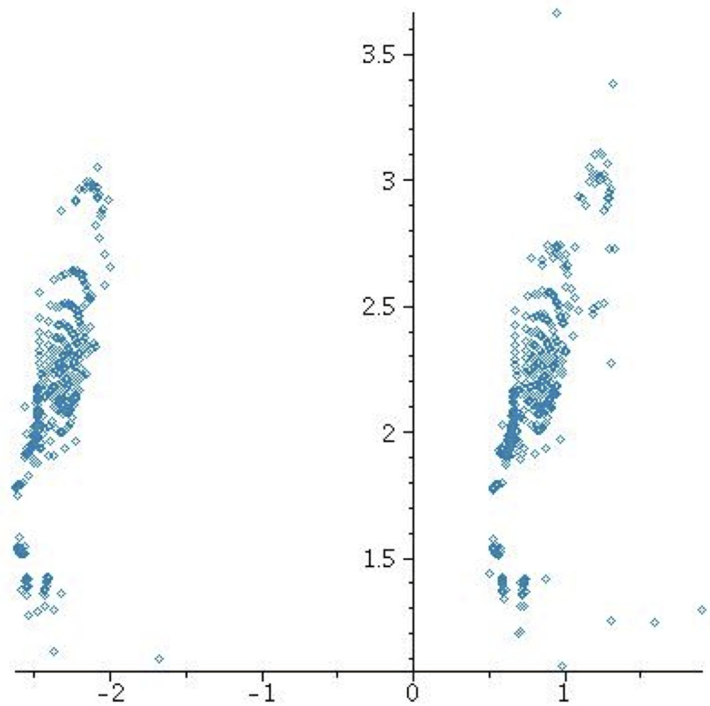

Figure 1. Periods of a Hyperelliptic Curve, Genus 39.

As further evidence that the distributions of periods of hyperelliptic curves have distinctive characteristics, consider the plot below of the *magnitudes* of the periods of a hyperelliptic curve $y^2 = f(x)$ where the degree of f is 96.

Notice that there are large "gaps" in the distribution of the magnitudes, which would be larger for the distribution of the squared magnitudes. All hyperelliptic curves investigated had such gaps, while no non-hyperelliptic curves investigated had comparable gaps. This is the evidence for 2.

5.4. **Analytical Evidence.** The examples shown (and many similar examples) motivate the conjecture above. Further evidence comes from the following Proposition, which indicates that the distribution of the periods of hyperelliptic curves maintains its basic shape as the curve varies in moduli space; in other words, the distributions seen in the examples hold, at least qualitatively, for "nearby" curves.

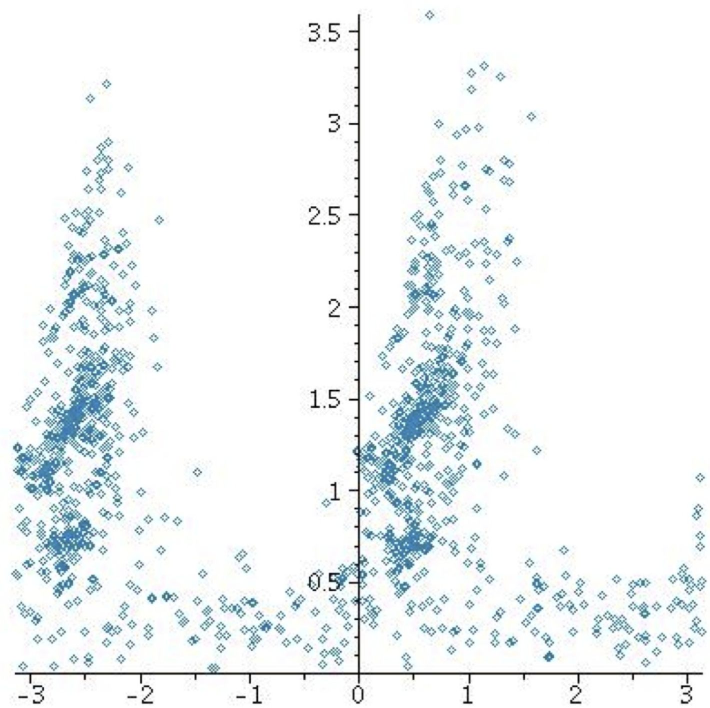

FIGURE 2. Periods of a Trigonal Curve, Genus 39.

Figure 4 shows the image of a canonical homology basis under the projection $(x, y) \mapsto x$ for a hyperelliptic curve X of genus 2 given by an equation $y^2 = f(x)$, where f has degree 6. The the branch cuts between the ramification points are black. The red (rectilinear) cycles represent the a–cycles, and the blue (curved) cycles represent the b–cycles. While some of the a–cycles appear to meet b–cycles twice, the second intersection does not occur on X itself. The images of the b–cycles cross the image of two branch cuts; this is necessary in order for the cycle to "jump" properly between sheets of X. Each cycle a_i encloses i branch cuts in such a way that the intersection products are correct.

The forms $\frac{x^a dx}{\sqrt{f}}$ $(0 \leq a \leq g-1)$ form a basis for $H^{(1,0)}(X)$. Normalize these to a basis $\{\omega_a\}$ such that $\int_{a_j} \omega_i = \delta_{ij}$.

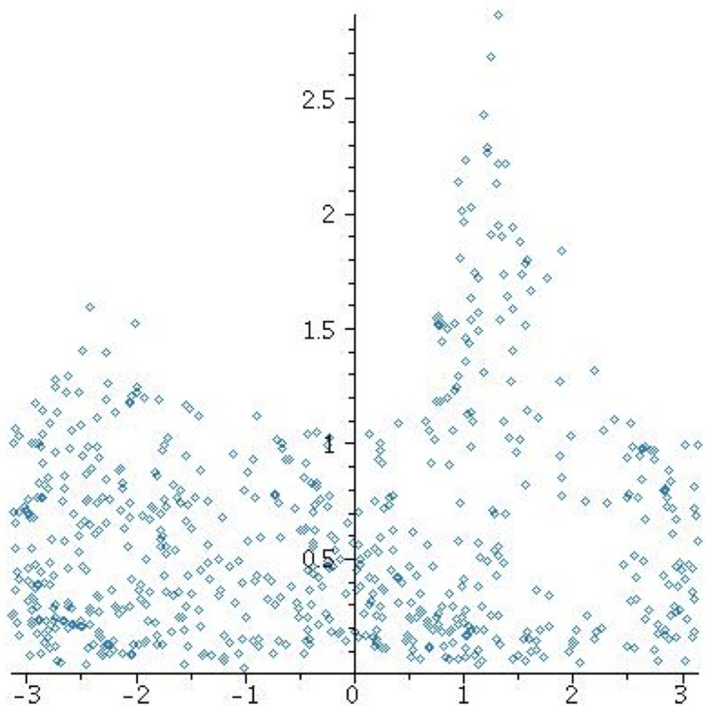

FIGURE 3. Periods of Fermat Curve of Degree 10

Lemma 1. *There is a canonical homology basis such that one of the branch points is not in the interior of any of the images of the a– or b–cycles in the x–plane. Call this the* free ramification point.

Proof. Examining Figure 4 shows that one of the branch points is not enclosed by any of the a– or b–cycles. This is because the number of branch cuts is one more than the genus, so one branch cut is not enclosed by any b-cycle, and since the image of each a–cycle encloses two points, two of the $2g+2$ branch points are not enclosed by such a cycle. □

The point labelled s in Figure 4 is a free ramification point. Clearly the choice of free ramification point depends on the choice of canonical homology basis.

Rewrite the equation for X as

$$y^2 = (x-s)f_\infty(x);$$

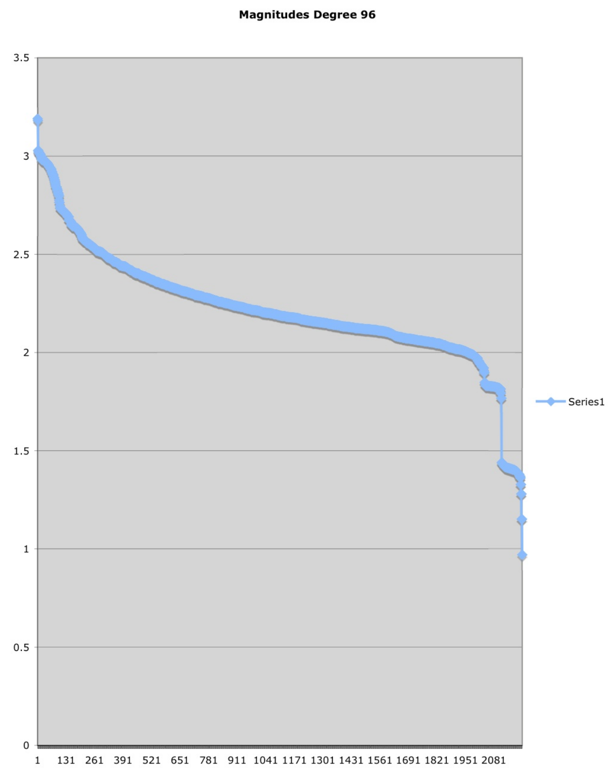

FIGURE 4. Magnitudes Periods of Hyperelliptic Curve of Degree 96

where f_∞ has degree $2g-1$. The curve $y^2 = f_\infty(x)$ has genus g but is ramified at infinity. Let X_s denote the curve $y^2 = (x-s)f_\infty(x)$, and X_∞ denote the curve $y^2 = f_\infty(x)$.

Proposition 1. *When the free ramification point on a hyperelliptic curve is large the period matrix is approximately that of X_∞.*

Proof. Let s denote the free ramification point. The key to the proof is to notice that the canonical homology basis stays the same as s becomes large, although the basis for $H^{(1,0)}(X)$ depends on s. For convenience, let $\sigma = \sqrt{x-s}$, so the equation for X_s is $y^2 = \sigma^2 f_\infty(x)$.

When s is large, σ is approximately constant, so the unnormalized period matrix is approximately $\frac{1}{\sigma}$ times the period matrix for X_∞. Tracing through the normalization computation leads immediately to the result. $\square$

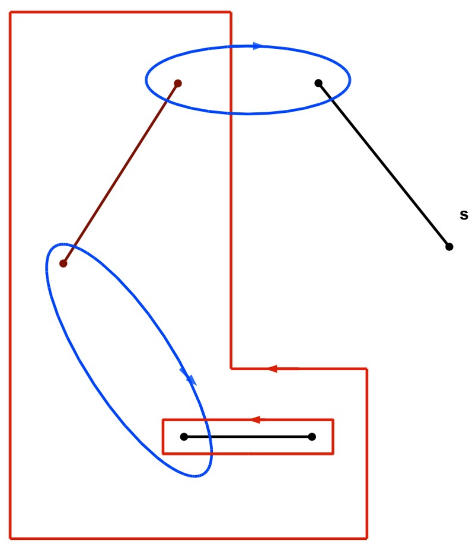

Figure 5. Canonical Homology Basis.

6. The General Case

Using numerical procedures similar to those described above, one can examine Maple-generated period matrices for arbitrary (within Maple's limitations) plane curves. These results deal with the modulus (absolute value) of the periods, rather than the arguments. This list was sorted in descending order.

Notice that it is possible to construct elements of $\mathcal{H}_g$ with any distribution by choice of the entries. In particular, the distribution may be concave down, concave up, have apparent discontinuities, or even be linear. But compare these possibilities with a typical result, the periods of the Fermat curve of degree 11, which appears below.

All of the period matrices of non-hyperelliptic surfaces examined have had a similar shape to the distribution, that is, generally concave up (which is consistent with a Zipfian distribution), hence conjecture 3.

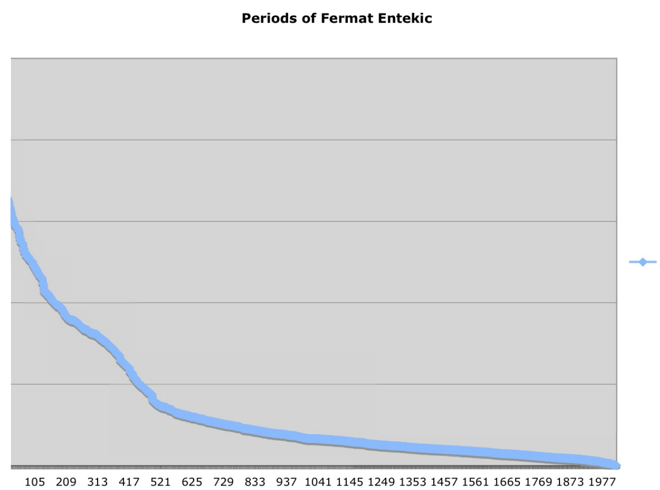

FIGURE 6. Periods of the Fermat Curve of degree 11

From the perspective of signal processing, this conjecture would imply that the period matrix of a Riemann surface has a fairly small number of *large* coefficients, while most are *small* – in effect, noise.

7. Conclusions

The numerical experiments outlined here demonstrate that the information-theoretic perspective leads to interesting new questions about the period matrices of compact Riemann surfaces.

In addition to proving or refuting the conjectures presented here, further work should explore the application of ideas from *Compressed Sensing* [6] [5], a technique from signal processing which can often find exact reconstruction of sparse signals from small samples. Sparsity occurs in natural signals such as image. A period matrix, whose size is $O(g^2)$ but which depends on only $O(g)$ parameters, is also a sparse signal. Rauch's 1954 result mentioned above suggests that this approach could be fruitful.

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TRACES OF ENTIRE FUNCTIONS ON ALGEBRAIC SUBVARIETIES

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Abstract. We show that, to some extent, the behavior of an entire function of several complex variables is reflected by its behavior on algebraic subvarieties.

1. Introduction

Let f be a holomorphic function on $\mathbb{C}^n$; given a family of algebraic subvarieties of $\mathbb{C}^n$, is it possible to determine the order of growth of f from the order of growth of the restriction of f along a general member of the family? For linear subspaces of $\mathbb{C}^n$, this problem was intensively studied by Pierre Lelong (see for example [7], [8]). The order of growth for entire functions can be defined in the following way.

Definition 1. *Let f be an entire function $f(z)$ on $\mathbb{C}^n$. We say that f is of finite order if if there exists a positive number t such that $|f(z)| = O(\exp(|z|^t)$. If f is of finite order, then the order of growth (or simply the order) of f is defined as*

$$\rho = \inf\{t : |f(z)| = O(\exp(|z|^t)\}.$$

If f is not of finite order, we say that f is of infinite order.

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The order of growth can be defined also for meromorphic functions, using Nevanlinna theory. However, since in this paper we consider only entire functions, we shall not state the definition, and refer instead to [5]. An interesting result concerning the order of meromorphic functions was proved by Gauthier and Hengartner.

Theorem 2. *Let S^{2n-1} denote the $(2n-1)$-sphere of radius 1 in $\mathbb{C}^n$ and, for $\theta \in S^{2n-1}$, let L_θ denote the line passing through θ and the origin. Let f be a meromorphic function on $\mathbb{C}^n$. Then, $f|L_\theta$ is of the same order as f, for almost all $\theta \in S^{2n-1}$. Moreover, if there exists a set of positive measure $S \subset S^{2n-1}$ such that $f|L_\theta$ is rational for all $\theta \in S$, then f is rational.*

In the present paper we investigate to what extent the order of growth of an entire function is reflected by its behavior along members of a family of algebraic subvarieties of $\mathbb{C}^n$. Since we cannot provide a general theory, we consider only some classical examples of families (Section 2 and Section 3). In Section 4 we show, by contrast, that the behavior of an entire function f along real curves does not provide any information about the order of growth of f.

2. Traces along subvarieties passing through the origin

2.1. **Grassmannians.** Let G_k denote the Grassmannian manifold of k-dimensional planes passing through the origin in $\mathbb{C}^n$. If E is a k-plane in $\mathbb{C}^n$, we denote by $[E]$ the corresponding point in G_k. Gauthier and Hengartner [5] showed that the order of a meromorphic function on $\mathbb{C}^n$ is determined by the order of the restriction of f along lines (or k-planes more generally) through the origin. In fact the following result, from which Theorem 2 follows as a particular case, is proved in [5].

Theorem 3. *Let f be a meromorphic function on $\mathbb{C}^n$. Then $f|X$ is of the same order as f for all $[X] \in G_k$ outside a set of measure zero.*

We wish to consider such matters with regards to topological rather than measure-theoretic genericity. To this end we introduce the following lemmas.

Lemma 4. *Let $p : Y \to X$ be a surjective map of irreducible quasi-projective varieties. If S is Zariski dense in X, then $p^{-1}(S)$ is Zariski dense in Y. If S' is Zariski dense in Y, then $p(S')$ is Zariski dense in X*

Proof. In this proof, all topological notions and notations are with respect to the Zariski topologies. Let $Z = \overline{p^{-1}(S)}$. Then, by Chevalley's theorem, its image $p(Z)$ is locally closed, that is, open in its closure. Since $p(Z)$ contains S, it follows that $p(Z)$ is open in X. Therefore the dimension on Z equals the sum of the dimension of X and the dimension of a general fiber of p, which is the same as the dimension of Y. Since Y irreducible, we must have $Y = Z$, which proves the first part.

As for the second part, if $p(S')$ is contained in a subvariety V, then $p^{-1}(V)$ is a subvariety of Y containing S'. Thus, $p^{-1}(V) = Y$ and $V = X$.

□

Lemma 5. *Let $S \subset G_k$ be a Zariski dense subset. Then $\widetilde{S} = \cup_{[E]\in S} E$ is Zariski dense in $\mathbb{C}^n$.*

Proof. Let $\mathfrak{S} \subset \mathbb{C}^n \times G_k$ be the set $\{(x, [E]) : x \in E\}$. As a product of quasi-projective varieties, it is also quasi-projective. Denote the projections p_1 (resp. p_2)

on the factors $\mathbb{C}^n$ (resp. G_k). Lemma 4 implies that $S' = p_2^{-1}(S)$ is Zariski dense in $\mathfrak{S}$. We note that $\tilde{S} = p_1(S')$, which is dense by the second part of Lemma 4.

□

Theorem 6. *Let f be an entire function on $\mathbb{C}^n$, such that $f \mid E$ is a polynomial of degree at most d, for all $[E] \in S$, where $S \subset G_k$ is Zariski dense. Then f is a polynomial of degree at most d.*

Proof. Let

$$f = \sum_{k=0}^{\infty} f_k$$

be the homogeneous expansion of f. For $k > d$, the polynomial f_k is zero on $\widetilde{S}$, which is Zariski dense. Therefore $f_k = 0$, which completes the proof.

□

Remark 7. *In Lemma 5 of [5] the authors show a similar result without assuming a uniform bound on the polynomials $f \mid E$. However, the set S in their case is non-polar. A Zariski dense set can be polar. For example, every countable subset of $\mathbb{C}$ having an accumulation point is both Zariski dense and polar. The following example shows that the conclusion of Theorem 6 fails, if we merely drop the restriction on the degrees of the polynomials.*

Example 8. *Let λ_j be a sequence of distinct non-zero complex numbers. For each j, set*

$$g_k(z, w) = \eta_k \prod_{j=1}^{k-1} (w - \lambda_j z),$$

where η_k is chosen so that $|g_k| < 2^{-k}$ on the ball centered at the origin and of radius k and moreover, $a_k = \eta_k (-1)^k \lambda_1 \lambda_2 \cdots \lambda_k > 0$. Then,

$$f(z, w) = \sum_{k=1}^{\infty} g_k(z, w)$$

is an entire function which is not a polynomial, since

$$f(z, 0) = \sum_{k=1}^{\infty} a_k z^k, \qquad a_k > 0,\ k = 1, 2, \cdots .$$

Now, for fixed j we write

$$f(z, w) = \sum_{k=1}^{j-1} p_k(z, w) + \sum_{k=j}^{\infty} p_k(z, w),$$

for all (z, w). In particular, on the line $w = \lambda_j z$, we have

$$f(z, w) = \sum_{k=1}^{j-1} p_k(z, \lambda_j z) + \sum_{k=j}^{\infty} p_k(z, \lambda_j z),$$

where the first term is a polynomial in z and the second term is zero, since $p_k(z, \lambda_j z) = 0$, for $j \le k$. Thus, on the line $w = \lambda_j z$, the function $f(z, w)$ is a polynomial in z.

2.2. **Weighted projective spaces.** Let $(a_1, \dots, a_n)$ be a vector, whose components a_k are positive integers, for $k = 1, \cdots, n$. Fix $z = (z_1, \dots z_n) \in \mathbb{C}^n$ a non-zero vector, and consider the curve C parametrized by $\gamma(t) = (t^{a_1} z_1, \dots t^{a_n} z_n), t \in \mathbb{C}$. The collection of all such curves, for z ranging over all non-zero vectors , is parametrized by the weighted projective space $\mathbb{P}_{(a_1,\dots a_n)}$, which is defined as follows.

Definition 9. *Given a vector $(a_1, \dots a_n)$ of positive integers, consider the action of the multiplicative group $\mathbb{C}^*$ on $\mathbb{C}^n \setminus \{0\}$ defined by*

$$t(z_1, \dots z_n) = (t^{a_1} z_1, \dots t^{a_n} z_n).$$

The quotient space $(\mathbb{C}^n \setminus \{0\}) / \mathbb{C}^* = \mathbb{P}_{(a_1,\dots a_n)}$ is called weighted projective space with weight vector $(a_1, \dots a_n)$.

The standard projective space is therefore $\mathbb{P}_{(1,\dots 1)} = \mathbb{P}^{n-1}$. We refer to [2] for the proof of the following well known facts: the weighted projective space $\mathbb{P}_{(a_1,\dots a_n)}$ is a projective variety of dimension $(n-1)$, and $\mathbb{P}_{(a_1,\dots a_n)}$ is smooth if and only if $\mathbb{P}_{(a_1,\dots a_n)}$ is isomorphic to the projective space of dimension $(n-1)$.

Example 10. *The map*

$$\begin{array}{rccl} f: & \mathbb{P}_{1,2} & \longrightarrow & \mathbb{P}^1 \\ & [w_0 : w_1] & \longmapsto & [w_0^2 : w_1] = [z_0 : z_1] \end{array}$$

is an isomorphism since $[\sqrt{z_0} : z_1] = [-\sqrt{z_0} : z_1]$ in $\mathbb{P}_{1,2}$.

Definition 11. *Given a vector $a = (a_1, \dots a_n)$, the a-degree of a polynomial is defined by letting the a-degree of the monomial w_k be a_k, and then extending the definition according to the definition of degree function. A polynomial is said to be weighted homogeneous of a-degree d, if all its monomial terms have a-degree d.*

The common zero locus of a finite collection of weighted homogeneous polynomials defines an algebraic subvariety of $\mathbb{P}_{(a_1,\dots a_n)}$. Let $z = (z_1, \dots z_n)$ be a non-zero vector in $\mathbb{C}^n$: we denote by $C = C_z$ the image of the curve $\gamma(t) = \gamma_z(t) = (t^{a_1} z_1, \dots t^{a_n} z_n)$ and by $[C] \in \mathbb{P}_{(a_1,\dots a_n)}$ the corresponding point on the weighted projective space. We say that such a curve C (and by abuse also $[C]$) is an a-curve.

Definition 12. *If $f(z)$ is an entire function on $\mathbb{C}^n$, the restriction to $C = C_z$ is an entire function of one variable t, and we say that $f|C$ is a polynomial of degree d if f is a polynomial of degree d in the variable t. We say that $f|C$ is of finite order if $f(tz^{a_1}, \cdots tz^{a_n})$ is of finite order as a function of t.*

Note that, if f is a polynomial of a-degree d, then it is a polynomial of degree d along every a-curve $C \in \mathbb{P}_{(a_1,\dots a_n)}$. The following analogue of Theorem 6 shows conversely that, if f is a polynomial of degree d along every a-curve $C \in \mathbb{P}_{(a_1,\dots a_n)}$, then f is a polynomial of a-degree d.

Theorem 13. *Let f be an entire function in $\mathbb{C}^n$. Assume that there exists a Zariski dense subset $S \subset \mathbb{P}_{(a_1,\dots a_n)}$, such that $f|C$ is a polynomial of degree at most d, for all a-curves $[C] \in S$. Then f is a polynomial of a-degree at most d and hence of degree at most $\sum(d/a_k)$.*

Proof. Since every monomial $z^m = z_1^{m_1} \cdots z_n^{m_n}$ is weighted homogeneous of a-degree $a \cdot m$, we may write f uniquely as a weighted homogeneous expansion $f =$

$\sum f_k$, where f_k is weighted homogeneous of a-degree k. Indeed, if $\sum p_m(z)$ is the homogeneous expansion of f, then

$$f(z) = \sum p_m(z) = \sum_{k=0}^{\infty} \left(\sum_{a \cdot m = k} p_m(z) \right) = \sum_{k=0}^{\infty} f_k(z).$$

Then, for all $k > d$, f_k is identically equal to zero on a Zariski dense subset $\tilde{S} = \bigcup_{[C] \in S} C$; hence it must be identically equal to 0. $\square$

In Nevanlinna theory, polynomial functions all have the same order, and that makes Theorem 3 uninteresting for polynomials. We shall now generalize Theorem 3 in non-trivial cases. In order to do this, we introduce some simple geometry that allows us to apply the results from [5]. Let $[w_1 : \cdots : w_n]$ be the homogeneous coordinates in $\mathbb{P}^{n-1}$ and $[z_1 : \cdots : z_n]$ be the homogeneous coordinates in $\mathbb{P}_{(a_1, \ldots a_n)}$. Let Γ_k be the group of a_k-th roots of 1, and let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$. Consider the action of Γ on $\mathbb{P}^{n-1}$ given by

$$(\epsilon_1, \ldots \epsilon_n)[w_1 : \cdots : w_n] = [\epsilon_1 w_1 : \cdots : \epsilon_n w_n].$$

Lemma 14. *$\mathbb{P}_{(a_1, \ldots a_n)}$ is isomorphic to the quotient $\mathbb{P}^{n-1}/\Gamma$ with respect to the action defined as above.*

Proof. Leaving the proof to the reader, we merely give the quotient map $\gamma' : \mathbb{P}^{n-1} \to \mathbb{P}_{(a_1, \ldots a_n)}$, since it will be used in the sequel:

$$\gamma'([w_1 : \cdots : w_n]) = [w_1^{a_1} : \cdots : w_n^{a_n}]. \tag{1}$$

$\square$

Let $\gamma : \mathbb{C}^n \backslash \{0\} \to \mathbb{C}^n \backslash \{0\}$ be the equivariant mapping defined by $\gamma(w_1, \ldots w_n) = (w_1^{a_1}, \ldots w_n^{a_n})$, and denote by $\pi : \mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ and $\pi' : \mathbb{C}^n - \{0\} \to \mathbb{P}_{(a_1, \ldots a_n)}$ the quotient maps. Then the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{C}^n \setminus \{0\} & \xrightarrow{\pi} & \mathbb{P}^{n-1} \\ \downarrow \gamma & & \downarrow \gamma' \\ \mathbb{C}^n \setminus \{0\} & \xrightarrow{\pi'} & \mathbb{P}_{(a_1, \ldots a_n)} \end{array}$$

It follows that a curve $(t^{a_1} z_1, \ldots t^{a_n} z_n)$ naturally lifts to a union of $a_1 a_2 \ldots a_n$ lines $(t z_1^{1/a_1}, \ldots, t z_n^{1/a_n})$ (counting multiplicity).

Lemma 15. *Let f be an entire function of finite order. Then for any $[C] \in \mathbb{P}_{a_1, \cdots a_n}$, the restriction $f|C$ is of finite order.*

Proof. Let $R > 1$, A and ρ positive numbers such that if $|(z_1, \ldots z_n)| > R$, then $|f(z_1, \ldots z_n)| < A \exp |(z_1, \ldots, z_n)|^\rho$. Clearly if $w = (w_1, \ldots, w_n)$ is sufficiently large, then $|(w_1^{a_1}, \ldots, w_n^{a_n})|$ is larger than R. Let $a = \max\{a_1, \ldots, a_n\}$. We may assume $\|w\| > 1$; so

$$|(f \circ \gamma)(w_1, \ldots, w_n)| = |f(w_1^{a_1}, \ldots, w_n^{a_n})| < A \exp(|(w_1^{a_1}, \ldots, w_n^{a_n})|^\rho) \leq$$
$$A \exp\left((\sum |w_k|^{2a_k})^{\rho/2} \right) < A \exp(n^{\rho/2} \|w\|^{a\rho}) < A' \exp(\|w\|^{a\rho+1}),$$

for large w which concludes the proof.

$\square$

The previous two lemmas show that the order of f along a twisted curve C, which is image of $(t^{a_1}z_1, \ldots t^{a_n}z_n)$, is determined by the order of $f \circ \gamma$ along the inverse image of this curve under γ, which consists of a finite union of lines. Moreover, the behavior of $f \circ \gamma$ is the same on each of those lines: if L is any line that projects to C (that is $\gamma(L) = C$), then $(f \circ \gamma)|L$ behaves like $f \mid C$.

We shall define a measure on the weighted projective spaces. Let μ be the standard measure on $\mathbb{P}^{n-1}$, induced by the Fubini-Study metric. If U is a measurable set, we define the average measure of U:

$$\hat{\mu}(U) = \sum_{g \in \Gamma} g(U).$$

Define a subset $O \subset \mathbb{P}_{(a_1, \ldots a_n)}$ to be measurable if and only if the inverse image $\gamma'^{-1}(O)$ is measurable in $\mathbb{P}^{n-1}$. The collection of such measurable subsets forms a sigma-algebra and the average measure $\hat{\mu}$ descends to a measure on $\mathbb{P}_{(a_1, \ldots a_n)}$. Since Γ is a finite group, a subset $O \subset \mathbb{P}_{(a_1, \ldots a_n)}$ has measure zero if and only if $\gamma'^{-1}(O) \subset \mathbb{P}^{n-1}$ has measure zero.

Theorem 16. *Let f be a a holomorphic function on $\mathbb{C}^n$ of order ρ. Then for all $[C] \in \mathbb{P}_{a_1, \cdots a_n}$ outside a set of measure zero, $f|C$ is of order ρ.*

Proof. Consider the pull-back $\tilde{f} = f \circ \gamma$. By the previous Lemma, $\tilde{f}$ is of finite order $\tilde{\rho}$. Hence, by Theorem 3 $\tilde{f}|L$ is of finite order $\tilde{\rho}$ for almost all lines. Moreover, $\tilde{f}|L$ is of order $\tilde{\rho}$ if and only if $\tilde{f}|gL$ is of order ρ for all $g \in \Gamma$. Fix a non zero vector $(z_1, \ldots z_n)$ and let $(w_1, \ldots w_n)$ be such that $\gamma(w_1, \ldots w_n) = (z_1, \ldots z_n)$. Let C be parametrized by $\gamma(t) = (t^{a_1}z_1, \ldots t^{a_n}z_n)$. Since $\tilde{f}(tw_1, \ldots tw_n) = f \circ \gamma(tw_1, \ldots tw_n) = f(t^{a_1}z_1, \ldots t^{a_n}z_n)$, it follows that f has order $\tilde{\rho}$ along the curve C. Since $\tilde{f}$ has order $\tilde{\rho}$ outside a set of lines of measure zero, f has the same order ρ outside a set of curves of measure zero. □

3. Traces along parallel translations of subvarieties

Let $V_0 \subset \mathbb{C}^n$ be a subvariety of $\mathbb{C}^n$ and $u \in \mathbb{C}^n$ be a fixed vector. In this section we study the traces of a holomorphic function along a family $\{V_c\}$, for $c \in \mathbb{C}$, where the varieties $V_c = V_0 + cu$ are obtained by translation in the direction u.. The space that parametrizes such a family is not compact.

3.1. **Parallel hyperplanes.** The analogue of Theorem 6 does not hold, as the function $f(x, y) = e^x - y$ shows. In the next subsection we shall show, however, that something can be said even in this case. The following example is an analogue of Example 8.

Example 17. *Given an increasing sequence $\{n_j : j = 1, 2, \cdots\}$ of positive integers, and given a sequence of distinct points $e_j \in \mathbb{C}$, there is an entire function $f(z, w)$ having the property that, for each j the function $f(e_j, \cdot)$ is a polynomial of degree n_j. Moreover, if $z \neq e_j$ for all j, then $f(z, \cdot)$ is not a polynomial.*

For each $k = 1, 2, \cdots$, we define a polynomial $g_k(z)$ as follows: set $g_1 = 1$ and for $k > 1$, set

$$g_k(z) = a_k \prod_{j=1}^{k-1} (z - e_j),$$

where $a_k \neq 0$ is chosen so small that

$$\max_{|z|\leq k} |g_k(z)| \leq \frac{1}{2^k k^{n_k}}.$$

The series

$$\sum_{k=1}^{\infty} g_k(z) w^{n_k}$$

converges uniformly on compacta. Indeed, writing

$$\sum_{k=1}^{\infty} g_k(z) w^{n_k} = \sum_{k=1}^{m} g_k(z) w^{n_k} + \sum_{k=m+1}^{\infty} g_k(z) w^{n_k},$$

and noting that, for $|z| \leq m, |w| \leq m$, and $k > m$, we have the estimate

$$|g_k(z) w^{n_k}| \leq \frac{1}{2^k k^{n_k}} k^{n_k} = \frac{1}{2^k},$$

we see that on $|z| \leq n, |w| \leq n$, the series is a polynomial plus a uniformly convergent series. Hence, the series converges uniformly on compacta and represents an entire function $f(z, w)$. Note that, for $z = e_m$,

$$f(e_m, w) = \sum_{k=1}^{m-1} g_k(e_k) w^{n_k} + a_m \prod_{j=1}^{m} (e_m - e_j) w^{n_m} + \sum_{k=m+1}^{\infty} g_k(e_m) w^{n_k},$$

is a polynomial of degree n_m (and no less), since each g_k is a polynomial of degree n_k and $g_k(e_m) = 0$, for $k > m$.

The second part of the statement follows immediately by looking at the construction of f. Indeed, if z is different from all e_j, then $g_k(z) \neq 0$ for all k. Thus,

$$f(z, w) = \sum_{k=1}^{\infty} g_k(z) w^k = \sum_{k=1}^{\infty} a_k w^k,$$

where $a_k \neq 0, k = 1, 2, \cdots$.

In the preceding example, we may choose the sequence $\{e_j\}$ to be dense so as to obtain a function $f(z, w)$ which is a polynomial in w for a dense set of z but for each n the set of z for which $f(z, \cdot)$ is a polynomial of degree n has no accumulation point.

3.2. **Tube domains.** Most studies on the order of growth of entire functions are concerned only with functions of finite order. Let $M_f(r) = \max_{|z \leq r|} |f(z)|$. Then the order of an entire function can be expressed as follows [7, page 9]

$$\rho(f) = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log r}.$$

If f is an entire function of a single complex variable, then the order of f can also be expressed [7, Theorem 1.9 a)] in terms of the MacLaurin coefficients $\{a_n\}$ of f:

$$\rho(f) = \limsup_{n \to \infty} \frac{n \log n}{\log(1/|a_n|)}.$$

Suppose $f(z, w)$ is holomorphic on a tube domain $D \times \mathbb{C}$, where D is a domain in $\mathbb{C}$. We may write

$$f(z, w) = \sum_{n=0}^{\infty} a_n(z) w^n, \quad z \in D,$$

where $a_n \in \mathcal{O}(D), n = 0, 1, \cdots$. For a compact subset $K \subset D$, we denote

$$M_n(K) = \max_{z \in K} |a_n(z)|.$$

Definition 18. *A holomorphic function $f(z,w)$ defined on a tube domain $D \times \mathbb{C}$ is said to be of order ρ_K in the w direction over K if*

$$\rho_K(f) = \limsup_{n \to \infty} \frac{n \log n}{\log(1/M_n(K))}.$$

A holomorphic function $f(z,w)$ in a tube domain $D \times \mathbb{C}$ is locally of finite (respectively infinite) order in the w direction if it is of finite (respectively infinite) order in the w direction over every relatively compact open subset of D.

From the Cauchy inequalities, it follows that $M_n(K) \to 0$, as $n \to \infty$. Thus, if $K_1 \subset K_2$, we have for large n that $0 \leq \log M_n(K_2)/\log M_n(K_1) \leq 1$. Consequently, if $K_1 \subset K_2$, then $\rho_{K_1}(f) \leq \rho_{K_2}(f)$. The following Harnack type principle is due to Pierre Lelong [8, Th. 33].

Lemma 19. *If f is holomorphic in a tube domain $D \times \mathbb{C}$ and D_1, D_2 are relatively compact open subsets of D, then for sufficiently large n, the ratio*

$$\frac{\log M_n(\overline{D}_1)}{\log M_n(\overline{D}_2)}$$

is bounded from below and from above by two positive numbers (we agree to disregard those n for which $M_n(\overline{D}_j) = 0$, since $M_n(\overline{D}_1) = 0$ if and only if $M_n(\overline{D}_2) = 0$).

Recall that a subset Y of a topological space S is said to be *residual* if $S \setminus Y$ is of first Baire category. If S is of second category, then we may think of a residual subset as topologically representing the 'majority' of points of S in the sense that, not only Y is 'large' but also $S \setminus Y$ is 'small'. As a consequence of the previous lemma we have the following.

Theorem 20. *If f is holomorphic in a 2-dimensional tube domain $D \times \mathbb{C}$, then either the order of f in the w direction is locally finite or it is locally infinite. In the first case, f_z is of course of finite order for every $z \in D$. In the second case, the order of f_z is infinite for most z, in the sense that the set of such z is residual in D.*

Proof. The first assertion in an immediate consequence of the previous lemma. The second assertion is trivial. To prove the third assertion, suppose that the order of f in the w direction is locally infinite. Let E the set of $z \in D$ for which $\rho(f_z) < +\infty$. Then

$$E = \cup_{j,k}\{z \in D : |f(z,w)| \leq k \exp(|w|^j)\} = \cup_{j,k} E_{j,k}.$$

Clearly, each $E_{j,k}$ is closed and it is nowhere dense since f is locally of infinite order in the w direction. This concludes the proof. □

Example 21. *Given an increasing sequence $\{n_j : j = 1, 2, \cdots\}$ of positive integers, and given a sequence of distinct points $e_j \in \mathbb{C}$, there is an entire function $F(z,w)$ having the property that, for each j the function $F(e_j, \cdot)$ is of order n_j (and no less).*

Proof. In Example 17, we construct an entire function $f(z,w)$, such that, for each e_j, the function $f(e_j, \cdot)$ is a polynomial of degree n_j (and no less). Put $F = e^f$. □

The following theorem is a consequence of Theorem 20.

Theorem 22. *Let f be an entire function. Then f is either of finite order, or it is of infinite order on a residual set of lines through the origin.*

Proof. Let $\sigma X :\to \mathbb{C}^2$ be the blow up of $\mathbb{C}^2$ at the origin. It is well known X is the total space of a line bundle $\pi : X \to \mathbb{P}^1$, and it follows that for all $u \in \mathbb{P}^1$ $X^0 = X - \pi^{-1}(u) \cong \mathbb{C}^2$. Therefore the the theorem follows if we apply Theorem 20 to $f \circ \sigma_{|X^0}$. □

3.3. **Traces along translations of submanifolds.** In this section we consider the case of families that are obtained by translating a given subvariety. The difficulty that arises is that members of the family can have non-empty intersection.

Lemma 23. *Let $p : X \to Y$ be a map of Stein manifolds, where $\dim X = m$ and $\dim Y = 1$. Assume further that, for each $y \in Y$, $p^{-1}(y)$ is a Stein manifold. Let f is a holomorphic function on Y. If there exists a subset S of Y with an accumulation point , such that $f|p^{-1}(y)$ is constant for all $y \in S$, then f is constant on each fiber of p.*

Proof. Given a point $y \in S$, there exists an open neighborhood O of y, and a system of local coordinates on O centered at y, say $(z, t_1, \dots t_{m-1})$, such that $p^{-1}(y) \cap O = \{z = 0\}$ (see [6]). We can construct the locally defined vector fields, for $k = 1, \dots m - n$:

$$v_k = \frac{\partial}{\partial t_k}$$

Hence $v_k(f)$ is identically zero on $p^{-1}(y) \cap O$, for all $y \in S$. Since $p^{-1}(y) \cap O$ is open in $p^{-1}(y)$, it follows that $v_k(f)(x) = 0$ for all $x \in p^{-1}(y)$ for all $y \in S$. The zero locus of $v_k(p)$ contains infinitely many irreducible components accumulating at $p^{-1}(y)$, which is possible only if $v_k(p)$ is identically zero. It follows that f is constant along each fiber over $p(O)$, and f descends to a function $\tilde{f}$ on $p(O)$. Consider now a maximal open set $U \subset Y$ over which $\tilde{f}$ extends as a holomorphic function. Suppose that $U \neq Y$: since $\tilde{f} \circ p = f$ on $p^{-1}(U)$, f itself would not be an entire function. Hence $U = Y$, and f descends to an entire function on Y. Therefore f descends to a function on Y.

□

Lemma 24. *Let f be an entire function on $\mathbb{C}^n$, and V_0 be a proper embedding of $\mathbb{C}^{n-1}$ into $\mathbb{C}^n$ containing the origin. For $z \in \mathbb{C}^n$, we write $z = (z_1, z')$, with $z_1 \in \mathbb{C}$ and $z' \in \mathbb{C}^{n-1}$. For a vector $(c, 0')$ let $V_c = V_0 + (c, 0')$. Suppose that there exists a sequence of vectors $(c_k, 0')$ converging to $(0, 0')$ such that f is bounded along the translate V_{c_k}, for all k. Then f is constant on each V_c*

Proof. First of all, by Liouville's theorem, f is constant on each V_{c_k}. Since V_0 is an analytic submanifold of $\mathbb{C}^n$, there exists an open polydisk D_0 containing $(0, 0')$ and a holomorphic function g on D_0 such that $V_0 \cap D_0$ coincides with the zero locus of g. Therefore, if we let $D_c = D_0 + (c, 0')$, $V_c \cap D_c$ is given by the zero locus of $g(z_1 - c, z')$. Consider the open set $O = \cup_c O_c \subset \mathbb{C}^{n+1}$, where $O_c = D_c \times \{c\}$. We may consider O as the image of the polydisc D_0 by the automorphism of $\mathbb{C}^{n+1}$ given by$(z, z', c) \mapsto (z + c, z', c)$. Hence, O is Stein. Consider the holomorphic function G defined on O by $G(z_1, z', c) = g(z_1 - c, z')$. Let $q : \mathbb{C}^{n+1} \to \mathbb{C}^n$ be the projection map $q(z_1, z', c) = (z_1, z')$, and p be the projection $p(z_1, z', c) = c$. Moreover, let

$\mathbb{V}$ be the n-dimensional Stein submanifold of O given by $G = 0$, and restrict the projections p and q to $\mathbb{V}$. The pull back $\tilde{f} = f \circ q$ is a holomorphic function on $\mathbb{V}$ with is constant on the fibers $p^{-1}(c_k) = (V_{c_k} \cap D_{c_k}) \times \{c_k\}$. Hence, by Lemma 23, $\tilde{f}$ is constant on each fiber of p. This implies that f is constant on $V_c \cap D_c$, for all $c \in \mathbb{C}$: by the identity principle, it must be constant on the entire submanifold V_c, for all c. $\square$

Remark 25. *If $V_c \cap V_{c'} \neq \emptyset$ for all (c, c') belonging to an open set of $\mathbb{C}^2$, then f must be a constant function.*

4. Traces along real curves

In the previous sections we established that in many interesting situations, the behavior of an entire function is dictated by the behavior of its restriction along complex subvarieties. In the following section we show that this is not true if we replace complex subvarieties by real subvarieties.

Definition 26. *For E a subset of $\mathbb{C}$, we denote by $A(E)$ the family of continuous functions on E which are holomorphic on the interior of E. We say that E is an approximation set if, for each function $g \in A(E)$ and each $\epsilon > 0$ on E, there exists an entire function f, such that $|f - g| < \epsilon$ on E.*

Approximation sets have been completely characterized by Norair U. Arakelian (see [3]). In fact, on approximation sets, we can do better than uniform approximation as the following lemma shows (see [3, p. 161]).

Lemma 27. *Let E be an approximation set in $\mathbb{C}$. Then, for each $g \in A(E)$ and each $\epsilon > 0$, there is an entire function f such that on E, not only $|f(z) - g(z)| < \epsilon$ but also $|f(z) - g(z)| \to 0$ as $z \to \infty$ on E.*

An asymptotic path in $\mathbb{C}^n$ is a continuous curve $\gamma : [0, +\infty) \to \mathbb{C}^n$, such that $\gamma(t) \to \infty$, as $t \to +\infty$. We assume that $\gamma(0) = 0$. An asymptotic path is said to be simple if γ is injective; it is said to be strictly monotonic if $|\gamma(t)|$ is strictly increasing.

Theorem 28. *(a) In $\mathbb{C}$, for every simple asymptotic path γ, there exist entire functions of arbitrarily fast growth which tend to zero along γ.*
(b) In $\mathbb{C}^2$, there exists a simple asymptotic path γ such that, if an entire function f tends to zero along γ, then $f \equiv 0$.

Proof. (a) Let γ be a simple asymptotic path in $\mathbb{C}$. We may construct another simple asymptotic path σ disjoint from γ. The union $E = \gamma \cup \sigma$ is an approximation set and so, by Lemma 27, if $\epsilon > 0$ and $\varphi : [0, +\infty)$ is an arbitrary continuous function, setting $g(z) = 0$ on γ and $g(z) = \varphi(|z|) + \epsilon$ on σ, we obtain an entire function f for which

$$f(z) \to 0, \quad z \in \gamma; \qquad |f(z)| > \varphi(|z|), \quad z \in \sigma.$$

This proves (a).

(b) Anatoliy Georgievich Vitushkin [10] showed the existence of a compact totally disconnected set K in $\mathbb{C}^2$, whose polynomial hull contains the bidisc. We may assume that K is contained in the bidisc. Given $r > 0$, by covering the sphere S_r of center 0 and radius r by finitely many bidiscs, we see that there exists a compact totally disconnected set K_r whose polynomial hull contains the sphere S_r. Since we may cover by arbitrarily small bidiscs, given $\epsilon > 0$, we may assume that K_r is contained in the shell

$$A(r,\epsilon) = \{(z,w) :< r - \epsilon < \sqrt{|z|^2 + |w|^2} < r + \epsilon\}.$$

Since the hull of K_r contains the sphere S_r, it also contains the closed ball $\overline{B}_r$.

In this fashion we may construct a sequence K_j, $j = 1, 2, \cdots$ of compact totally disconnected sets, such that for each j, the polynomial hull of K_j contains the ball $\overline{B}_j$ of radius j, and K_j is contained in the shell $A(j, \epsilon)$. Now, by a theorem of Louis Antoine [1], for each j, there exists a simple arc γ_j, which passes through each point of K_j and we may assume that $\gamma_j \subset A(j, \epsilon)$. It is easy to construct a simple asymptotic path γ, which contains each γ_j. Suppose f is an entire function which tends to zero on γ. Fix $r > 0$ and $\delta > 0$. Choose j so large that $|f| < \delta$ on $\gamma \cap K_j$ and the hull of K_j contains $\overline{B}_r$. Since the polynomial hull is the same as the holomorphic hull, $|f| < \delta$ on B_r. Since r and δ were arbitrary positive numbers, $f = 0$. This completes the proof of (b). $\square$

If γ is an asymptotic path in $\mathbb{C}$ and θ is a rotation of $\mathbb{C}^n$, we denote by γ_θ the asymptotic path obtained by the corresponding rotation of γ.

Theorem 29. *Let γ be a strictly monotonic simple asymptotic path in $\mathbb{C}$, and let φ be a positive continuous function on $[0, +\infty)$. Then, there exists an entire function f on $\mathbb{C}$ such that, for every γ_θ, $f(z) \to 0$ as $z \to \infty$ on γ_θ and moreover, $\max_{|z|=r} |f(z)| > \varphi(r)$, for each $r \geq 0$.*

Proof. Thus, there exist entire functions tending to zero on every rotation of γ and having arbitrarily fast growth. To obtain such a function f we may first construct a strip U containing γ such that $E = \overline{\mathbb{C}} \setminus U$ is connected and locally connected. Thus E is a set of uniform approximation. We may construct U close enough to γ that every rotation γ_θ of γ is eventually in E. We may construct a simple asymptotic path $\sigma \subset U \setminus \gamma$. The set $F = E \cup \sigma \cup \gamma$ is also a set of approximation. We define a function $g \in A(F)$ by setting $g = 0$ on E and γ and g some continuous function on σ which grows so quickly that $\varphi(|z|) = o(g(z))$ as $z \to \infty$ on σ. Since F is a set of approximation, there exists an entire function f such that $|f(z) - g(z)| \to 0$, as $z \to \infty$ on F. In particular, for $\epsilon > 0$ and sufficiently large r, if $z \in \sigma$, with $|z| = r$

$$|f(z)| > |g(z)| - o(1) > \frac{\varphi(|z|)}{\epsilon} - o(1) > \varphi(r).$$

$\square$

Corollary 30. *Let φ be a positive continuous function on $[0, +\infty)$. Then, there exists an entire function f on $\mathbb{C}^2$ such that, $f(z) \to 0$ along every real ray from the origin and moreover,*

$$\max_{|z|^2+|w|^2=r^2} |f(z,w)| > \varphi(r), \quad \forall r \geq 0.$$

Proof. Set $\varphi_1(r) = \sqrt{\varphi(r\sqrt{2})}$. By Theorem 29 , there is an entire function f_1 of one complex variable tending to zero on each ray and such that $\max_{|z|=r} |f_1(z)| > \varphi_1(r)$. Set $f(z,w) = f_1(z)f_1(w)$. Let X_ζ be the real ray in $\mathbb{C}^2$ passing through a point $\zeta \in S^3$, where $\zeta = (r_1 e^{i\theta_1}, r_2 e^{i\theta_2})$, $r_1^2 + r_2^2 = 1$. We may assume $r_1 \neq 0$. A point of X_ζ has the form $(\rho r_1 e^{i\theta_1}, \rho r_2 e^{i\theta_2})$ and $f(z,w) = f_1(\rho r_1 e^{i\theta_1}) f_1(\rho r_2 e^{i\theta_2})$. As $\rho \to +\infty$, $\rho r_j e^{i\theta_j}$ remain respectively on the rays $\arg z = \theta_j, j = 1, 2$. Thus, $f_1(z) \to 0$ and

$f_2(w)$ remains bounded. Hence $f(z,w) \to 0$. We have shown that f tends to zero along each real ray from the origin.

Now we check the growth of f.

$$\max_{|z|^2+|w|^2=r^2} |f(z,w)| \geq \max_{|z|^2=|w|^2=r^2/2} |f_1(z)||f_1(w)|$$
$$\geq \max_{|u|=r/\sqrt{2}} |f_1(u)|^2 > (\varphi_1(r/\sqrt{2}))^2 = \varphi(r).$$

Thus, f has the required growth. □

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ALGEBRAIC CHARACTERIZATIONS FOR REDUCTION SYSTEMS

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Abstract. We give algebraic characterizations for a reduction system to be respectively Noetherian and confluent, and for a Noetherian reduction system to be confluent. The characterization of a confluent reduction system $(A, \to)$ is based on a relationship between the confluence of the system and the exactness of the colimit functor colim : $Ab^{\mathcal{A}} \to Ab$ where $\mathcal{A}$ is the small category with objects the elements of A and arrows $a \to b$ whenever b is a consequence of a in the system.

1. Introduction

The aim of this paper is to express in algebraic terms the termination and confluence of a reduction system. Three algebraic characterizations for a reduction system $(A, \to)$ are given, theorems 2 and 3 which give sufficient and necessary conditions for $(A, \to)$ to be respectively noetherian and confluent, and theorem 5 which gives sufficient and necessary conditions under which a Noetherian reduction system is confluent. For the first two characterizations we use a few results from [8] regarding a unitary ring $[\mathcal{C}]$ which is always associated to a small additive category $\mathcal{C}$. First we explain how the ring is defined and than we mention the results. The underlying set of $[\mathcal{C}]$ is the

Key words and phrases. Reduction system, confluence, λ-confluence, termination, monoid presentation, exact colimit, filtered category.

set of $|\mathcal{C}| \times |\mathcal{C}|$ matrices of the form $[\alpha_{p,q}]$ where $\alpha_{p,q} \in \mathcal{C}(p,q)$ and each row and column has finitely many nonzero entries. The addition and multiplication in $[\mathcal{C}]$ are defined by using the addition and composition in $\mathcal{C}$ in the following way

$$[\alpha_{p,q}] + [\beta_{p,q}] = [\alpha_{p,q} + \beta_{p,q}] \text{ and } [\alpha_{p,q}] \cdot [\beta_{p,q}] = [\gamma_{a,b}] \text{ where } \gamma_{a,b} = \sum_{c \in |\mathcal{C}|} \alpha_{a,c} \cdot \beta_{c,b}.$$

It is shown in theorems 7.1 and 7.1* of [8] that the category of right modules $Ab^{[\mathcal{C}]}$ is related to the category of covariant additive functors $Ab^{\mathcal{C}}$ via exact functors

$$Ab^{[\mathcal{C}]} \underset{R,\ S}{\overset{T}{\rightleftarrows}} Ab^{\mathcal{C}} \tag{1}$$

where R and S are respectively right and left adjoint for T. Likewise, for the contravariant case there are adjoint pairs

$$Ab^{[\mathcal{C}]^*} \underset{R^*,\ S^*}{\overset{T^*}{\rightleftarrows}} Ab^{\mathcal{C}^*} . \tag{2}$$

As we make use of S^* and R, we recall here briefly that for any $F \in Ab^{\mathcal{C}^*}$, $S^*(F) = \underset{q \in |\mathcal{C}|}{\oplus} F(q)$ and the action of $\alpha = [\alpha_{p,q}]$ on $S^*(F)$ is given by

$$\alpha u_q = \sum_{p \in |\mathcal{C}|} u_p F(\alpha_{p,q}),$$

where u_q is the coproduct injection. Similarly, $R(F) = \prod_{q \in |\mathcal{C}|} F(q)$ with the action of matrices on the right defined in a similar fashion to the above. Using the above adjunctions, it is shown that for every $F \in Ab^{\mathcal{C}}$ and $G \in Ab^{\mathcal{C}^*}$ there is a natural equivalence

$$SF \otimes_{[\mathcal{C}]} S^*G \simeq F \otimes_{\mathcal{C}} G. \tag{3}$$

For the second characterization, apart from the above results from [8], we use a result of Isbell and Mitchell in [6] which states that categories $\mathcal{C}$ for which the colimit functor colim : $Ab^{\mathcal{C}} \to Ab$ is exact, are precisely those categories whose affinization aff $\mathcal{C}$ has filtered components. Here the affinization of $\mathcal{C}$ is the (nonadditive) subcategory of $\mathbb{Z}\mathcal{C}$ consisting of those morphisms whose integer coefficients sum to 1. In general we have $\mathcal{C} \subseteq \text{aff } \mathcal{C}$, with the equality if and only if $\mathcal{C}$ is a preordered set. Being filtered means two things, first any pair of objects map to a common object, and secondly, for any two morphism α_1, α_2 with the same domain and codomain, there exists β such that $\beta\alpha_1 = \beta\alpha_2$. For preordered sets the second condition is superfluous.

A special case of reduction systems are those arising from monoid presentations. If $\mathcal{P} = \langle \mathbf{x}, \mathbf{r} \rangle$ is a presentation of a monoid S, then associated to it there is the reduction system $(\mathbf{x}^*, \to)$ where $\mathbf{x}^*$ is the free monoid on $\mathbf{x}$ and $\to$ is made of pairs $(u\alpha v, u\beta v)$ where $(\alpha, \beta) \in \mathbf{r}$ and $u, v \in \mathbf{x}^*$. In fact this reduction system can be regarded as a disjoint union $\bigsqcup_{s \in S} \langle \mathcal{P}_s, \to_s \rangle$ of reduction systems $\langle \mathcal{P}_s, \to_s \rangle$, where $\mathcal{P}_s$ is the subset of those elements from $\mathbf{x}^*$ which represent the element $s \in S$, and $\to_s$ consists of those pairs of $\to$ with both coordinates inside $\mathcal{P}_s$. If it happens that $\mathcal{P}$ represents a group G, it is interesting to ask if the confluence of $\mathcal{P}_e$ (e is the unit of G) implies that of $\mathcal{P}_g$ for every $g \in G$. We call the presentation $\mathcal{P} = \langle \mathbf{x}, \mathbf{r} \rangle$ of G a λ-confluent presentation whenever the reduction system $\mathcal{P}_e$ is confluent. Here λ is the empty word representing the unit e. We use the result of theorem 3 and proposition 4.1.2, p. 117 of [4] to give a sufficient and necessary condition under which the λ-confluence of a monoid presentation $\mathcal{P} = \langle \mathbf{x}, \mathbf{r} \rangle$ of a group implies the confluence of $\mathcal{P}$ in the special case when $(\mathcal{P}_e, \to_e)$ is complete, that is confluent and Noetherian. In fact the termination of $\mathcal{P}_e$ implies that of $\mathcal{P}_g$ for if $\tilde{x}$ is a word representing g^{-1}, and if ρ is an infinite sequence of reductions in $\mathcal{P}_g$, then $\tilde{x}\rho$ will be an infinite sequence of reductions in $\mathcal{P}_e$. Proposition 4.1.2 states that if $\varphi : \Lambda \to \Gamma$ is a ring homomorphism, A a right Γ module, C a left Λ module, and if $\mathrm{Tor}_p^{\Lambda}(\Gamma, C) = 0$ for every $p > 0$, then

$$\mathrm{Tor}_n^{\Lambda}(A, C) \simeq \mathrm{Tor}_n^{\Gamma}(A, {}_{(\varphi)}C) \tag{4}$$

where ${}_{(\varphi)}C = \Gamma \otimes_\Lambda C$. Note that the isomorphism (4) holds true even in the case when Γ and A, regarded as right Λ modules, are not unitary, therefore we do not need to assume that $\varphi : \Lambda \to \Gamma$ is a ring homomorphism which sends the unit 1_Λ of Λ to the unit 1_Γ of Γ. Along the proof of proposition 4, we use from [4] (see on page 149) the following definition. If Λ and Γ are two augmented rings with respective augmentations $\varepsilon_\Lambda : \Lambda \to Q_\Lambda$, $\varepsilon_\Gamma : \Gamma \to Q_\Gamma$, and $\varphi : \Gamma \to \Lambda$ a map of augmented rings, then there is a map

$$\Psi : \Gamma \otimes_\Lambda Q_\Lambda \to Q_\Gamma$$

defined by $\Psi(\gamma \otimes x) = \gamma \cdot \psi(x)$ where $\psi : Q_\Lambda \to Q_\Gamma$ is the map induced by φ. Again, the definition of Ψ is still possible under the assumption that Γ is non unitary as a right Λ module via φ.

As we mentioned at the beginning of the introduction, our main objective is to characterize algebraically important notions of the theory of rewriting systems such as confluence and termination proving that such notions are in fact algebraic in nature. Before we explain below the significance of our results, we recall that associated to a reduction system $(A, \to)$ there is the reduction graph Γ_A with vertex set $V(\Gamma_A) = A$ and set of edges

$$E(\Gamma_A) = \{(a, b) : a \in A, b \in A \text{ if there is a reduction rule } a \to b\}.$$

Further, we denote by $F\Gamma_A$ the free category generated by Γ_A, by $\mathbb{Z}F\Gamma_A$ the additive category arising from $F\Gamma_A$ and finally by $[\mathbb{Z}F\Gamma_A]$ the ring associated with $\mathbb{Z}F\Gamma_A$.

Theorem 2 identifies the termination of a reduction system to a chain condition of principal right ideals in a semigroup arising from the system. More specifically, given a finitely branching reduction system $(A, \to)$, that is a system in which every element has finitely many descendants, we denote by RA the multiplicative subsemigroup of the monoid $([\mathbb{Z}F\Gamma_A], \cdot)$ consisting of all those matrices E with finitely many nonzero entries with the additional property that for any $a \in A$, $E_{a,a} \neq z \cdot 1_a$ where 1_a is the identity on a in $F\Gamma_A$ and $z \in \mathbb{Z}$. Our theorem then states that the system is Noetherian if and only if every descending chain of principal right ideals of $(RA, \cdot)$ terminates.

Theorem 3 identifies the confluence of a reduction system $(A, \to)$ with the flatness of a certain module arising from the system. More specifically we let $\mathcal{A}$ be the preorder arising from $F\Gamma_A$ by identifying the parallel arrows and then consider the adjoint situation (2). Theorem 3 then states that $(A, \to)$ is confluent if and only if $S^*\Delta\mathbb{Z}$ is a flat $[\mathbb{Z}\mathcal{A}]^*$ module, where $\Delta\mathbb{Z}$ is the constant functor at $\mathbb{Z}$ over $\mathbb{Z}\mathcal{A}^*$. Proposition 4 is an attempt to get an application of Theorem 3 to the specific situation where the reduction system arises from a monoid presentation $\mathcal{P} = \langle \mathbf{x}, \mathbf{r} \rangle$ of a group G. The problem we try to shade some light on in this proposition, is the so called the problem of λ-confluence which asks under what conditions the confluence of $\mathcal{P}_e$ implies the confluence of $\mathcal{P}_g$ for any $g \in G$. We prove, under the assumption that $(\mathcal{P}_e, \to_e)$ is complete, that for every $g \in G$, $(\mathcal{P}_g, \to_g)$ is confluent if and only if there is an irreducible $v \in \mathcal{P}_g$ such that ${}_{(\varphi_v)}S_e^*\Delta\mathbb{Z} \cong S_g^*\Delta\mathbb{Z}$.

Our final result is Theorem 5 which relates the confluence of a Noetherian reduction system $(A, \to)$ to the algebraic structure of the so called reduction monoid P arising from the system. The monoid P is defined as follows

$$P = \{\tau \in \mathcal{T}(A) : \tau(u) = v \text{ only if } v \text{ is a descendant of } u \text{ or } u = v\},$$

where $\mathcal{T}(A)$ is the full transformation monoid on A, and the operation in P is the usual composition of transformations. Our theorem then states that $(A, \rightarrow)$ is complete if and only if the reduction monoid P has a zero element.
We note that the above results connect different areas of pure mathematics, respectively the theory of semigroups and homological algebra to that of rewriting systems. We hope that these connections we have presented here will become helpful in the future in treating problems of rewriting systems.

2. Noetherian reduction systems

Let $(A, \rightarrow)$ be a reduction system which is finitely branching. This class of reduction systems is of interest because it includes for example the reduction systems arising from semigroup presentations with finitely many relations.

Lemma 1. *If $(A, \rightarrow)$ is a Noetherian and finitely branching reduction system, then for every $a \in A$, there is $n_a \in \mathbb{N}$ such that the length of any path in Γ_A from a to a successor of a does not exceed n_a.*

Proof. Since in such systems every element has finitely many successors, then for every $a \in A$ there are finitely many paths in Γ_A from a to successors of a. If we take n_a to be the maximum of the lengths of those paths, this will do. □

Denote by RA the multiplicative subsemigroup of the monoid $([\mathbb{Z}F\Gamma_A], \cdot)$ consisting of all those matrices E with finitely many nonzero entries with the additional property that for any $a \in A$, $E_{a,a} \neq z \cdot 1_a$ where 1_a is the identity on a in $F\Gamma_A$ and $z \in \mathbb{Z}$.

Theorem 2. *A finitely branching reduction system $(A, \rightarrow)$ is Noetherian if and only if every descending chain of principal right ideals of $(RA, \cdot)$ terminates.*

Proof. Let

$$E^{(0)} \cup E^{(0)} \cdot RA \supseteq E^{(1)} \cup E^{(1)} \cdot RA \supseteq \cdots \supseteq E^{(n)} \cup E^{(n)} \cdot RA \supseteq E^{(n+1)} \cup E^{(n+1)} \cdot RA \supseteq \cdots$$

be a descending chain of principal right ideals of RA. For $i \in \mathbb{N}$ we let $Q^{(i)} \in RA$ be such that

$$E^{(i)} = E^{(i-1)} \cdot Q^{(i)}. \tag{5}$$

Using (5), one can show by recursion that

$$E^{(n)}_{a,b} = \sum_{a_n \in A} \cdots \sum_{a_1 \in A} E^{(0)}_{a,a_1} \cdot Q^{(1)}_{a_1,a_2} \cdots Q^{(n-1)}_{a_{n-1},a_n} \cdot Q^{(n)}_{a_n,b}. \tag{6}$$

From the definition of RA, each of the $n+1$ factors of a nonzero term of (6) is made of linear combinations of respectively elements from $\mathrm{Hom}_{F\Gamma_A}(a, a_1)$, $\mathrm{Hom}_{F\Gamma_A}(a_i, a_{i+1})$ and $\mathrm{Hom}_{F\Gamma_A}(a_n, b)$ with integers. If we take one component from each of the above hom-sets that take part in the formation of the terms of (6) we obtain a path in Γ_A of the form

$$a \rightarrow a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a_n \rightarrow b \tag{7}$$

whose length is $n+1$ since none of the arrows represented in (7) arises from an identity morphism in $F\Gamma_A$. Let $\mathcal{E}_i$ be the subset of A corresponding to the rows of $E^{(i)}$ which contain nonzero elements. It is easy to see from (5) that $\mathcal{E}_i \subseteq \mathcal{E}_0$. We claim that

$$\forall a \in \mathcal{E}_0, \exists n_a \in \mathbb{N}, \text{ such that } \forall b \in A \text{ with } E^{(n_a)}_{a,b} = 0.$$

Indeed, take n_a as in Lemma 1. Then, path (7) have length $n_a+1>n_a$, therefore the term that gave rise (7) could not have been nonzero. Let now $n=\max\{n_a : a\in\mathcal{E}_0\}$. From the above claim we get that $E^{(n)}_{a,b}=0$ for all $a\in\mathcal{E}_0$ and $b\in A$, therefore $E^{(n)}$ is the zero matrix.

Conversely, suppose that every descending chain of principal right ideals of RA terminates and assume that there exist an infinite path

$$e_1\cdot e_2\cdots e_n\cdot e_{n+1}\cdots$$

in Γ_A. Define matrices from RA as follows

$$E^{(n)}=[\varepsilon_{a,b}] \text{ where } \varepsilon_{a,b}=\begin{cases} e_1\cdots e_n & \text{if} \quad a=\iota(e_1) \text{ and } b=\tau(e_n)\\ 0 & \text{otherwise}\end{cases}$$

It is now obvious that $E^{(n+1)}\subseteq E^{(n)}\cdot RA$ and that $E^{(n)}\notin E^{(n+1)}\cdot RA$ which shows that the descending chain

$$E^{(1)}\cup E^{(1)}\cdot RA\supseteq\cdots\supseteq E^{(n)}\cup E^{(n)}\cdot RA\supseteq E^{(n+1)}\cup E^{(n+1)}\cdot RA\supseteq\cdots$$

does not terminate, a contradiction. □

3. Confluent reduction systems

As before we let $F\Gamma_A$ be the free category generated by the reduction graph associated with the reduction system $(A,\to)$. Let $\mathcal{A}$ be the quotient $F\Gamma_A/\sim$ where $\sim$ is the congruence generated by all the pairs (α,β) with $\alpha,\beta\in F\Gamma_A(a,b)$ and a,b varying in A. In this way $\mathcal{A}$ becomes a preordered set and in this case the elements of $[\mathbb{Z}\mathcal{A}]$ have a simple form: they are $A\times A$ matrices $[\alpha_{p,q}]$ where $\alpha_{p,q}\in\mathbb{Z}$ and each row and column has finitely many nonzero entries. If an entry $\alpha_{p,q}$ is nonzero, then q is a consequence of p in the system $(A,\to)$. Remark here that the confluence of $(A,\to)$ is equivalent to aff $\mathcal{A}$ having filtered components, therefore instead of looking directly for the confluence of $(A,\to)$, one should look for conditions under which aff $\mathcal{A}$ has filtered components. We give such a condition in terms of $[\mathbb{Z}\mathcal{A}]$ and for this purpose we note first that the adjunctions (1) and (2) to the case of the small additive category $\mathbb{Z}\mathcal{A}$ become

(8)
$$Ab^{[\mathbb{Z}\mathcal{A}]} \underset{R,\ S}{\overset{T}{\rightleftarrows}} Ab^{\mathbb{Z}\mathcal{A}}$$

and

(9)
$$Ab^{[\mathbb{Z}\mathcal{A}]^*} \underset{R^*,\ S^*}{\overset{T^*}{\rightleftarrows}} Ab^{\mathbb{Z}\mathcal{A}^*}\ .$$

Theorem 3. *The reduction system $(A,\to)$ is confluent if and only if $S^*\Delta\mathbb{Z}$ is a flat $[\mathbb{Z}\mathcal{A}]^*$ module, where $\Delta\mathbb{Z}$ is the constant functor at $\mathbb{Z}$ over $\mathbb{Z}\mathcal{A}^*$.*

Proof. For every $G\in Ab^{\mathbb{Z}\mathcal{A}^*}$, every $M\in Ab^{[\mathbb{Z}\mathcal{A}]}$ and every abelian group B, the following natural equivalences hold true

$$\begin{aligned} Ab(S^*G\otimes_{[\mathbb{Z}\mathcal{A}]^*}M,B) &\simeq Ab^{[\mathbb{Z}\mathcal{A}]}(M,Ab(S^*G,B))\\ &\simeq Ab^{[\mathbb{Z}\mathcal{A}]}(M,RAb(G,B))\\ &\simeq Ab^{\mathbb{Z}\mathcal{A}}(TM,Ab(G,B))\\ &\simeq Ab(G\otimes_{\mathbb{Z}\mathcal{A}^*}TM,B).\end{aligned}$$

Then from Yoneda lemma there must be a natural equivalence

$$S^*G \otimes_{[\mathbb{Z}\mathcal{A}]^*} M \simeq G \otimes_{\mathbb{Z}\mathcal{A}^*} TM. \tag{10}$$

If we assume now that $(A, \to)$ is confluent, then aff $\mathcal{A}$ has filtered components and then from [6] the functor $\Delta\mathbb{Z}$ is flat as a right $\mathbb{Z}\mathcal{A}^*$ module. We want to show that $S^*\Delta\mathbb{Z}$ is flat, that is, if $M \to N$ is an injection in $Ab^{[\mathbb{Z}\mathcal{A}]}$, then the induced morphism $S^*\Delta\mathbb{Z}\otimes_{[\mathbb{Z}\mathcal{A}]^*} M \to S^*\Delta\mathbb{Z}\otimes_{[\mathbb{Z}\mathcal{A}]^*} N$ is an injection in Ab. To see this we can use the naturality of (10) by replacing G with $\Delta\mathbb{Z}$ and then obtaining the commutative diagram

$$\begin{tikzcd}
S^*\Delta\mathbb{Z} \otimes_{[\mathbb{Z}\mathcal{A}]^*} M \arrow[r] \arrow[d, equal] & S^*\Delta\mathbb{Z} \otimes_{[\mathbb{Z}\mathcal{A}]^*} N \arrow[d, equal] \\
\Delta\mathbb{Z} \otimes_{\mathbb{Z}\mathcal{A}^*} TM \arrow[r] & \Delta\mathbb{Z} \otimes_{\mathbb{Z}\mathcal{A}^*} TN
\end{tikzcd}$$

whose vertical arrows are isomorphisms and the bottom arrow is an injection since T preserves injections and $\Delta\mathbb{Z}$ is flat.

Conversely, suppose that $S^*\Delta\mathbb{Z}$ is flat and want to show that for any injection $F_1 \to F_2$ in $Ab^{\mathbb{Z}\mathcal{A}}$, the induced morphism $\Delta\mathbb{Z} \otimes_{\mathbb{Z}\mathcal{A}^*} F_1 \to \Delta\mathbb{Z} \otimes_{\mathbb{Z}\mathcal{A}^*} F_2$ is an injection in Ab. If we apply the natural equivalence (3) to the case when $\mathcal{C} = \mathbb{Z}\mathcal{A}$ and $G = \Delta\mathbb{Z}$, we get the commutative square

$$\begin{tikzcd}
S^*\Delta\mathbb{Z} \otimes_{[\mathbb{Z}\mathcal{A}]^*} SF_1 \arrow[r] \arrow[d, equal] & S^*\Delta\mathbb{Z} \otimes_{[\mathbb{Z}\mathcal{A}]^*} SF_2 \arrow[d, equal] \\
\Delta\mathbb{Z} \otimes_{\mathbb{Z}\mathcal{A}^*} F_1 \arrow[r] & \Delta\mathbb{Z} \otimes_{\mathbb{Z}\mathcal{A}^*} F_2
\end{tikzcd}$$

with the vertical arrows being isomorphisms. Since S is an exact functor and $S^*\Delta\mathbb{Z}$ is flat, the top arrow is an injection, therefore the bottom one will be an injection too proving the flatness of $\Delta\mathbb{Z}$. □

4. A discussion on λ-confluence

As we characterized the confluence of a reduction system in terms of the flatness of a certain module arising from the system, then it is natural to use this characterization to obtain information on the confluence of some particular reduction systems. We focus on those reduction systems arising from monoid presentations which give groups with the intention to find conditions under which the corresponding reduction system is confluent.

Let $\mathcal{P} = \langle \mathbf{x}, \mathbf{r} \rangle$ be a monoid presentation for a group G and let $(\mathcal{P}_e, \to_e)$ and $(\mathcal{P}_g, \to_g)$ be the reduction systems corresponding to the unit element e and to some $g \in G$, $g \neq e$. Assume throughout that $(\mathcal{P}_e, \to_e)$ is complete, therefore as we mentioned in the introduction, $(\mathcal{P}_g, \to_g)$ will be terminating. We let $\mathcal{I}_g$ be the set of irreducible words representing g. Denote by Λ the ring $[\mathbb{Z}\left(F\Gamma_{\mathcal{P}_e}/\sim\right)]$ and by Γ the ring $[\mathbb{Z}\left(F\Gamma_{\mathcal{P}_g}/\sim\right)]$ where $\sim$ is defined as before. Let for any $g \in G$ denote by $R_g^*\Delta\mathbb{Z}$, $S_g^*\Delta\mathbb{Z}$ the modules from Ab^{Γ^*} defined in (9). Define

$$\varepsilon_\Lambda : \Lambda \to R_e^*\Delta\mathbb{Z}$$

by setting for every matrix $\alpha = [\alpha_{p,q}] \in \Lambda$,

$$\pi_p \varepsilon_\Lambda(\alpha) = \sum_{q \in \mathcal{P}_e} \alpha_{p,q}$$

where π_p is the p'th projection. This definition makes sense since α is row finite. It is obviously a group homomorphism and surjective since for every $b \in R_e^*\Delta\mathbb{Z}$ if we take $\alpha \in \Lambda$ such that $\alpha_{p,p} = \pi_p(b)$ for every $p \in \mathcal{P}_e$, and $\alpha_{p,q} = 0$ for $p \neq q$, then from the definition of ε_Λ we see that $\varepsilon_\Lambda(\alpha) = b$. To see that ε_Λ is a homomorphism of left modules we must prove that for every $\alpha, \beta \in \Lambda$ and every $p \in \mathcal{P}_e$, $\pi_p\varepsilon_\Lambda(\alpha \cdot \beta) = \pi_p(\alpha \cdot \varepsilon_\Lambda(\beta))$. Indeed,

$$\pi_p\varepsilon_\Lambda(\alpha \cdot \beta) = \sum_{s\in\mathcal{P}_e}\sum_{q\in\mathcal{P}_e} \alpha_{p,q}\beta_{q,s},$$

and

$$\pi_p(\alpha \cdot \varepsilon_\Lambda(\beta)) = \sum_{q\in\mathcal{P}_e} \alpha_{p,q} \sum_{s\in\mathcal{P}_e} \beta_{q,s},$$

which are equal to each other. We note that the augmentation ideal I_Λ consists in those matrices whose row elements sum to zero. In a similar fashion we define an augmentation homomorphism $\varepsilon_\Gamma : \Gamma \to R_g^*\Delta\mathbb{Z}$ whose augmentation ideal I_Γ again consists in those matrices from Γ whose row elements sum to zero. For every $[\alpha_{p,q}] \in \Lambda$ and every $v \in \mathcal{I}_g$, denote by $v \cdot [\alpha_{p,q}]$ the matrix from Γ whose only nonzero entries are those $\alpha_{vp,vq} = \alpha_{p,q}$ whenever $\alpha_{p,q}$ is nonzero. For each $v \in \mathcal{I}_g$ define

$$\varphi_v : \Lambda \to \Gamma$$

by

$$\varphi_v(\alpha) = v \cdot \alpha.$$

It is easy to see that φ_v is a homomorphism of rings. Also from the definition of φ_v we see that $\varphi_v(I_\Lambda) \subseteq I_\Gamma$ hence it is a map of augmented rings and therefore it induces a Λ module homomorphism $\psi_v : R_e^*\Delta\mathbb{Z} \to R_g^*\Delta\mathbb{Z}$ if we regard $R_g^*\Delta\mathbb{Z}$ as a left Λ module via φ_v. As mentioned in the introduction, ψ_v induces a homomorphism of left Γ modules

$$\Psi_v : \ _{(\varphi_v)}R_e^*\Delta\mathbb{Z} \to R_g^*\Delta\mathbb{Z}$$

defined by

$$\Psi_v(\gamma \otimes a) = \gamma \cdot \psi_v(a).$$

Now using the fact that $S_e^*\Delta\mathbb{Z}$ and $S_g^*\Delta\mathbb{Z}$ are submodules of respectively $R_e^*\Delta\mathbb{Z}$ and $R_g^*\Delta\mathbb{Z}$, and the (easily checking) fact that for every $v \in \mathcal{I}_g$ the only nonzero coordinates of $\psi_v(a)$ are those indexed by vu whenever $\pi_u(a) \neq 0$, one can see that Ψ_v induces a homomorphism of left Γ modules

$$\tilde{\Psi}_v : \ _{(\varphi_v)}S_e^*\Delta\mathbb{Z} \to S_g^*\Delta\mathbb{Z}.$$

Note that for different $v \in \mathcal{I}_g$, modules $_{(\varphi_v)}S_e^*\Delta\mathbb{Z}$ may be different. For their coproduct $\bigoplus_{v\in\mathcal{I}_g} {}_{(\varphi_v)}S_e^*\Delta\mathbb{Z}$ we let

$$\tilde{\Psi} : \bigoplus_{v\in\mathcal{I}_g} {}_{(\varphi_v)}S_e^*\Delta\mathbb{Z} \to S_g^*\Delta\mathbb{Z} \tag{11}$$

be the homomorphism of left Γ modules arising from the universal property of coproducts. We show that $\tilde{\Psi}$ is surjective. For this we need only to prove that any generator of the abelian group $S_g^*\Delta\mathbb{Z}$, that is any $d \in S_g^*\Delta\mathbb{Z}$ with $\pi_w(d) = 1$ for a unique $w \in \mathcal{P}_g$, is in the image of $\tilde{\Psi}$. Let $v \in \mathcal{I}_g$ be an irreducible descendant of w. Denote by γ^w the matrix from Γ whose only nonzero entry is $\gamma^w_{w,v} = 1$ and let $c^w \in \bigoplus_{v\in\mathcal{I}_g} {}_{(\varphi_v)}S_e^*\Delta\mathbb{Z}$ such that $\pi_\xi c^w \neq 0$ if and only only if $\xi = v$ and that $\pi_v c^w = \gamma^w \otimes a^w$ where the only nonzero coordinate of a^w is $\pi_\lambda a^w = 1$. The definition of $\tilde{\Psi}$ implies that $\tilde{\Psi}(c^w) = \tilde{\Psi}_v(\gamma^w \otimes a^w) = d$. With the notations established above we prove the following.

Proposition 4. *Let $\mathcal{P} = \langle \mathbf{x}, \mathbf{r} \rangle$ be a monoid presentation for a group G such that the reduction system $(\mathcal{P}_e, \to_e)$ is complete, then for every $g \in G$, $(\mathcal{P}_g, \to_g)$ is confluent if and only if there is $v \in \mathcal{I}_g$ such that ${}_{(\varphi_v)}S_e^*\Delta\mathbb{Z} \cong S_g^*\Delta\mathbb{Z}$.*

Proof. If $(\mathcal{P}_g, \to_g)$ is confluent, then $\mathcal{I}_g$ is a singleton, let say $\mathcal{I}_g = \{v\}$ and in this case (11) has the form

$$\tilde{\Psi} = \Psi_v : {}_{(\varphi_v)}S_e^*\Delta\mathbb{Z} \to S_g^*\Delta\mathbb{Z}.$$

Next we show that $\tilde{\Psi}$ is a split epimorphism of Γ modules. For this we define for each $w \in \mathcal{P}_g$ the family of group homomorphisms

$$\theta_w : \mathbb{Z}_w \to {}_{(\varphi_v)}S_e^*\Delta\mathbb{Z},$$

where $\mathbb{Z}_w$ is an isomorphic copy of the additive group $\mathbb{Z}$, by

$$\theta_w(1) = c^w$$

with $c^w = \gamma^w \otimes a$ where the only nonzero coordinate of a is $\pi_\lambda a = 1$ and the only non zero entry of γ^w is $\gamma^w_{w,v} = 1$. Since $S_g^*\Delta\mathbb{Z}$ as an abelian group is isomorphic to $\underset{w\in\mathcal{P}_g}{\oplus} \mathbb{Z}_w$, the family θ_w yields an abelian group homomorphism

$$\Theta : S_g^*\Delta\mathbb{Z} \to {}_{(\varphi_v)}S_e^*\Delta\mathbb{Z}$$

such that $\Theta u_w = \theta_w$ where u_w is the coproduct injection. It is easy to see that Θ is a homomorphism of Γ modules. To see that Θ splits, let $d \in S_g^*\Delta\mathbb{Z}$ be any generator with $\pi_w(d) = 1$, then we have

$$\begin{aligned}\tilde{\Psi}\Theta(d) &= \tilde{\Psi}\Theta u_w(1)\\ &= \tilde{\Psi}(\theta_w(1)) = \tilde{\Psi}(c^w) = d.\end{aligned}$$

As a result we get the direct sum of Γ modules ${}_{(\varphi_v)}S_e^*\Delta\mathbb{Z} \cong S_g^*\Delta\mathbb{Z} \oplus K$ where K is the kernel of $\tilde{\Psi}$. But, as we just saw, any generator $\gamma^w \otimes a$ of ${}_{(\varphi_v)}S_e^*\Delta\mathbb{Z}$ is in the image of Θ, therefore Θ is surjective, $K = 0$ and ${}_{(\varphi_v)}S_e^*\Delta\mathbb{Z} \cong S_g^*\Delta\mathbb{Z}$.
For the converse, since $(\mathcal{P}_e, \to_e)$ is confluent, then from theorem 3 $S_e^*\Delta\mathbb{Z}$ is a flat left Λ module, hence if we regard Γ as a right Λ module via φ_v for the given $v \in \mathcal{I}_g$, we have that $\mathrm{Tor}_p^\Lambda(\Gamma, S_e^*\Delta\mathbb{Z}) = 0$ for every $p > 0$, then from proposition 4.1.2 of [4] we obtain the isomorphisms $\mathrm{Tor}_n^\Lambda(A, S_e^*\Delta\mathbb{Z}) \simeq \mathrm{Tor}_n^\Gamma(A, {}_{(\varphi_v)}S_e^*\Delta\mathbb{Z})$ for every $n > 0$ and every right Γ module A. Since $\mathrm{Tor}_n^\Lambda(A, S_e^*\Delta\mathbb{Z}) = 0$, we get the flatness of the left Γ module ${}_{(\varphi_v)}S_e^*\Delta\mathbb{Z}$ hence the flatness $S_g^*\Delta\mathbb{Z}$. Theorem 3 implies that $(\mathcal{P}_g, \to_g)$ is confluent. □

5. Complete reduction systems

In this section we give an algebraic characterization for a Noetherian reduction system to be confluent. Differently from the Newman's lemma ([9]) which states that Noetherian reduction systems are confluent if and only if they are locally confluent, our characterization translates the confluence purely in terms of semigroup theory. First, for every reduction system $(A, \to)$, we construct a submonoid P of the full transformation monoid $\mathcal{T}(A)$ on the set A as follows:

$$P = \{\tau \in \mathcal{T}(A) : \tau(u) = v \text{ only if } v \text{ is a descendant of } u \text{ or } u = v\}.$$

It is clear that, under the usual composition of transformations, P forms a submonoid of $\mathcal{T}(A)$. We call P the reduction monoid of $(A, \to)$. Before we give our characterization, we recall that a Noetherian reduction systems $(A, \to)$ is complete if and only if every element from A has a unique irreducible descendant.

Theorem 5. *Let $(A, \to)$ be a Noetherian reduction systems. Then, $(A, \to)$ is complete if and only if the reduction monoid P has a zero element.*

Proof. If $(A, \to)$ is complete, then, for every $\omega \in A$, the respective congruence class $[\omega]$ has a unique irreducible element, say $irr([\omega])$. Let $\theta \in P$ be the element which sends every $\omega \in A$ to its corresponding $irr([\omega])$. It is easy to show that θ is the zero of P.

Conversely, suppose that P has a zero element θ. Denote by $Irr(\omega)$ the set of irreducibles which are descendants of ω, and write $Irr = \cup_{\omega \in A} Irr(\omega)$. If we think of θ as a $2 \times \infty$ matrix, then we first show that the second row of θ consists only of elements from Irr. Indeed, if there is $u \in A$ such that $\theta(u) = v$ and $v \notin Irr$, then for τ which sends v to some corresponding successor v', we would have $\tau\theta(u) = v'$, which means that $\tau\theta \neq \theta$. Note also that in the second row we always have represented all the elements from Irr because they are not transformed under any element of P. Hence the second row of θ consists only of all the elements of Irr. Next we show that every $\omega \in A$ has a unique irreducible descendant. Suppose by way of contradiction that there is some $u \in A$ which has a set $\{i_\lambda | \lambda \in \Lambda\}$ of distinct irreducible descendants. Let K_λ for $\lambda \in \Lambda$ be respectively $\theta^{-1}(i_\lambda)$. Suppose that $u \in K_\lambda$. Since i_ν with $\nu \neq \lambda$ is a descendant of u too, then there will be some v such that v is a successor of u and i_ν is a descendant of v or $i_\nu = v$. Distinguish between two cases.

(1) $v \notin K_\lambda$. Let $\tau \in P$ be such that it sends u to v. Then $\theta\tau(u) = \theta(v) \neq i_\lambda$ which contradicts the fact that θ is the zero.

(2) $v \in K_\lambda$. Let $\tau \in P$ be such that it sends v to i_ν. Then $\theta\tau(v) = \theta(i_\nu) = i_\nu \neq i_\lambda$ which again contradicts the fact that θ is the zero.

So it remains that u can not have more than one irreducible descendant and hence the system is complete. □

Corollary 6. *A Noetherian reduction system $(A, \to)$ is complete if and only if the monoid P constructed as above, has cohomological dimension 0.*

Proof. This follows immediately from Theorem 5 and from [5]. □

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ON THE DISTRIBUTION OF THE VALUES OF ADDITIVE FUNCTIONS OVER INTEGERS WITH A FIXED NUMBER OF DISTINCT PRIME DIVISORS

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Abstract. We study the distribution of the values of certain additive functions restricted to those integers with a fixed number of prime divisors.

1. Introduction

We study the distribution of the values of certain additive functions restricted to those integers with a fixed number of prime divisors.

Given an additive function f for which there exists a real number $C > 0$ such that $|f(p^a)| < C$ for all prime powers p^a, we let

$$A_x = \sum_{p \le x} \frac{f(p)}{p}$$

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and we let f^* be the additive function (which depends on x) defined on prime powers by $f^*(p^a) = f(p^a) - \dfrac{a}{x_2} A_x$, where $x_2 = \log\log x$. Let

$$B_x = \sqrt{\sum_{p \le x} \frac{(f^*(p))^2}{p}}$$

and assume that $B_x \to \infty$. For each integer $k \ge 1$, let

$$\xi_{k,x} := \frac{k}{x_2}, \qquad \wp_k := \{n \in \mathbb{N} : \omega(n) = k\}, \qquad \pi_k(x) := \#\{n \le x : n \in \wp_k\}.$$

Finally, let $\delta < \frac{1}{2}$ be a fixed positive number. Then, we prove that

$$\lim_{x\to\infty} \max_{\substack{k \\ \xi_{k,x} \in [\delta, 2-\delta]}} \max_{y \in \mathbb{R}} \left| \frac{1}{\pi_k(x)} \# \left\{ n \le x : n \in \wp_k,\ \frac{f^*(n)}{B_x \sqrt{\xi_{k,x}}} < y \right\} - \Phi(y) \right| = 0,$$

where

$$\Phi(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-u^2/2}\, du. \tag{1.1}$$

We also establish a result concerning the distribution of $\omega(\varphi_\ell(n))$ where $\omega(m)$ stands for the number of distinct prime factors of $m \ge 2$ (with $\omega(1) = 0$) and φ_ℓ stands for the ℓ-th iterate of the Euler φ-function.

2. Notations

For each integer $n \ge 2$, $\Omega(n)$ stand for the number of prime divisors of n counting their multiplicity, setting $\Omega(1) = 0$. Let also $p(n)$ and $P(n)$ stand for the smallest and largest prime factors of $n \ge 2$, with $p(1) = P(1) = 1$.

We shall use the notations $x_1 = \log x$, $x_2 = \log\log x$, and so on.

For every positive integers k and D, let us further set

$$\begin{aligned}
\wp_k &:= \{n \in \mathbb{N} : \omega(n) = k\},\\
\pi_k(x) &:= \#\{n \le x : n \in \wp_k\},\\
\mathcal{N}_k &:= \{n \in \mathbb{N} : \Omega(n) = k\},\\
N_k(x) &:= \#\{n \le x : n \in \mathcal{N}_k\},\\
\pi_k(x|D) &:= \#\{n \le x : (n,D) = 1,\ n \in \wp_k\},\\
N_k(x|D) &:= \#\{n \le x : (n,D) = 1,\ n \in \mathcal{N}_k\},\\
\xi_{k,x} &:= \frac{k}{x_2}.
\end{aligned}$$

Let Φ be the standard Gaussian law defined above in (1.1). We also write $\psi(t)$ for the characteristic function of the Gaussian law, that is,

$$\psi(t) := e^{-t^2/2} \qquad (t \in \mathbb{R}).$$

We shall also be using the two sequences of integers

$$a_\ell = \frac{1}{(\ell+1)!} \quad \text{and} \quad b_\ell = \frac{1}{\sqrt{2\ell+1}} \cdot \frac{1}{\ell!} \qquad (\ell = 1, 2, \ldots). \tag{2.1}$$

Throughout this paper, the letters c and C always denote positive constants, but not necessarily the same at each occurrence.

3. Main results

Theorem 1. *Let f be an additive function for which there exists a real number $C > 0$ such that $|f(p^a)| < C$ for all prime powers p^a. Let $A_x = \sum_{p \le x} \frac{f(p)}{p}$. Let $f^* = f^*_x$ be the additive function defined on prime powers by*

$$f^*(p^a) = \begin{cases} f(p^a) - \frac{a}{x_2} A_x & \text{if } p^a \le x, \\ 0 & \text{otherwise .} \end{cases}$$

Further set

$$B_x = \sqrt{\sum_{p \le x} \frac{(f^*(p))^2}{p}}$$

and assume that $B_x \to \infty$ as $x \to \infty$. Then, given an arbitrary positive real number $\delta < \frac{1}{2}$,

$$\lim_{x\to\infty} \max_{\substack{k \\ \xi_{k,x} \in [\delta, 2-\delta]}} \max_{y \in \mathbb{R}} \left| \frac{1}{\pi_k(x)} \# \left\{ n \le x : n \in \wp_k, \ \frac{f^*(n)}{B_x \sqrt{\xi_{k,x}}} < y \right\} - \Phi(y) \right| = 0.$$

Let us add that in 2008, Kátai and Subbarao [3] proved the following result.

Theorem A. *With the notations of Theorem 1, we have*

$$\lim_{x\to\infty} \max_{\xi_{k,x} \in [\delta, 2-\delta]} \max_{y \in \mathbb{R}} \left| \frac{1}{N_k(x)} \# \left\{ n \le x : n \in \mathcal{N}_k, \ \frac{f^*(n)}{B_x \sqrt{\xi_{k,x}}} < y \right\} - \Phi(y) \right| = 0.$$

Theorem 2. *Let a_ℓ and b_ℓ be the two sequences defined in (2.1). Let $\xi = \xi_{k,x}$ and assume that ℓ is fixed. Setting*

$$s_\xi(n) := \frac{\omega(\varphi_\ell(n)) - a_\ell \xi x_2^{\ell+1}}{b_\ell \sqrt{\xi} x_2^{\ell+\frac{1}{2}}},$$

then, given an arbitrary positive real number $\delta < \frac{1}{2}$,

$$\lim_{x\to\infty} \max_{z \in \mathbb{R}} \max_{\xi \in [\delta, 2-\delta]} \left| \frac{1}{N_k(x)} \# \{ n \le x : n \in \mathcal{N}_k, \ s_\xi(n) < z \} - \Phi(z) \right| = 0 \tag{3.1}$$

and

$$\lim_{x\to\infty} \max_{z \in \mathbb{R}} \max_{\xi \in [\delta, 2-\delta]} \left| \frac{1}{\pi_k(x)} \# \{ n \le x : n \in \wp_k, \ s_\xi(n) < z \} - \Phi(z) \right| = 0. \tag{3.2}$$

4. An appropriate estimate for $\pi_k(x|D)$

As a preliminary result to be used in the proof of Theorem 1, we will show the following lemma.

Lemma 1. *Given a positive number $\delta < \frac{1}{2}$, then, as $x \to \infty$, we have, uniformly for $\xi \in [\delta, 2-\delta]$,*

$$\pi_k(x|D) = (1 + o(1))\pi_k(x) \cdot \prod_{p|D} \left(1 - \frac{\xi}{p\left(1 - \frac{1-\xi}{p}\right)} \right). \tag{4.1}$$

Proof. It is well known that

$$\pi_k(x) = F(\xi)\frac{x}{x_1}\frac{x_2^{k-1}}{(k-1)!}\left(1+O\left(\frac{1}{x_2}\right)\right),$$

where

$$F(z) = \frac{1}{\Gamma(z+1)}\prod_p\left(1+\frac{z}{p-1}\right)\left(1-\frac{1}{p}\right)^z.$$

(see for instance the classical paper of Selberg [4]).

Given an integer $D \geq 2$, let $\mathcal{B}_D$ stand for the multiplicative semigroup generated by the prime divisors of D.

Let us first write

$$\sum_{\substack{n=1\\(n,D)=1}}^{\infty}\frac{z^{\omega(n)}}{n^s} = \left\{\sum_{n=1}^{\infty}\frac{z^{\omega(n)}}{n^s}\right\}\Lambda_k(s,z), \tag{4.2}$$

where

$$\Lambda_k(s,z) = \prod_{p|D}\left(1+\frac{z}{p^s}+\frac{z}{p^{2s}}+\dots\right)^{-1} = \prod_{p|D}\left(1-\sum_{\ell=1}^{\infty}\frac{z(1-z)^{\ell-1}}{p^{\ell s}}\right) = \sum_{\substack{m=1\\m\in\mathcal{B}_D}}^{\infty}\frac{E(m)}{m^s},$$

where $E(m)$ is a multiplicative function defined on the set $\mathcal{B}_D$ by $E(1) = 1$, $E(p) = -z$, $E(p^\ell) = -z(1-z)^{\ell-1}$ for each $\ell \geq 2$ and each $p \in \mathcal{B}_D$.

Now, let us write each positive integer m as $m = MR$, where M is the squarefull part of m and where R is squarefree. Then, clearly,

$$E(M) = (-z)^{\omega(M)}(1-z)^{\Omega(M)-\omega(M)}, \quad E(R) = (-z)^{\omega(R)},$$

implying that if we set $\Delta(m) = \Omega(m) - \omega(m)$, then

$$E(m) = (-z)^{\omega(m)}(1-z)^{\Delta(m)} = \sum_{\nu=0}^{\Delta(m)}\binom{\Delta(m)}{\nu}(-1)^{\nu+\omega(m)}z^{\nu+\omega(m)},$$

so that it follows from (4.2) that

$$\pi_k(x|D) = \sum_{m\in\mathcal{B}_D}\sum_{\nu=0}^{\Delta(m)}(-1)^{\nu+\omega(m)}\binom{\Delta(m)}{\nu}\pi_{k-(\omega(m)+\nu)}\left(\frac{x}{m}\right). \tag{4.3}$$

Given a fixed positive real number $\delta < 1/2$, then it is easy to prove that, uniformly for $k \in [\delta x_2, (2-\delta)x_2]$, we have

$$\frac{\pi_{k-(\omega(m)+\nu)}\left(\frac{x}{m}\right)}{\pi_k(x)} = (1+\varepsilon_2(x))\xi^{\omega(m)+\nu} \qquad (x\to\infty), \tag{4.4}$$

where $\varepsilon_2(x) \to 0$ as $x \to \infty$.

Using (4.4) in (4.3), we get

$$\begin{aligned}\frac{\pi_k(x|D)}{\pi_k(x)} &= (1+\varepsilon_2(x))\sum_{m\in\mathcal{B}_D}\frac{1}{m}\sum_{\nu=0}^{\Delta(m)}(-1)^{\nu+\omega(m)}\binom{\Delta(m)}{\nu}\xi^{\omega(m)+\nu}\\ &= (1+\varepsilon_2(x))\sum_{m\in\mathcal{B}_D}\frac{(-\xi)^{\omega(m)}}{m}(1-\xi)^{\Delta(m)}\end{aligned}$$

$$
\begin{aligned}
&= (1+\varepsilon_2(x))\prod_{p|D}\left(1+\frac{-\xi}{p}+\frac{-\xi(1-\xi)}{p^2}+\cdots\right)\\
&= (1+\varepsilon_2(x))\prod_{p|D}\left(1-\frac{\xi}{p\left(1-\frac{1-\xi}{p}\right)}\right),
\end{aligned}
$$

which proves (4.1), thus completing the proof of Lemma 1. □

5. Proof of Theorem 1

From the identity

$$\sum_{\substack{n=1\\(n,D)=1}}^{\infty}\frac{z^{\Omega(n)}}{n^s}=\sum_{n=1}^{\infty}\frac{z^{\Omega(n)}}{n^s}\times\prod_{p|D}\left(1-\frac{z}{p^s}\right)\qquad (z\in\mathbb{C},|z|<2,s>1),$$

it follows that

$$N_k(x|D)=\sum_{d|D}\mu(d)N_{k-\Omega(d)}\left(\frac{x}{d}\right). \tag{5.1}$$

Let w_x be a function which tends to $+\infty$ (as $x\to\infty$), but slowly enough so that $w_x/\log B_x\to\infty$ as $x\to\infty$.

Let $\nu(n)=\dfrac{f^*(n)}{B_x\sqrt{\xi}}$ and let us introduce the additive function ν^* defined on prime powers by

$$\nu^*(p^a)=\begin{cases}0 & \text{if } p\le w_x,\\ \nu(p^a) & \text{otherwise.}\end{cases}$$

We further introduce the functions $g(n)=e^{i\tau\nu(n)}$ and $g^*(n)=e^{i\tau\nu^*(n)}$. It follows from these definitions that

$$\max_{n\le x}|g(n)-g^*(n)|\le c|\tau|\max_{P(m)\le w_x}|\nu(m)|\to 0\qquad \text{as } x\to\infty \tag{5.2}$$

uniformly for $\xi\in[\delta,2-\delta]$.

Assume that $P(D)\le w_x$. Then $g^*(d)=1$ for every $d\in\mathcal{B}_D$ and therefore

$$\sum_{\substack{n=1\\(n,D)=1}}^{\infty}\frac{g^*(n)z^{\Omega(n)}}{n^s}=\sum_{n=1}^{\infty}\frac{g(n)z^{\Omega(n)}}{n^s}\times\prod_{p|D}\left(1-\frac{z}{p^s}\right)\qquad (z\in\mathbb{C},\ |z|<2,\ s>1).$$

Let

$$M_k(x|D):=\sum_{\substack{n\le x\\(n,D)=1\\\Omega(n)=k}}g^*(n);\qquad M_k(x|1)=M_k(x).$$

Then

$$M_k(x|D)=\sum_{d|D}\mu(d)M_{k-\Omega(d)}\left(\frac{x}{d}\right). \tag{5.3}$$

Using Theorem 3 of Kátai and Subbarao [3], we obtain that

$$\lim_{x\to\infty}\sup_{\xi\in[\delta,2-\delta]}\left|\frac{1}{N_k(x)}\sum_{\substack{n\le x\\n\in\mathcal{N}_k}}g(n)-\psi(\tau)\right|=0$$

uniformly for $\tau \in [-C, C]$, where C is a positive constant depending only on g (that is, on f^*), implying that, in light of (5.2),

$$\lim_{x\to\infty} \sup_{\xi\in[\delta,2-\delta]} \left| \frac{1}{N_k(x)} \sum_{\substack{n\le x\\ n\in\mathcal{N}_k}} g^*(n) - \psi(\tau) \right| = 0. \tag{5.4}$$

Now, for each divisor d in (5.3), we have

$$\frac{k-\Omega(d)}{x_2} \ge \delta - \frac{\Omega(d)}{x_2} \ge \frac{\delta}{2} \quad \text{provided } x \text{ is large enough.}$$

Thus, applying (5.4) with $\delta/2$ in place of δ, we get that

$$M_{k-\Omega(d)}\left(\frac{x}{d}\right) = (1+o(1))N_k(x|D)\psi(\tau) \qquad (x\to\infty) \tag{5.5}$$

uniformly as D runs over the integers satisfying $P(D) \le w_x$ and $|\tau| \le C$.

Now, for $Y \ge 2$, let Q_Y stand for $\displaystyle\prod_{p\le Y} p$ and $\mathcal{B}_Y$ for the multiplicative semigroup generated by $\{p \in \wp : p \le Y\}$.

Observe that

$$\pi_k(x) = \sum_{d\in\mathcal{B}_Y} \pi_{k-\omega(d)}\left(\frac{x}{d}|Q_Y\right). \tag{5.6}$$

Now split the right hand side of (5.6) as follows:

$$\pi_k(x) = \sum_{d\le Y^Y} + \sum_{d>Y^Y} = \Sigma_1 + \Sigma_2, \tag{5.7}$$

say. First, we have, using the Hardy-Ramanujan inequality $\pi_k(x) \le C\dfrac{x}{x_1}\dfrac{(x_2+c)^{k-1}}{(k-1)!}$ uniform in k (see Hardy and Ramanujan [2]),

$$\begin{aligned}
\Sigma_2 &\le \sum_{\substack{Y^Y\le d\le\sqrt{x}\\ d\in\mathcal{B}_Y}} \frac{x}{dx_1}\frac{(x_2+c)^{k-\omega(d)-1}}{(k-\omega(d)-1)!} + O\left(x\sum_{\substack{d>\sqrt{x}\\ d\in\mathcal{B}_Y}}\frac{1}{d}\right)\\
&\le C\pi_k(x)\sum_{\substack{Y^Y\le d\le\sqrt{x}\\ d\in\mathcal{B}_Y}}\frac{1}{d}\left(\frac{k+c}{x_2}\right)^{\omega(d)} + O\left(\frac{x}{x^{1/4}}\sum_{d\in\mathcal{B}_Y}\frac{1}{\sqrt{d}}\right).
\end{aligned} \tag{5.8}$$

Clearly,

$$\sum_{d\in\mathcal{B}_Y}\frac{1}{\sqrt{d}} = \prod_{p\le Y}\left(1-\frac{1}{\sqrt{p}}\right)^{-1} < \exp\left\{\sum_{p\le Y}\frac{1}{\sqrt{p}}\right\} < \exp\{c\sqrt{Y}\}. \tag{5.9}$$

Thus, it follows from (5.9) that if $E(x)$ stands for the error term in (5.8) and if we choose $Y = Y(x) = x_3$, then we clearly have

$$E(x) \ll x^{4/5}, \tag{5.10}$$

say. On the other hand,

$$\sum_{\substack{Y^Y\le d\le\sqrt{x}\\ d\in\mathcal{B}_Y}}\frac{1}{d}\left(\frac{k+c}{x_2}\right)^{\omega(d)} \le \frac{1}{Y^{Y/2}}\prod_{p\le Y}\left(1+\left(\frac{k+c}{x_2}\right)\left(1+\frac{1}{\sqrt{p}}+\frac{1}{p}+\dots\right)\right)$$

$$= \frac{1}{Y^{Y/2}} \prod_{p \le y} \left(1 + \frac{(k+c)/x_2}{\sqrt{p}-1}\right) \to 0 \text{ as } Y = Y(x) \to \infty. \tag{5.11}$$

Hence, using (5.10) and (5.11) in (5.8) and in light of (5.7), we can replace (5.6) by

$$\pi_k(x) = (1+o(1)) \sum_{\substack{d \in \mathcal{B}_Y \\ d \le Y^Y}} \pi_{k-\omega(d)}\left(\frac{x}{d}|Q_Y\right) \qquad (x \to \infty). \tag{5.12}$$

Now, consider the two expressions

$$S_k(x) = \sum_{\substack{n \le x \\ n \in \wp_k}} e^{i\tau\nu^*(n)} \qquad \text{and} \qquad S_k(x|D) = \sum_{\substack{n \le x \\ n \in \wp_k \\ (n,D)=1}} e^{i\tau\nu^*(n)}.$$

Then, by choosing $Y = w_x$, we may write

$$\begin{aligned} S_k(x) &= \sum_{m \in \mathcal{B}_{w_x}} \sum_{\substack{n \le x/m \\ n \in \wp_{k-\omega(m)} \\ (n,Q_{w_x})=1}} e^{i\tau\nu^*(n)} \\ &= \sum_{\substack{m \in \mathcal{B}_{w_x} \\ m < w_x^{w_x}}} S_{k-\omega(m)}\left(\frac{x}{m}|Q\right) + o(\pi_k(x)) \qquad (x \to \infty). \end{aligned} \tag{5.13}$$

We will now show that the proportion of the non-squarefree integers belonging to the set $\{n \le x : \omega(n) = k,\ p(n) > w_x\}$ is small.

Setting

$$Q := Q_{w_x} \qquad \text{and} \qquad h(n) := \sum_{\substack{p^a \| n \\ a \ge 2}} 1,$$

then we may write

$$\sum_{\substack{n \le x \\ (n,Q)=1,\ n \in \mathcal{N}_k}} h(n) = \sum_{\substack{p^a \le \sqrt{x} \\ a \ge 2 \\ p > Y}} N_{k-a}\left(\frac{x}{p^a}|Q\right) + O\left(x \sum_{\substack{p^a > \sqrt{x} \\ a \ge 2}} \frac{1}{p^a}\right). \tag{5.14}$$

Now, using Theorem 5 from the book of Tenenbaum ([5], page 205) and using relation (5.1), one can prove that

$$\begin{aligned} N_k(x|D) &= \left(\sum_{\delta | D} \frac{\mu(\delta)}{\delta} \xi^{\Omega(\delta)}\right) N_k(x)(1+o(1)) \\ &= \prod_{p|D} \left(1 - \frac{\xi}{p}\right) N_k(x)(1+o(1)), \end{aligned} \tag{5.15}$$

as $x \to \infty$. Using (5.15) in (5.14), we obtain that

$$\begin{aligned} \sum_{\substack{n \le x \\ (n,Q)=1,\ n \in \mathcal{N}_k}} h(n) &\le cN_k(x|Q) \sum_{\substack{p > w_x \\ \alpha \ge 2}} \frac{1}{p^\alpha} + O(x^{3/4}) \\ &\le \varepsilon_1(x) N_k(x|Q), \end{aligned} \tag{5.16}$$

where $\varepsilon_1(x) \to 0$ as $x \to \infty$.

Then, from Lemma 1, we have

$$
\begin{aligned}
\sum_{\substack{n\le x\\ n\in\wp_k,\ (n,Q)=1}} h(n) &\le \sum_{\substack{p^\alpha\\ p>Y}} \sum_{\substack{p^\alpha\|n\\ n=p^\alpha m\le x\\ m\in\wp_{k-1},\ (m,Q)=1}} 1\\
&\le \sum_{\substack{p^\alpha,p>Y\\ \alpha\ge 2}} \pi_{k-1}(\frac{x}{p^\alpha}|Q)+O(x^{3/4})\\
&\le c\prod_{p|Q}\left(1-\frac{\xi}{p(1-\frac{1-\xi}{p})}\right)\sum_{p^\alpha\le x}\pi_{k-1}\left(\frac{x}{p^\alpha}\right)+O(x^{3/4})\\
&\ll \pi_k(x|Q)\sum_{p>Y,\ \alpha\ge 2}\frac{1}{p^\alpha}+O(x^{3/4})\\
&= \pi_k(x|Q)\varepsilon_3(x)+O(x^{3/4}), \qquad (5.17)
\end{aligned}
$$

where $\varepsilon_3(x)\to 0$ as $x\to\infty$, thus proving our claim that we may ignore those non-squarefree integers for which $\omega(n)=k$ and $p(n)>w_x$.

Hence, from (5.17), we get that

$$
\begin{aligned}
N_k(x|Q) &= \sum_{\substack{n\le x\\ n\in\mathcal{N}_k,\ (n,Q)=1}} |\mu(n)| + \sum_{\substack{n\le x\\ n\in\mathcal{N}_k,\ (n,Q)=1\\ n \text{ not squarefree}}} 1\\
&= \sum_{\substack{n\le x\\ n\in\mathcal{N}_k,\ (n,Q)=1}} |\mu(n)| + O(\varepsilon_3(x)\pi_k(x|Q)), \qquad (5.18)
\end{aligned}
$$

say, where we used (5.17).

We can now move to estimate the main term on the right hand side of (5.13). To do so, we make use of (5.5), which, in light of (5.18), yields. as $x\to\infty$,

$$
\begin{aligned}
S_{k-\omega(m)}\left(\frac{x}{m}|Q\right) &= M_{k-\omega(m)}\left(\frac{x}{m}|Q\right)+o\left(\pi_{k-\omega(m)}\left(\frac{x}{m}|Q\right)\right)\\
&= (1+o(1))N_{k-\omega(m)}\left(\frac{x}{m}|Q\right)\psi(\tau)\\
&= (1+o(1))\pi_{k-\omega(m)}\left(\frac{x}{m}|Q\right)\psi(\tau), \qquad (5.19)
\end{aligned}
$$

since $|\psi(\tau)|>c$ for some positive constant c on every finite interval $|\tau|<B$.

Substituting (5.19) in (5.13), we get that, uniformly for $\dfrac{k}{x_2}\in[\delta,2-\delta]$,

$$
S_k(x)=(1+o(1))\pi_k(x)\psi(\tau) \qquad (x\to\infty),
$$

thus completing the proof of Theorem 1.

6. Proof of Theorem 2

We will use the method developed in the paper of Bassily, Kátai and Wijsmuller [1].

We first introduce the sequence of completely multiplicative functions τ_ℓ, $\ell=0,1,\ldots$, which we define on primes p by

$$
\tau_0(p)=1, \qquad \tau_\ell(p)=\sum_{q|p-1}\tau_{\ell-1}(q) \quad \text{for each } \ell\ge 1.
$$

From this definition, it is clear that

$$0 \le \omega(\varphi_\ell(n)) \le \tau_\ell(n) \qquad \text{for all integers } n \ge 1,\ \ell \ge 0.$$

Note also that Kátai and Subbarao [3] proved that

$$A_\ell(x) = \sum_{p \le x} \frac{\tau_\ell(p)}{p} = \frac{1}{(\ell+1)!} x_2^{\ell+1} + O(x_2^\ell), \tag{6.1}$$

$$B_\ell^2(x) = \sum_{p \le x} \frac{\tau_\ell^2(p)}{p} = \frac{x_2^{2\ell+1}}{(2\ell+1)(\ell!)^2} + O\left(\frac{x_2^{2\ell+1/2}}{2\ell+1/2}\right). \tag{6.2}$$

Definition. *We say that the primes $q_0, q_1, \ldots, q_\ell$ constitute an ℓ-chain if $q_{i-1} | q_i - 1$ for $i = 1, \ldots, \ell$. We denote by $Q_\ell(n)$ those ℓ-chains such that $q_\ell | n$ and by $Q_\ell(n, q_0)$ those ℓ-chains with $q_\ell | n$ and starting with q_0, in which case we write*

$$q_0 \to q_1 \to \ldots \to q_\ell, \quad q_\ell | n.$$

Given a positive integer $n \in \mathcal{N}_k$ with $n \le x$, we will now count the number of those ℓ-chains $q_0 \to q_1 \to \ldots \to q_\ell$, $q_\ell | n$, for which $x^{1/4} < q_\ell < x$. To do so, let us choose $U \in [x^{1/4}, x]$ and let us count those positive integers $n \in \mathcal{N}_k$ for which there is an ℓ-chain with $q_\ell | n$ and $q_\ell \in [U, 2U]$. For such a q_ℓ to exist, we must have $q_0 | q_1 - 1$, $q_1 | q_2 - 1$, $\ldots$, $q_{\ell-1} | q_\ell - 1$, thus implying that any prime q_ℓ can be considered at most $\tau_{\ell-1}(q_\ell - 1)$ times. Now, for a given prime q_ℓ, if $n = q_\ell m \le x$ with $n \in \mathcal{N}_k$, then we have $m \le x/U$, $m \in \mathcal{N}_{k-1}$, implying that the number of such m's is at most $cN_{k-1}(x/U)$. We have thus established that the number of such chains is

$$\ll \left\{ \sum_{U < q \le 2U} \tau_{\ell-1}(q-1) \right\} N_{k-1}\left(\frac{x}{U}\right).$$

Let us now introduce another definition. Let x be a large number and let $\overline{Q_\ell}(n, q_0)$ stand for the set of ℓ-chains with $q_\ell | n$ which starts at q_0 and such that $q_\ell \le x$. Then, since $|\overline{Q_\ell}(n, q_0)| \ge 1$, we have

$$\begin{aligned} L_k^{(1)} &:= \sum_{n \in \mathcal{N}_k} \sum_{q_0 < y} \left(|\overline{Q_\ell}(n, q_0)| - 1 \right) \\ &\le \sum_{n \in \mathcal{N}_k} \sum_{q_0 < y} |\overline{Q_\ell}(n, q_0)| \\ &\le \sum_{\substack{q_0 \to \ldots \to q_\ell \\ q_\ell < x^{1/4}}} N_{k-1}\left(\frac{x}{q_\ell}\right) \\ &\le cN_{k-1}(x) \sum_{\substack{q_0 \to \ldots \to q_\ell \\ q_\ell < x^{1/4},\ q_0 < y}} \frac{1}{q_\ell} \\ &= cN_{k-1}(x) E_\ell, \end{aligned} \tag{6.3}$$

say.

Now, using Lemma 2.5 of Bassily, Kátai and Wijsmuller [1], we have that

$$E_\ell \le c \sum_{\substack{q_0 \to \ldots \to q_\ell \\ q_0 < y}} \frac{x_2}{q_{\ell-1}} \le cE_{\ell-1}x_2 \le \ldots < c^\ell x_2^\ell E_0, \tag{6.4}$$

where

$$E_0 = \sum_{q_0<y} \frac{1}{q_0} < c \log\log y.$$

Substituting (6.4) in (6.3), and since $N_{k-1}(x) \ll_\delta N_k(x)$, we obtain that

$$L_k^{(1)} < c_1 x_2^\ell \log\log y \cdot N_k(x). \tag{6.5}$$

Now, let

$$L_k^{(2)} := \sum_{n\in\mathcal{N}_k} \sum_{q_0 \ge y} \left(|\overline{Q_\ell}(n,q_0)| - 1\right).$$

Since $|\overline{Q_\ell}(n,q_0)| \ne 1$, it follows that there are at least two chains

$$\begin{aligned} &q_0 \to q_1 \to \ldots \to q_\ell \\ &q_0' \to q_1' \to \ldots \to q_\ell' \end{aligned}$$

such that $q_\ell | n$, $q_\ell' | n$. Using the argument displayed in [1], one can establish that

$$L_k^{(2)} \ll N_k(x) \frac{x_2^{2\ell+1}}{y},$$

so that choosing $y = \log^2 x$, we obtain that

$$L_k^{(2)} = o(N_k(x)) \qquad (x\to\infty). \tag{6.6}$$

It follows from (6.5) and (6.6) that, in order to prove Theorem 2, it is enough to prove it with $\tau_\ell(n)$ instead of $\omega(\varphi_\ell(n))$. Hence we shall prove that, if $\ell \ge 1, a_\ell, b_\ell, \xi = \xi_{k,x}$ are as in Theorem 2 and if we set

$$t_\xi(n) := \frac{\tau_\ell(n) - a_\ell \xi x_2^{\ell+1}}{b_\ell \cdot \sqrt{\xi} \cdot x_2^{\ell+1/2}},$$

then

$$\lim_{x\to\infty} \max_{z\in\mathbb{R}} \max_{\xi\in[\delta,2-\delta]} \left| \frac{1}{N_k(x)} \# \{ n \le x : n \in \mathcal{N}_k,\ t_\xi(n) < z \} - \Phi(z) \right| = 0. \tag{6.7}$$

In order to prove relation (3.1) of Theorem 2, we use Theorem 1, while to prove relation (3.2) of Theorem 2, we use the above Theorem A.

We start by choosing the strongly additive function f defined on primes p by $f(p) = \dfrac{\tau_\ell(p)\cdot(\ell+1)!}{x_2^\ell}$. Then, in light of (6.1),

$$A_x = \sum_{p\le x} \frac{f(p)}{p} = \frac{(\ell+1)!}{x_2^\ell} \sum_{p \le x} \frac{\tau_\ell(p)}{p} = x_2 + O(1).$$

With the additive function f^* defined on primes p by $f^*(p) = f(p) - \dfrac{A_x}{x_2}$, we have, using (6.2),

$$B_x^2 \;=\; \sum_{p\le x} \frac{f^*(p)^2}{p} = \sum_{p\le x} \frac{f(p)^2}{p} - 2\frac{A_x}{x_2} \sum_{p\le x} \frac{f(p)}{p} + \left(\sum_{p\le x} \frac{1}{p} \right) \frac{A_x^2}{x_2^2}$$

$$
\begin{aligned}
&= \frac{(\ell+1)!^2}{x_2^{2\ell}} \sum_{p\le x} \frac{\tau_\ell(p)^2}{p} - 2\frac{A_x}{x_2}\sum_{p\le x}\frac{f(p)}{p} + \left(\sum_{p\le x}\frac{1}{p}\right)\frac{A_x^2}{x_2^2}\\
&= \frac{(\ell+1)!^2}{x_2^{2\ell}} \frac{x_2^{2\ell+1}}{(2\ell+1)(\ell!)^2} + O(x_2^{1/2}) - 2\frac{A_x}{x_2}\sum_{p\le x}\frac{f(p)}{p} + \left(\sum_{p\le x}\frac{1}{p}\right)\frac{A_x^2}{x_2^2}\\
&= \frac{(\ell+1)^2}{2\ell+1}x_2 - 2\left(1+O\left(\frac{1}{x_2}\right)\right)(x_2+O(1)) + (x_2+O(1))\left(1+O\left(\frac{1}{x_2}\right)\right)\\
&= \left(\frac{(\ell+1)^2}{2\ell+1} - 1\right)x_2 + O(\sqrt{x_2}) = \frac{\ell^2}{2\ell+1}x_2 + O(\sqrt{x_2}),
\end{aligned}
$$

thereby satisfying the conditions of Theorem 1 (respectively, Theorem A), thus completing the proof of Theorem 2.

7. Further remarks

Using Theorem 1 and Theorem A along with the method elaborated in the paper of Kátai and Subbarao [3], it is possible to deduce theorems of the same type as that of Theorem 2. For instance, it would be possible to prove the following assertion.

Theorem 3. *Let $a \ge 1$ and $b \ne 0$ be fixed integers. Consider the multiplicative function g defined on primes p by $g(p) = \max(ap+b, 1)$ and assume that there exists a positive constant c such that $g(p^a) < cp^a$ for all prime powers p^a. Assume also that $g(n)$ only takes integer positive values. Further let g_ℓ stand for the ℓ-fold iterate of g. Then, there exist computable positive constants c_ℓ and d_ℓ for which the function*

$$\mu_\ell(n) := \frac{\omega(g_\ell(n)) - c_\ell \xi x_2^{\ell+1}}{d_\ell \xi x_2^{\ell+1/2}}$$

satisfies

$$
\begin{aligned}
\lim_{x\to\infty} \max_{z\in\mathbb{R}} \max_{\xi\in[\delta,2-\delta]} \left| \frac{1}{N_k(x)} \#\{n \le x : n \in \mathcal{N}_k,\ \mu_\ell(n) < z\} - \Phi(z)\right| &= 0,\\
\lim_{x\to\infty} \max_{z\in\mathbb{R}} \max_{\xi\in[\delta,2-\delta]} \left| \frac{1}{\pi_k(x)} \#\{n \le x : n \in \wp_k,\ \mu_\ell(n) < z\} - \Phi(z)\right| &= 0.
\end{aligned}
$$

In particular, Theorem 3 can be applied to the function $g = \sigma$, the sum of the divisors function. It also applies to the multiplicative functions P, P^* and $\widetilde{P}$ defined on prime powers p^a by $P(p^a) = (a+1)p^a - ap^{a-1}$, $P^*(p^a) = 2p^a - 1$ and $\widetilde{P}(p^a) = 2p^a - p^{a-1}$, which were introduced and studied by L. Toth [6].

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ON GENERALIZED GAUSSIAN NUMBERS

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Abstract. We establish some new properties and identities of Generalized Gaussian Numbers (GGN) which are defined recently in [10, 11] parallel to those of Gaussian coefficients. We present generating functions and some properties which are very useful for GGN. We obtain some family of sequences which are unimodal and present the log-concavity property of GGN. Finally, we give a connection of GGN to the Rogers-Szego polynomials.

1. Introduction

In enumerative combinatorics, binomial coefficients and Gaussian coefficients are very important class of fields of studies. While binomial coefficients have interpretation in terms of subset selection, Gaussian coefficients have a classical interpretation related to counting subspaces of a finite vector space. Binomial and Gaussian coefficients are already well studied and well discussed in the past (see [3, 6, 7, 13]).

Recently, Generalized Gaussian Numbers (GGN) which in a special case give Gaussian coefficients are defined and some of their properties parallel to those of Gaussian coefficients are established. Moreover, GGN have an interpretation related to the counting of submodules of a finite module [10, 11]. In [10, 11], the

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authors enumerate the codes over finite rings and define new number sequences by GGN and give some properties.

On the other hand, many integer sequences arising from enumerative combinatorics turn out to be unimodal, or even log-concave. Although proving these properties seem natural, it is sometimes difficult to overcome. A very good survey for these properties is given in [12]. Such as Gaussian coefficients, GGN play an important role in enumerative combinatorics since it presents the number of submodules of a finite module and in a special case GGN give binomial coefficients which are frequently involved in constructing properties of some special numbers.

We summarize some of the fundamental properties of binomial and Gaussian coefficients. Binomial coefficients, $\binom{n}{k}$, satisfy the well known relations: $\binom{n}{k} = \binom{n}{n-k}$, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ and (the identity) $\sum_{k-even} \binom{n}{k} = \sum_{k-odd} \binom{n}{k}$.

The construction of properties and identities of some special numbers is done by binomial coefficients. Hence binomial and Gaussian coefficients (or q-binomials) play an important role in Number Theory. From [3], Gaussian coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{q^{n-i+1}-1}{q^i-1}, \quad \begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1, \quad q \neq 1. \tag{1}$$

Unlike the binomial coefficients Gaussian coefficients have only a limited number of properties. The following properties of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are due to [3]. We have the triangular recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n \\ k-1 \end{bmatrix}_q, \tag{2}$$

the identity

$$\sum_{keven} \begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{kodd} \begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{3}$$

Some properties parallel to those Eq. (1) and Eq. (2) are given in [10] and [11].

The Rogers-Szego polynomial in a single variable, $H_n(t)$, is defined as

$$H_n(t) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k.$$

For nonnegative integers $k_1, k_2, \ldots, k_m$ such that $k_1+k_2+\ldots+k_m = n$, Gaussian multinomial coefficient (or q-multinomial coefficient) of length m is defined in [14] as

$$\begin{bmatrix} n \\ k_1, k_2, \ldots, k_m \end{bmatrix}_q = \frac{(q)_n}{(q)_{k_1}(q)_{k_2}\ldots(q)_{k_m}}$$

where $(q)_n = (1-q)(1-q^2)\ldots(1-q^n)$.

The homogeneous Rogers-Szego polynomial in m variables for $m \geq 2$, denoted by $\hat{H}_n(t_1, t_2, \ldots, t_m)$, is defined in [14] as

$$\hat{H}_n(t_1, t_2, \ldots, t_m) = \sum_{k_1+k_2+\ldots+k_m=n} \left[\begin{array}{c} n \\ k_1, k_2, \ldots, k_m \end{array} \right]_q t_1^{k_1} \ldots t_m^{k_m} \tag{4}$$

and Rogers-Szego polynomial in $m-1$ variables, denoted $H_n(t_1, t_2, \ldots, t_{m-1})$, is defined as

$$H_n(t_1, t_2, \ldots, t_{m-1}) = \hat{H}_n(t_1, t_2, \ldots, t_{m-1}, 1).$$

For any $a, r \neq 1$ and $n \geq 1$, $(a, r)_n = (a; r)_n$ is defined in [14] by

$$(a, r)_n = (1-a)(1-ar)\ldots(1-ar^{n-1}),$$

and $(a, r)_0 = 1$.

When $a = r$, then,

$$(a, a)_n = (1-a)(1-a^2)\ldots(1-a^n).$$

Denote this number by $(a)_n$ shortly.

Rogers-Szego polynomials were first defined by Rogers [8] in terms of their generating function and some researchers have also studied [1, 2, 5] them. They have an important role in combinatorial number theory, symmetric function theory and orthogonal polynomials.

In this paper, we prove the log-concavity and unimodality of a sequence defined by GGN. Moreover, some more properties and identities of the GGN are given. Finally, we point out Rogers-Szego polynomial [8, 9, 14] and its relation to GGN.

2. Recurrence Relations and Generating Functions

In this section, some properties and generating functions of GGN are given. As we already know that the binomial coefficients stand for the number of subsets of a finite set of a particular size. Gaussian binomial coefficients stand for the number of vector spaces of a particular dimension of a finite dimensional space. Recently, a direct calculation of the number of submodules of a particular type of a finite module is introduced and these numbers are called Generalize Gaussian Numbers [11].

Theorem 2.1. [11] *The number of $\mathbb{Z}_q$-submodules of type $(k_1, k_2, \ldots, k_m)$ of the finite module $\mathbb{Z}_q^n$ where $q = p^m$ (p prime and m a positive integer) is*

$$\frac{\prod_{t=1}^{m}\prod_{i=0}^{k_t-1}\left((p^{m-t+1})^n - (p^{m-t})^n \cdot p^{\sum_{j=0}^{t-1} k_j} \cdot p^i\right)}{\prod_{s=1}^{m}\prod_{r=0}^{k_s-1} \cdot A}$$

where

$$A = \prod_{a=1}^{s}(p^{m-s+1})^{k_a}\prod_{j=s+1}^{m}(p^{m-j+1})^{k_j} - \left(\prod_{a=1}^{s}(p^{m-s})^{k_a}\right)(p^{m-s+1})^{k_{s+1}} \cdot \prod_{t=s+2}^{m}(p^{m-t+1})^{k_t} \cdot p^r$$

Denote the number given in the previous theorem by

$$N^q_{k_1,k_2,\ldots,k_m}(n) = \left[\begin{array}{c} n \\ k_1, k_2, \ldots, k_m \end{array} \right]_{\mathbf{Z}_q}$$

where q is a prime power and m is the nilpotency of the generator matrix of the maximal ideal.

Definition 2.1. For $m = 2$ and $q = p^2$, we call the number of $\mathbb{Z}_q$-submodules of type (k_1, k_2) of the finite module$\mathbb{Z}_q^n$ as Generalized Gaussian Numbers (GGN) and denote it by $N^q_{k_1,k_2}(n) = \begin{bmatrix} n \\ k_1, k_2 \end{bmatrix}_{\mathbf{Z}_q}$.

Then, we write by [11]

$$N^q_{k_1,k_2}(n) = \begin{bmatrix} n \\ k_1, k_2 \end{bmatrix}_{\mathbf{Z}_q} =$$

(5)
$$\frac{(p^{2n} - p^n)\dots(p^{2n} - p^{n+k_1-1})(p^n - p^{k_1})\dots(p^n - p^{k_1+k_2-1})}{(p^{2k_1}p^{k_2} - p^{k_1+k_2})\dots(p^{2k_1}p^{k_2} - p^{k_1+k_2+k_1-1})(p^{k_1+k_2} - p^{k_1})\dots(p^{k_1+k_2} - p^{k_1+k_2-1})}.$$

In short, Eq. (5) is written as

$$(6) \qquad \frac{p^{nk_1}\prod_{i=0}^{k_1-1}(p^n - p^i)\prod_{j=0}^{k_2-1}(p^n - p^{k_1+j})}{p^{{k_1}^2+2k_1k_2}\prod_{i=0}^{k_1-1}(p^{k_1} - p^i)\prod_{j=0}^{k_2-1}(p^{k_2} - p^j)}.$$

Theorem 2.2. *GGN satisfies the following triangular recurrence relation*

$$(7) \qquad N^q_{n-(k-1),k}(n+1) = N^q_{n-(k-1),k-1}(n) + p^k N^q_{n-k,k}(n)$$

where $q = p^2$.

Proof. From Eq. (6), we obtain the following

$$(8) \quad N^q_{n-(k-1),k}(n+1) = \frac{p^{(n+1)(n-k+1)}\prod_{i=0}^{n-k}(p^{n+1} - p^i)\prod_{j=0}^{k-1}(p^{n+1} - p^{n-k+1+j})}{p^{(n-k+1)^2+2(n-k+1)k}\prod_{i=0}^{n-k}(p^{n-k+1} - p^i)\prod_{j=0}^{k-1}(p^k - p^j)}$$

(9)
$$N^q_{n-(k-1),k-1}(n) = \frac{p^{n^2-nk+n}\prod_{i=0}^{n-k}(p^n - p^i)\prod_{j=0}^{k-2}(p^n - p^{n-k+1+j})}{p^{(n-k+1)^2+2(n-k+1)(k-1)}\prod_{i=0}^{n-k}(p^{n-k+1} - p^i)\prod_{j=0}^{k-2}(p^{k-1} - p^j)}$$

and

$$(10) \qquad N^q_{n-k,k}(n) = \frac{p^{n(n-k)}\prod_{i=0}^{n-k-1}(p^n - p^i)\prod_{j=0}^{k-1}(p^n - p^{n-k+j})}{p^{(n-k)^2+2(n-k)k}\prod_{i=0}^{n-k-1}(p^{n-k} - p^i)\prod_{j=0}^{k-1}(p^k - p^j)}.$$

In order to carry out the equality, we put the denominators of the equalities (9) and (10) into a common one similar to the denominator of the equality (8). Multiplying both numerator and denominator parts of equation (9) by

$$p^{-2(n-k+1)}p^{k-1}(p^k - 1),$$

we obtain

(11)
$$N^q_{n-(k-1),k-1}(n) = \frac{p^{n^2-nk+3n-k+1}(p^k - 1)\prod_{i=0}^{n-k}(p^n - p^i)\prod_{j=0}^{k-2}(p^n - p^{n-k+1+j})}{p^{(n-k+1)^2+2(n-k+1)k}\prod_{i=0}^{n-k}(p^{n-k+1} - p^i)\prod_{j=0}^{k-1}(p^k - p^j)}$$

Multiplying both numerator and denominator parts of equation (10) by

$$p^{2n+1}p^{n-k}(p^{n-k+1}),$$

we obtain
(12)

$$N^q_{n-k,k}(n+1) = \frac{p^{n^2-nk+1+3n-k}(p^{n-k+1}-1)\prod_{i=0}^{n-k-1}(p^n-p^i)\prod_{j=0}^{k-1}(p^n-p^{n-k+j})}{p^{(n-k+1)^2+2(n-k+1)k}\prod_{i=0}^{n-k}(p^{n-k+1}-p^i)\prod_{j=0}^{k-1}(p^k-p^j)}.$$

By multiplying (12) with p^k and then summing the result with (11) and by applying the necessary operations we obtain the result.

□

Next, we illustrate with a couple of examples.

Example 1. *For values $k=4, n=5, q=4$ we have*

$$N^4_{2,4}(6) = N^4_{2,3}(5) + 2^4 \cdot N^4_{1,4}(5).$$

$$651 = 155 + 2^4 31.$$

□

Example 2. *For values $k=5, n=6, q=4$ we have*

$$N^4_{2,5}(7) = N^4_{2,4}(6) + 2^5 \cdot N^4_{1,5}(6).$$

$$2667 = 651 + 2^5 \cdot 63.$$

□

Theorem 2.3. *GGN satisfies the following relation as a generating function*

$$(p-1)N^q_{k,1}(n) = (p^n-1)N^q_{k,0}(n-1) \tag{13}$$

where $q=p^2$.

Proof.

$$\begin{aligned}
(p-1)N^q_{k,1}(n) &= \frac{(p-1)p^{nk}(p^n-1)(p^n-p)\dots(p^n-p^{k-1})(p^n-p^k)}{p^{k^2+2k}(p^k-1)(p^k-p)\dots(p^k-p^{k-1})(p-1)} \\
&= \frac{p^{nk-2k}(p^n-1)p(p^{n-1}-1)\dots p(p^{n-1}-p^{k-2})p(p^n-p^{k-1})}{p^{k^2}(p^k-1)(p^k-p)\dots(p^k-p^{k-1})} \\
&= \frac{p^{nk-k}(p^n-1)(p^{n-1}-1)\dots(p^{n-1}-p^{k-2})(p^n-p^{k-1})}{p^{k^2}(p^k-1)(p^k-p)\dots(p^k-p^{k-1})} \\
&= \frac{(p^n-1)p^{(n-1)k}(p^{n-1}-1)(p^{n-1}-p)\dots(p^{n-1}-p^{k-1})}{p^{k^2}(p^k-1)(p^k-p)\dots(p^k-p^{k-1})} \\
&= (p^n-1)N^q_{k,0}(n-1).
\end{aligned}$$

This completes the proof. □

Example 3. $n=6, q=p^2=4:$

$$N^4_{3,1}(6) = (2^6-1)N^4_{3,0}(5) \qquad (k=3)$$

$$624960 = (2^6-1)9920,$$

$$N^4_{4,1}(6) = (2^6 - 1)N^4_{4,0}(5) \qquad (k = 4)$$

$$31248 = (2^6 - 1)496, \ \textit{and}$$

$$N^4_{5,1}(6) = (2^6 - 1)N^4_{5,0}(5) \qquad (k = 5)$$

$$63 = (2^6 - 1).$$

□

Example 4. $n = 4, q = p^2 = 9:$

$$2N^9_{3,1}(5) = (3^5 - 1)N^9_{3,0}(4) \qquad (k = 3)$$

$$2.130680 = (3^5 - 1)1080$$

$$2N^9_{4,1}(5) = (3^5 - 1)N^9_{4,0}(4) \qquad (k = 4)$$

$$2.121 = 3^5 - 1.$$

□

Theorem 2.4. [10] *Some of the properties of GGN which are used for generating Gaussian Numbers are given.* $(q = p^2)$

$$\begin{aligned} N^q_{0,k}(n) &= N^q_{0,n-k}(n), \quad N^q_{k,0}(n) = N^q_{n-k,0}(n) \\ N^q_{k_1,k_2}(n) &= N^q_{k_2,k_1}(n), \quad (k_1 + k_2 = n) \\ N^q_{0,n}(n) &= N^q_{n,0}(n) = 1 \\ N^q_{k,0}(n) &= (p^k)^{n-k} N^q_{0,k}(n) \\ N^q_{k_1,k_2}(n) &= N^q_{n-(k_1+k_2),k_2}(n). \end{aligned}$$

Proof. In order to prove the equalities above, we apply Eq. (6) to all of them similar to those we did in Theorem 2.2 and Theorem 2.3. □

$n/(0,k)$	$(\mathbf{0,1})$	$(\mathbf{0,2})$	$(\mathbf{0,3})$	$(\mathbf{0,4})$
1	$S_0(p)$			
2	$S_1(p)$	1		
3	$S_2(p)$	p^2+p+1	1	
4	$S_3(p)$	$p^4+p^3+2p^2+p+1$	p^3+p^2+p+1	1
5	$S_4(p)$	$p^6+p^5+2p^4$ $+2p^3+2p^2+p+1$	$p^6+p^5+2p^4$ $+2p^3+2p^2+p+1$	p^4+p^3 $+p^2+p+1$

TABLE 1. Table of values for $N^{p^2}_{0,k}(n)$ and $S_k(p) = \sum_{i=0}^{k} p^i$.

$n/(k,1)$	$(\mathbf{0},\mathbf{1})$	$(\mathbf{1},\mathbf{1})$	$(\mathbf{2},\mathbf{1})$	$(\mathbf{3},\mathbf{1})$
1	$S_0(p)$			
2	$S_1(p)$	$p+1$		
3	$S_2(p)$	$p^4+2p^3+2p^2+p$	p^2+p+1	
4	$S_3(p)$	$p^7+2p^6+3p^5+$ $+3p^4+2p^3+p^2$	$p^7+2p^6+3p^5+$ $+3p^4+2p^3+p^2$	p^3+p^2+p+1
5	$S_4(p)$	$p^{10}+2p^9+3p^8$ $+4p^7+4p^6+$ $3p^5+2p^4\ +p^3$	$p^{12}+2p^{11}+4p^{10}$ $+5p^9+6p^8+5p^7$ $+4p^6+2p^5+p^4+$ $3p^3+2p^2+2p+1$	$p^{10}+2p^9+3p^8$ $+4p^7+4p^6+$ $3p^5+2p^4\ +p^3$

Table 2. Table of values for $N^{p^2}_{k,1}(n)$

Using the properties given by Eq. (7), Eq. (13) and the properties given in Theorem 2.4 we can quickly generate the values of $N^q_{k_1,k_2}(n)$ as shown in the following table where $q=p^2$.

One may try to obtain the numbers by different primes p and n using Table 1 or Table 2. They may be generalized to any n by Eq. (7), Eq. (13) and Theorem 2.4. Of course these tables are only two small examples of those sequences given by GGN. On the other hand, we can easily see some properties if we write the entries of (for example) Table 1 in the following way:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & p+1 & & 1 & & & \\
 & & 1 & & p^2+p+1 & & p^2+p+1 & & 1 & & \\
 & 1 & & p^3+p^2+p+1 & & p^4+p^3+2p^2+p+1 & & p^3+p^2+p+1 & & 1 & \\
1 & & p^4+p^3+p^2+p+1 & & p^6+p^5+2p^4+2p^3+2p^2+p+1 & & p^6+p^5+2p^4+2p^3+2p^2+p+1 & & p^4+p^3+p^2+p+1 & & 1
\end{array}$$

Figure 1. Pascal's Type triangle

The figure is analogous to Pascal's Triangle of the usual binomial coefficients. It has a symmetry like Pascal's Triangle. This figure is also the same as Gaussian coefficients. The properties are similar to those in Gaussian coefficients. However, the properties which are obtained by Table 2 are not exactly the same as the usual Gaussian numbers or binomial coeeficients since GGN is more general than the others.

3. Loq-Concavity, Unimodality of The Sequences and Rogers-Szego Polynomials

We prove that some of the number sequences which are obtained from GGN are log-concave. We present a family of sequences which are unimodal. We also generalize these sequences. Here q is always equal to p^2.

Definition 3.1. [12] A sequence is said to be unimodal if for some $0 \leq j \leq n$ we have $a_0 \leq a_1 \leq \ldots \leq a_j \geq a_{j+1} \geq \ldots \geq a_n$ and is said to be logarithmically concave (or log-concave for short) if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $1 \leq i \leq n-1$.

Theorem 3.1. *The sequence $N_{k,0}^q(n)$ satisfies the log-concavity property.*

Proof. In order to prove, we have to show that the fraction

$$\frac{(N_{k,0}^q(n))^2}{N_{k-1,0}^q(n)N_{k+1,0}^q(n)}$$

is larger than 1.

$$\begin{aligned}
\left(N_{k,0}^q(n)\right)^2 &= \left[\frac{(p^{2n}-p^n)(p^{2n}-p^{n+1})\ldots(p^{2n}-p^{n+k-1})}{(p^{2k}-p^k)(p^{2k}-p^{k+1})\ldots(p^{2k}-p^{k+k-1})}\right]^2 \\
N_{k-1,0}^q(n) &= \frac{(p^{2n}-p^n)(p^{2n}-p^{n+1})\ldots(p^{2n}-p^{n+k-1-1})}{(p^{2(k-1)}-p^{k-1})(p^{2(k-1)}-p^k)\ldots(p^{2k}-p^{k-1+k-1-1})} \qquad (14) \\
N_{k+1,0}^q(n) &= \frac{(p^{2n}-p^n)(p^{2n}-p^{n+1})\ldots(p^{2n}-p^{n+k+1-1})}{(p^{2(k+1)}-p^{k+1})(p^{2(k+1)}-p^{k+2})\ldots(p^{2(k+1)}-p^{k+1+k+1-1})}.
\end{aligned}$$

From the equalities given by Eq. (14), we write

$$\begin{aligned}
&\frac{(N_{k,0}^q(n))^2}{N_{k-1,0}^q(n)N_{k+1,0}^q(n)} = \\
&= \frac{\frac{(p^{2n}-p^n)(p^{2n}-p^{n+1})\ldots(p^{2n}-p^{n+k-1})}{(p^{2k}-p^k)(p^{2k}-p^{k+1})\ldots(p^{2k}-p^{k+k-1})}\,\frac{(p^{2n}-p^n)\ldots(p^{2n}-p^{n+k-1})}{(p^{2k}-p^k)(p^{2k}-p^{k+1})\ldots(p^{2k}-p^{k+k-1})}}{\frac{p^{2n}-p^{n+k-1}}{p^{2n}-p^{n+k-1}}\,\frac{(p^{2n}-p^n)(p^{2n}-p^{n+1})\ldots(p^{2n}-p^{n+k-1-1})}{(p^{2(k-1)}-p^{k-1})\ldots(p^{2k}-p^{k-1+k-1-1})}\,\frac{(p^{2n}-p^n)\ldots(p^{2n}-p^{n+k+1-1})}{(p^{2(k+1)}-p^{k+1})\ldots(p^{2(k+1)}-p^{k+1+k+1-1})}} \\
&= \frac{\frac{1}{(p^{2k}-p^k)(p^{2k}-p^{k+1})\ldots(p^{2k}-p^{2k-1})}}{\frac{p^{2n}-p^{n+k}}{(p^{2(k-1)}-p^{k-1})\ldots(p^{2(k-1)}-p^{2k-3})(p^{2(k+1)}-p^{k+1})\ldots(p^{2(k+1)}-p^{2k+1})(p^{2n}-p^{n+k-1})}} \\
&= \frac{1}{(p^{2k}-p^k)\ldots(p^k-p^{2k-1})(p^{2k}-p^k)\ldots(p^{2k}-p^{2k-1})(p^{2n}-p^{n+k})} \\
&\quad \cdots(p^{2(k-1)}-p^{k-1})(p^{2(k-1)}-p^k)\ldots(p^{2(k-1)}-p^{2k-3})(p^{2(k+1)}-p^{k+1})\ldots \\
&\quad \cdots(p^{2(k+1)}-p^{2k+1})(p^{2n}-p^{n+k-1})
\end{aligned}$$

We multiply both numerator and denominator parts of the last equality by k^2. Then the last equality is equal to

$$k^2\frac{(p^{2(k+1)}-p^{k+1})(p^{2n}-p^{n+k-1})}{(p^{2(k-1)}-p^{k-2})(p^{2n}-p^{n+k})}. \qquad (15)$$

Multiplying the Eq. (15) by p^4, we obtain

$$\frac{p^4k^2(p^{2(k+1)}-p^{k+1})(p^{2n}-p^{n+k-1})}{(p^{2(k+1)}-p^{k+2})(p^{2n}-p^{n+k})} \tag{16}$$

Now it is obvious that Eq. (16) is greater than 1. □

Example 5. *$q=4, n=4$:*

$(N^4_{2,1}(4))^2 = 420^2, N^4_{1,1}(4) = 420, N^4_{3,1}(4) = 15,\quad \frac{420^2}{420.15} = 28 > 1,$

$q=4, n=5$:

$(N^4_{1,3}(5))^2 = 930^2, N^4_{0,3}(5) = 155, N^4_{2,3}(5) = 155,\quad \frac{930^2}{155.155} = 36 > 1,$

$q=4, n=6$:

$(N^4_{2,2}(6))^2 = 364560^2, N^4_{1,2}(6) = 78120, N^4_{0,2}(6) = 651,\quad \frac{364560^2}{78120.651} \sim 2614 > 1.$ □

Example 6. *$q=9, n=3$:*

$(N^9_{1,1}(3))^2 = 156^2, N^4_{0,1}(3) = 13, N^4_{2,1}(3) = 13,\quad \frac{156^2}{13.13} = 144 > 1$

$q=9, n=5$:

$(N^9_{2,0}(5))^2 = 882090^2, N^4_{1,0}(5) = 980, N^4_{3,0}(5) = 882090,\quad \frac{882090^2}{980.882090} \sim 900 > 1.$ □

Lemma 3.2. *The sequence $N^q_{k,1}(n)$ is unimodal where $k \leq n-1$.*

Proof. We write below the sequence for all possible values of $k = 0, 1, 2, \ldots, n-1$ and any $q = p^2$:

$N^q_{0,1}(n) \qquad N^q_{1,1}(n) \qquad N^q_{2,1}(n) \qquad \ldots \qquad N^q_{n-2,1}(n) \qquad N^q_{n-1,1}(n)$

We apply Eq. (5) to the entries of the sequence:

(17)

$$N^q_{0,1}(n) = \frac{p^n - p^0}{p - p^0} = p^{n-1} + p^{n-2} + \ldots + p + 1,$$

$$\begin{aligned} N^q_{1,1}(n) &= \frac{(p^{2n} - p^n)(p^n - p)}{(p^3 - p^2)(p^2 - p)} = \frac{p^n(p^n - 1)p(p^{n-1} - 1)}{p^2(p-1)p(p-1)} \\ &= p^{n-2}(p^{n-1} + p^{n-2} + \ldots + 1)(p^{n-2} + p^{n-3} + \ldots + 1) \end{aligned}$$

$$\begin{aligned} N^q_{2,1}(n) &= \frac{(p^{2n} - p^n)(p^{2n} - p^{n+1})(p^n - p^2)}{(p^5 - p^3)(p^5 - p^4)(p^3 - p^2)} = \frac{p^n(p^n - 1)p^{n+1}(p^{n-1} - 1)p^2(p^{n-2} - 1)}{p^3(p^2 - 1)p^4(p-1)p^2(p-1)} \\ &= \frac{p^{2n-6}(p^{n-1} + p^{n-2} + \ldots + p + 1)(p^{n-2} + \ldots + p + 1)(p^{n-3} + \ldots + p + 1)}{p+1} \end{aligned}$$

$$\begin{aligned} N^q_{3,1}(n) &= \frac{(p^{2n} - p^n)(p^{2n} - p^{n+1})(p^{2n} - p^{n+2})(p^n - p^3)}{(p^7 - p^4)(p^7 - p^5)(p^7 - p^6)(p^4 - p^3)} \\ &= \frac{1}{(p+1)^2} p^{3n-12}(p^{n-1} + \ldots + p + 1)(p^{n-2} + \ldots + p + 1)(p^{n-3} + \ldots + p + 1) \cdot \\ &\quad \cdot (p^{n-4} + \ldots + p + 1) \end{aligned}$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$N^q_{n-2,1}(n) = \frac{(p^{2n} - p^n)(p^{2n} - p^{n+1}) \ldots (p^{2n} - p^{2n-3})(p^n - p^{n-2})}{(p^{2n-3} - p^{n-1})(p^{2n-3} - p^n) \ldots (p^{2n-3} - p^{2n-4})(p^{n-1} - p^{n-2})}.$$

We multiply both numerator and denominator parts of the equality (17) by $p^{3(n-2)}$ and have

$$\begin{aligned} N^q_{n-2,1}(n) &= \frac{(p^{2n} - p^n)(p^{2n} - p^{n+1}) \ldots (p^{2n} - p^{2n-3})(p^n - p^{n-2})}{(p^{2n} - p^{n+2})(p^{2n} - p^{n+3}) \ldots (p^{2n} - p^{2n-1})(p^{n-1} - p^{n-2})} \\ &= \frac{p^n(p^n - 1)p^{n+1}(p^{n-1} - 1)(p^n - p^{n-2})p^{3n-6}}{p^{2n-2}(p^2 - 1)p^{2n-1}(p-1)(p^{n-1} - p^{n-2})} \\ &= p^{n-2}(p^{n-1} + \ldots + p + 1)(p^{n-2} + \ldots + p + 1). \end{aligned} \tag{18}$$

Finally, in a similar way we obtain

$$N^q_{n-1,1}(n) = p^{n-2} + p^{n-1} + \ldots + p + 1)(p^{n-3} + \ldots + p + 1. \tag{19}$$

If n is odd then the middle term, $\left(\frac{n+1}{2}\right)$–th term, is $N^q_{\frac{n+1}{2},1}(n)$. The sequence is increasing until the middle term and decreasing after it.

If n is even then we have two middle terms, $N^q_{\frac{n}{2},1}(n)$, $N^q_{\frac{n}{2}+1,1}(n)$, whose values are the same. This may be ensured by the equalities given by (17)-(18). Moreover, $5th$ equality of Theorem 2.3 guarantees this result.

Hence the sequence $N^q_{k,1}(n)$ is unimodal.

For odd n

$$\begin{array}{ccccccccccc} N^q_{0,1}(n) & & N^q_{1,1}(n) & & N^q_{2,1}(n) & \dots & N^q_{\frac{n+1}{2},1}(n) & \dots & N^q_{n-2,1}(n) & & N^q_{n-1,1}(n) \\ & \nearrow & & \nearrow & & \nearrow & & \searrow & \searrow & & \searrow \end{array}$$

For even n

$$\begin{array}{ccccccccccc} N^q_{0,1}(n) & & N^q_{1,1}(n) & & N^q_{2,1}(n) & \dots & N^q_{\frac{n}{2},1}(n) & & N^q_{\frac{n}{2}+1,1}(n) & \dots & N^q_{n-2,1}(n) \\ & \nearrow & & \nearrow & & \nearrow & & \rightarrow & \searrow & \searrow & \searrow \end{array}$$

□

Example 7. *$q = 4, n = 6$:*

$N^4_{0,1}(6)$	$N^4_{1,1}(6)$	$N^4_{2,1}(6)$	$N^4_{3,1}(6)$	$N^4_{4,1}(6)$	$N^4_{5,1}(6)$
63	31248	624960	624960	31248	63

$q = 4, n = 7$:

$N^4_{0,1}(7)$	$N^4_{1,1}(7)$	$N^4_{2,1}(7)$	$N^4_{3,1}(7)$	$N^4_{4,1}(7)$	$N^4_{5,1}(7)$	$N^4_{6,1}(7)$
127	256032	21165312	90708480	21165312	256032	127

$q = 9, n = 3$:

$N^9_{0,1}(3)$	$N^9_{1,1}(3)$	$N^9_{2,1}(3)$
13	156	13

$q = 9, n = 4$:

$N^9_{0,1}(4)$	$N^9_{1,1}(4)$	$N^9_{2,1}(4)$	$N^9_{3,1}(4)$
40	4680	4680	40.

□

Lemma 3.3. *The sequence $N^q_{k,2}(n)$ is unimodal where $k \leq n-2$.*

Proof. We can give the proof in the similar way as in Lemma 3.2. We write below the sequence for all possible values of $k = 0, 1, 2, \dots, n-2$ and any $q = p^2$:

$$N^q_{0,2}(n) \quad N^q_{1,2}(n) \quad N^q_{2,2}(n) \quad \dots \quad N^q_{n-3,2}(n) \quad N^q_{n-2,2}(n)$$

We apply Eq. (5) to the entries of the sequence:

$$N^q_{0,2}(n) = \frac{(p^n-1)(p^n-p)}{(p^2-1)(p^2-p)} = \frac{(p^{n-1}+p^{n-2}+\dots+1)(p^{n-1}+p^{n-2}+\dots+1)}{p+1} \tag{20}$$

(21)
$$\begin{aligned} N^q_{1,2}(n) &= \frac{(p^{2n}-p^n)(p^n-p)(p^n-p^2)}{(p^4-p^3)(p^3-p)(p^3-p^2)} \\ &= \frac{p^{n-3}(p^{n-1}+p^{n-2}+\ldots+1)(p^{n-2}+p^{n-3}+\ldots+1)(p^{n-3}+p^{n-4}+\ldots+1)}{p+1} \end{aligned}$$

(22)
$$N^q_{2,2}(n) = \frac{p^{2n-8}(\sum_{i=0}^{n-1}p^i)(\sum_{i=0}^{n-2}p^i)(\sum_{i=0}^{n-3}p^i)(\sum_{i=0}^{n-4}p^i)}{(p+1)^2}$$

(23)
$$\begin{aligned} N^q_{n-3,2}(n) &= \frac{(p^{2n}-p^n)(p^{2n}-p^{n+1})\ldots(p^{2n}-p^{2n-4})(p^n-p^{n-3})(p^n-p^{n-2})}{(p^{2n-4}-p^{n-1})(p^{2n-4}-p^n)\ldots(p^{2n-4}-p^{2n-5})(p^{n-1}-p^{n-3})(p^{n-1}-p^{n-2})} \\ &= \frac{p^{n-3}(p^{n-1}+p^{n-2}+\ldots+1)(p^{n-2}+p^{n-3}+\ldots+1)(p^{n-3}+p^{n-4}+\ldots+1)}{p+1}. \end{aligned}$$

(24)
$$\begin{aligned} N^q_{n-2,2}(n) &= \frac{(p^{2n}-p^n)(p^{2n}-p^{n+1})\ldots(p^{2n}-p^{2n-4})(p^n-p^{n-3})(p^n-p^{n-2})}{(p^{2n-4}-p^{n-1})(p^{2n-4}-p^n)\ldots(p^{2n-4}-p^{2n-5})(p^{n-1}-p^{n-3})(p^{n-1}-p^{n-2})} \\ &= \frac{(p^{n-1}+p^{n-2}+\ldots+1)(p^{n-1}+p^{n-2}+\ldots+1)}{p+1}. \end{aligned}$$

Hence, the expressions in the above equations have the following relations

$$Eq.\ (20) < Eq.\ (21) < Eq.\ (22) < \ldots$$

The inequality continues in the same way until the term $N^q_{\frac{n}{2},2}(n)$ for even n and $N^q_{\frac{n+1}{2},2}(n)$ (also $N^q_{\frac{n+1}{2}+1,2}(n)$) for odd n. After the terms $N^q_{\frac{n}{2},2}(n)$ and $N^q_{\frac{n+1}{2}+1,2}(n)$, the sequence is decreasing because of the equalities given by (20)-(24). This proves the Lemma. □

Example 8. $q = 4, n = 6$:

$N^4_{0,2}(6)$	$N^4_{1,2}(6)$	$N^4_{2,2}(6)$	$N^4_{3,2}(6)$	$N^4_{4,2}(6)$
651	78120	36456	78120	651

$q = 4, n = 7$:

$N^4_{0,2}(7)$	$N^4_{1,2}(7)$	$N^4_{2,2}(7)$	$N^4_{3,2}(7)$	$N^4_{4,2}(7)$	$N^4_{5,2}(7)$,
2667	1322832	26456640	26456640	1322832	2667

$q = 25, n = 3$:

$N^{25}_{0,2}(4)$	$N^{25}_{1,2}(4)$	$N^{25}_{2,2}(4)$
806	4030	806.

□

Theorem 3.4. *The sequence $N^q_{k_1,k_2}(n)$ is unimodal where $k_1 + k_2 \le n$ and $q = p^2$.*

Proof. By Theorem 2.4, Lemma 3.2 and Lemma 3.3 we obtain the result. $\square$

In the following theorem, we give a connection between GGN and Rogers-Szego polynomials.

Prior to the theorem, we first write (6) as

$$\left[\begin{array}{c} n \\ k_1, k_2 \end{array}\right]_{\mathcal{Z}_q} = \frac{\overbrace{p^{nk_1}\prod_{i=0}^{k_1-1}(p^n - p^i)}^{A}\,\overbrace{p^{k_1k_2}\prod_{j=0}^{k_2-1}(p^{n-k_1} - p^j)}^{B}}{\underbrace{p^{{k_1}^2+k_1k_2}\prod_{l=0}^{k_1-1}(p^{k_1} - p^l)}_{C}\,\underbrace{p^{k_1k_2}\prod_{m=0}^{k_2-1}(p^{k_2} - p^m)}_{D}}. \tag{25}$$

Theorem 3.5. *The following equality holds:*

$$A \cdot \hat{H}_n(t_1, t_2, t_3) = \sum_{k_1+k_2+k_3=n} N \cdot t_1^{k_1} t_2^{k_2} t_3^{k_3} \tag{26}$$

where $A = p^{2nk_1+nk_2-2{k_1}^2-{k_2}^2-2k_1k_2}$ *and* $N = \left[\begin{array}{c} n \\ k_1, k_2 \end{array}\right]_{\mathcal{Z}_q}$ $(q = p^2)$.

Proof. By using the notation introduced in Eq. (25), part A can be rewritten as

$$p^{nk_1}\prod_{i=0}^{k_1-1}(p^n - p^i) = p^{nk_1}p^{nk_1}(1-(\frac{1}{p})^n)(1-(\frac{1}{p})^{n-1})\ldots(1-(\frac{1}{p})^{n-k_1+1})$$

$$A = p^{2nk_1}\frac{(1-(\frac{1}{p}))(1-(\frac{1}{p})^2)\ldots(1-(\frac{1}{p})^n)}{(1-(\frac{1}{p}))(1-(\frac{1}{p})^2)\ldots(1-(\frac{1}{p})^{n-k_1})} = p^{2nk_1}\frac{(\frac{1}{p},\frac{1}{p})_n}{(\frac{1}{p},\frac{1}{p})_{n-k_1}}, \tag{27}$$

similarly part B can be rewritten as

$$p^{k_1k_2}\prod_{j=0}^{k_2-1}(p^{n-k_1} - p^j) =$$

$$p^{k_1k_2}p^{(n-k_1)k_2}(1-(\frac{1}{p})^{n-k_1})(1-(\frac{1}{p})^{n-k_1-1})\ldots(1-(\frac{1}{p})^{n-k_1-k_2+1})$$

$$B = p^{nk_2}\frac{(1-(\frac{1}{p}))(1-(\frac{1}{p})^2)\ldots(1-(\frac{1}{p})^{n-k_1})}{(1-(\frac{1}{p}))(1-(\frac{1}{p})^2)\ldots(1-(\frac{1}{p})^{n-k_1-k_2})} = p^{nk_2}\frac{(\frac{1}{p},\frac{1}{p})_{n-k_1}}{(\frac{1}{p},\frac{1}{p})_{n-k_1-k_2}}, \tag{28}$$

part C can be rewritten as

$$p^{{k_1}^2+k_1k_2}\prod_{l=0}^{k_1-1}(p^{k_1} - p^l) =$$

$$p^{{k_1}^2+k_1k_2}p^{{k_1}^2}(1-(\frac{1}{p})^{k_1})(1-(\frac{1}{p})^{k_1-1})\ldots(1-(\frac{1}{p})^{k_1-k_1+1})$$

$$C = p^{2{k_1}^2+k_1k_2}(\frac{1}{p},\frac{1}{p})_{k_1}, \tag{29}$$

and finally part D can be rewritten as

$$p^{k_1k_2}\prod_{m=0}^{k_2-1}(p^{k_1}-p^m) = p^{k_1k_2}p^{{k_2}^2}(1-(\frac{1}{p})^{k_2})(1-(\frac{1}{p})^{k_2-1})\dots(1-(\frac{1}{p})^{k_2-k_2+1})$$

$$D = p^{k_1k_2+{k_2}^2}(\frac{1}{p},\frac{1}{p})_{k_2}. \tag{30}$$

Then, by using the new expressions of A, B, C, D in (25)

$$\begin{bmatrix} n \\ k_1, k_2 \end{bmatrix}_{\mathcal{Z}_q} = \frac{p^{2nk_1}\frac{(\frac{1}{p},\frac{1}{p})_n}{(\frac{1}{p},\frac{1}{p})_{n-k_1}}p^{nk_2}\frac{(\frac{1}{p},\frac{1}{p})_{n-k_1}}{(\frac{1}{p},\frac{1}{p})_{n-k_1-k_2}}}{p^{2{k_1}^2+k_1k_2}(\frac{1}{p},\frac{1}{p})_{k_1}p^{k_1k_2+{k_2}^2}(\frac{1}{p},\frac{1}{p})_{k_2}} \tag{31}$$

Take $\frac{1}{p}=q$:

$$\begin{bmatrix} n \\ k_1, k_2 \end{bmatrix}_{\mathcal{Z}_q} = p^{2nk_1+nk_2-2{k_1}^2-{k_2}^2-2k_1k_2}\frac{(q,q)_n(q,q)_{n-k_1}}{(q,q)_{n-k_1}(q,q)_{k_1}(q,q)_{n-k_1-k_2}(q,q)_{k_2}} \tag{32}$$

For $(q)=(q,q)_n$ in (32),

$$\begin{bmatrix} n \\ k_1, k_2 \end{bmatrix}_{\mathcal{Z}_q} = p^{2nk_1+nk_2-2{k_1}^2-{k_2}^2-2k_1k_2}\frac{(q)_n(q)_{n-k_1}}{(q)_{n-k_1}(q)_{k_1}(q)_{n-k_1-k_2}(q)_{k_2}} \tag{33}$$

The following two equalities are well known:

$$\begin{bmatrix} n \\ k_1 \end{bmatrix}_q = \frac{(q)_n}{(q)_{n-k_1}(q)_{k_1}}, \quad \begin{bmatrix} n-k_1 \\ k_2 \end{bmatrix}_q = \frac{(q)_{n-k_1}}{(q)_{n-k_1-k_2}(q)_{k_2}}.$$

Then, (33) is equal to the number

$$p^{2nk_1+nk_2-2{k_1}^2-{k_2}^2-2k_1k_2}\begin{bmatrix} n \\ k_1 \end{bmatrix}_q\begin{bmatrix} n-k_1 \\ k_2 \end{bmatrix}_q. \tag{34}$$

Moreover, if we do some abbreviation in (33), we obtain (35):

$$p^{2nk_1+nk_2-2{k_1}^2-{k_2}^2-2k_1k_2}\frac{(q)_n}{(q)_{k_1}(q)_{k_2}(q)_{n-k_1-k_2}} \tag{35}$$

$$= p^{2nk_1+nk_2-2{k_1}^2-{k_2}^2-2k_1k_2} \underbrace{\begin{bmatrix} n \\ k_1, k_2, n-k_1-k_2 \end{bmatrix}_q} \tag{36}$$

$(k_3 = n - k_1 - k_2, \quad A = p^{2nk_1+nk_2-2{k_1}^2-{k_2}^2-2k_1k_2})$

$$\begin{bmatrix} n \\ k_1, k_2 \end{bmatrix}_{\mathcal{Z}_q} = A \quad \underbrace{\begin{bmatrix} n \\ k_1, k_2, k_3 \end{bmatrix}_q}_{q-multinomial \quad coefficient} \tag{37}$$

We let $N = \begin{bmatrix} n \\ k_1, k_2 \end{bmatrix}_{\mathcal{Z}_q}$. Then $\begin{bmatrix} n \\ k_1, k_2, k_3 \end{bmatrix}_q = \frac{N}{A}$.

The underlined expression in (36) and (37) is due to the definition given in [14]. The homogeneous Rogers-Szego polynomial in 3 variables is the following:

$$\hat{H}_n(t_1, t_2, t_3) = \sum_{k_1+k_2+k_3=n} \underbrace{\begin{bmatrix} n \\ k_1, k_2, k_3 \end{bmatrix}_q}_{\frac{N}{A}} t_1^{k_1} t_2^{k_2} t_3^{k_3},$$

where $N = \begin{bmatrix} n \\ k_1, k_2 \end{bmatrix}_q$ is GGN and A is equal to the number

$$A = p^{2nk_1+nk_2-2{k_1}^2-{k_2}^2-2k_1k_2}.$$

Then the result is obvious:

$$A \cdot \hat{H}_n(t_1, t_2, t_3) = \sum_{k_1+k_2+k_3=n} N \cdot t_1^{k_1} t_2^{k_2} t_3^{k_3} \tag{38}$$

□

4. Conclusion

In this paper, we have developed some functions and properties for Generalized Gaussian Numbers which are related to the number of submodules of a finite module. We present that some families of sequences which are obtained via GGN are log-concave and unimodal and we give some examples. As a future work, these studies may be generalized to GGN for any m and some further relations may be obtained between GGN and Rogers-Szego polynomials.

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ON SOME ORDER 6 NON-SYMPLECTIC AUTOMORPHISMS OF ELLIPTIC K3 SURFACES.

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Abstract. We classify non-symplectic automorphisms of order 6 on elliptic K3 surfaces which commute with a given elliptic fibration. We show how their study can be reduced to the study of non-symplectic automorphisms of order 3 and to a local analysis of the fibers. In particular, we determine the possible fixed loci and give their location on the singular fibers. When the Picard lattice is fixed, we show that K3 surfaces come in mirror pairs.

1. Introduction

An automorphism of a K3 surface is called non-symplectic when the induced action on the holomorphic 2-form is non-trivial. The study of non-symplectic automorphisms was pioneered by Nikulin [Nik81] who analyzed the case of involutions. Since then, these automorphisms have been extensively studied by several authors. Let us mention Vorontsov [Vor83], Kondō [Kon86, Kon92], Xiao Gang [Xia96], Machida and Oguiso [MO98], Oguiso and Zhang [OZ98, OZ00], Zhang [Zha07], Artebani and Sarti [AS08], and Artebani, Sarti and Taki [AST11]. From these works, we now know that if a K3 surface admits a non-symplectic automorphism, then the surface is algebraic and the Euler totient function evaluated at the order of the automorphism is at most 66. Moreover, non-symplectic automorphisms of prime order have been classified, a synthetic classification can be found in [AST11], and some authors have started to investigate the simulatenous existence of symplectic and non-symplectic automorphisms [Fra11].
One of the reasons behind the interest in non-symplectic involutions is the mirror

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construction of Borcea [Bor97] and Voisin [Voi93]. They construct Calabi-Yau and an explicit mirror map using, as building blocks K3 surfaces with non-symplectic involutions, and elliptic curves. This construction can be extended to K3 surfaces with non-symplectic automorphisms of order 3, 4 and 6 [Dil06].

In this paper, using the classification of non-symplectic automorphisms of order 3 [AS08], we study order 6 automorphisms of elliptic K3 surfaces commuting with the fibration by performing a combinatorial analysis of the action on the fixed locus.

2. Plan

In Section 3, we define primitive non-symplectic automorphisms and fix the notation for the rest of the paper. In Section 4, we give the final classification of all possible fixed loci. In Section 5, we show that the fixed locus of a primitive non-symplectic automorphism of order 6 consists in a disjoint union of points, rational curves, and possibly one genus one curve. Our cases fall thus in two distinct situations which are analyzed in Sections 6 and 7. In the last Section, 8, we focus on the special case where ζ fixes the Picard lattice.

3. Notation

Let X be a smooth projective K3 surface and ζ an automorphism of X. The induced action of ζ on $H(X, \Omega^2) \simeq \mathbb{C}$ gives rise to a character χ. An automorphism is called *symplectic*, if it lies in the kernel of χ, and *non-symplectic* otherwise. If the order of ζ and $\chi(\zeta)$ agree, then ζ is called *primitive*. In the rest of the article, ζ will be a primitive non-symplectic automorphism of order 6 acting on X.

As suggested by Cartan [Car57], given a fixed point P of ζ, we can linearize the action around it. Since ζ is of order 6 and primitive, the linearized action can be written as

$$\begin{pmatrix} \xi_6^k & 0 \\ 0 & \xi_6^{k'} \end{pmatrix}$$

where $(k,k') \in \{(0,1); (2,5); (3,4)\}$ and ξ_6 is a primitive 6^{th} root of unity. While the first case corresponds to P lying on a fixed smooth curve, the last two options correspond to P being isolated. We will use the standard notation and say that P is of type $\frac{1}{6}(k,k')$. Since ζ is primitive, its iterates will also be non-symplectic. We will denote their fixed locus by $X^{[i]} = \{x \in X \text{ s.t. } \zeta^i x = x\}$. The components of the $X^{[i]}$ will be described by the following variables:

- $p_{\frac{1}{n}(k,k')}$: number of isolated fixed points of type $\frac{1}{n}(k,k')$ in $X^{\left[\frac{6}{n}\right]}$, for $n \in \{6,3,2\}$.
- $l^{[i]}$: number of rational curves in $X^{[i]}$.
- $g^{[i]}$: maximal genus among the curves in $X^{[i]}$.
- $g_M = \max\{1, g^{[1]}\}$.

When referring to [AS08], we will use their notation, namely:

- g: highest genus of the curves in $X^{[2]}$.
- n: number of fixed points in $X^{[2]}$ (all are of type $\frac{1}{3}(2,2)$).
- k: total number of curves in $X^{[2]}$.

4. Results

Our first result is a global description of the fixed locus of ζ.

Theorem 4.1. *The fixed locus $X^{[1]}$ consists of one of the two following collections:*

1. *a smooth genus 1 curve and three isolated fixed points of type $\frac{1}{6}(2,5)$.*
2. *a disjoint union of smooth rational curves and points, $C_1 \sqcup \ldots C_l \sqcup P_1 \sqcup \ldots \sqcup P_{p_{\frac{1}{6}(3,4)}+p_{\frac{1}{6}(2,5)}}$, satisfying*

$$p_{\frac{1}{6}(3,4)} + 2p_{\frac{1}{6}(2,5)} - 6l^{[1]} = 6. \tag{4.1}$$

The proof of this Theorem follows from Section 5.
From our analysis in Sections 7 and 6 we obtain the following:

Classification 4.2. *Let $X \to \mathbb{P}^1$ be an elliptic K3 surface and ζ a primitive non-symplectic automorphism of order 6 preserving the elliptic fibration. The fixed locus of ζ is one of the configurations listed in Table 1 or consists of the disjoint union of a genus 1 curve and three isolated points.*

Reading Table 1: First note that since ζ preserves the fibration, we have an induced action, ψ, on the basis. The order of ψ is either one or two.
In the Table, each row begins by a description of $X^{[2]}$, the fixed locus of ζ^2. After that comes a list of the singular fibers of the fibration (x is the number of fibers of type X). Then, we give a description of $X^{[1]}$ when ψ is the identity. Finally, the last two groups refer to the case where ψ is an involution; we list the fibers above the two fixed points, and the components of $X^{[1]}$.

5. Study of the fixed locus

Lemma 5.1. *The fixed locus of ζ consists of a disjoint union of smooth curves and points*

$$X^{[1]} = C_0 \sqcup \ldots C_m \sqcup P_1 \sqcup \ldots \sqcup P_{p_{\frac{1}{6}(3,4)}+p_{\frac{1}{6}(2,5)}}$$

with $g(C_0) \geq 0 = g(C_1) = \ldots = g(C_m)$.

Proof. The first part of the statement follows from the Hodge Index Theorem. The argument is analogue to those found in [Nik81, Voi93, Dil06, AS08]:
A disjoint union of smooth curves on a K3 surface can have at most one element with strictly positive self-intersection. By adjunction, that is a curve of genus at least 2.
If a curve has self-intersection 0, then it is an elliptic curve and induces an elliptic fibration $\pi : X \to \mathbb{P}^1$. Since the action is non-symplectic, it descends non-trivially to the base and fixes two points. The fixed locus of ζ is thus a component of the fibers above these two points. One of the fibers is the original fixed curve. The remaining curves of the fixed locus are either a smooth elliptic curve or a disjoint union of rational components of one of Kodaira's singular fibers. So either the fixed locus is as the one described in the statement, or it consists exactly in the disjoint union of two genus 1 curves. However, if $X^{[1]}$ were to contain two genus 1 curves, then so would $X^{[2]}$ and this option was ruled out in [AS08]. □

Lemma 5.2. *The components of $X^{[1]}$ satisfy*

$$p_{\frac{1}{6}(3,4)} + 2p_{\frac{1}{6}(2,5)} - 6l^{[1]} + 6g_M = 12. \tag{5.1}$$

Table 1. Fixed locus when X is ζ-elliptic.

#	g	n	k	ii	iv	ii^*	iv^*	$p_{(3,4)}$	$p_{(2,5)}$	$l^{[1]}-1$	F_0	F_∞	$p_{(3,4)}$	$p_{(2,5)}$	$l^{[1]}$
1	5	0	2	12	0	0	0	12	0	0	I_0	I_0	6	0	0
2	4	1	2	10	1	0	0	10	1	0	I_0	IV	4	1	0
3	3	2	2	8	2	0	0	8	2	0	I_0	I_0	6	0	0
4	2	3	2	6	3	0	0	6	3	0	I_0	IV	4	1	0
5	3	3	3	8		0	1	10	1	0	I_0	IV^*	6	3	1
6	1	4	2	4	4	0	0	4	4	0	I_0	I_0	6	0	0
7	2	4	3	6	1	0	1	8	2	0	IV	IV^*	4	4	1
8	3	4	4	7	0	1	0	10	4	1					
9	0	5	2	2	5	0	0	2	5	0	I_0	IV	4	1	0
10	1	5	3	4	2	0	1	6	3	0	I_0	IV^*	6	3	1
11	2	5	4	5	1	1	0	8	5	1					
12	0	6	3	2	3	0	1	4	4	0	IV	IV^*	4	4	1
13	1	6	4	3	2	1	0	6	6	1					
14	0	7	4	1	3	1	0	4	7	1					
15	1	7	5	3	0	1	1	8	5	1					
16	0	8	5	1	1	1	1	6	6	1					
17	1	8	6	2	0	2	0	8	8	2	I_0	I_0	6	0	0
18	0	9	6	0	1	2	0	6	9	2	I_0	I_0	4	1	0
3'	3	2	2	8	2	0	0	8	2	0	IV	IV	2	2	0
6'	1	4	2	4	4	0	0	4	4	0	IV	IV	2	2	0

Proof. The formula is simply the Lefschetz holomorphic formula, see [TT75], applied to ζ. $\square$

We will use the classification, determined by Artebani and Sarti [AS08], of non-symplectic automorphism of order 3 to find information on $X^{[2]}$, which in turn will yield us data on the nature of $X^{[1]}$. We first recapitulate the results of [AS08], which we will use later, and then relate the fixed loci $X^{[1]}$ and $X^{[2]}$.

Theorem 5.3. [AS08, Proposition 4.2] *Let σ be a non-symplectic automorphism of order 3 acting on a K3 surface X. If the fixed locus of σ contains two or more curves, then X is isomorphic to an elliptic K3 surface whose Weierstrass equation is*

$$y^2 = x^3 + p_{12}(t)$$

and on which σ acts as $(x, y, t) \mapsto (\zeta_3^2 x, y, t)$.

Proposition 5.4. [AS08, Corollary 4.3] *Let σ and X be as in the statement of Theorem 5.3. If $X^{[1]}$ contains a curve C of positive genus, then C is a double section of the Weierstrass fibration, i.e. C is hyperelliptic.*

Lemma 5.5. *If $P \in X^{[1]}$ is of type $\frac{1}{6}(2,5)$, then it is also an isolated point in $X^{[2]}$. If $P \in X^{[1]}$ is of type $\frac{1}{6}(3,4)$, then it lies on a smooth curve in $X^{[2]}$. Moreover, one has the following inequalities $p_{\frac{1}{3}(2,2)} \geq p_{\frac{1}{6}(2,5)}$ and $l^{[2]} \geq l^{[1]}$.*

Proof. The first two statements are obvious after one takes the square of the matrix giving the localized action of ζ at P. The inequalities ensue. $\square$

Corollary 5.6. *If $X^{[1]}$ contains at least two distinct curves or a curve and an isolated point of type $\frac{1}{6}(3,4)$, or more generally, if $X^{[2]}$ contains at least two distinct curves, then X is isomorphic to an elliptic K3 surface whose Weierstrass equation is*

$$y^2 = x^3 + p_{12}(t).$$

Proof. Lemma 5.5 implies that $X^{[2]}$ contains at least two distinct curves. The first part of the statement follows thus directly from Theorem 5.3. $\square$

In the rest of the paper, we will focus on elliptic K3 surfaces. The following two Lemmas show that this is a somewhat mild restriction.

Lemma 5.7. *Let X be a K3 surface and τ a non-symplectic automorphism of order 3 which preserves the fibration. Then there exists a primitive non-symplectic automorphism ζ of X such that $\tau = \zeta^2$ and X is ζ-elliptic.*

Proof. We know from 5.3 that X is of the form $y^2 = x^3 + p_{12}(t)$ and τ acts as $(x, y, t) \mapsto (\xi_3 x, y, t)$. It is easy to see that $\sigma : (x, y, t) \mapsto (\xi_3 x, -y, t)$ acts on X and has the required properties. $\square$

Lemma 5.8. *If the fixed locus of a non-symplectic automorphism of order 6 contains a rational curve then X is elliptic.*

Proof. When the fixed locus contains at least one curve, formula 5.1 reduces to $p_{\frac{1}{6}(3,4)} + 2p_{\frac{1}{6}(2,5)} - 6l^{[1]} = 6$. If $p_{\frac{1}{6}(3,4)}$ is strictly positive, then Corollary 5.6 implies that X is elliptic. Otherwise, $p_{(2,5)} \geq 6$ and thus n is an odd number greater than or equal to 6. From [AS08, Table 2] one can see that all cases where $X^{[2]}$ contains a rational curve and where n is an odd number larger than 6 are elliptic. $\square$

While Lemma 5.7 shows that any automorphism of order 3 preserving the fibration factors through an automorphism of order 6, this automorphism does not have to be unique. It is possible that a generic automorphism of order 6 when applied twice gives an automorphim of order 3 commuting with the fibration. From now on we will focus on automorphims of order 6 that actually do preserve the elliptic fibration:

Definition 5.9. *If X, ζ are as in the statement of Corollary 5.6, we will say that X is ζ-*elliptic *if ζ preserves the elliptic fibration.*

In this situation, we have an induced action on the basis which we will denote by $\psi = \pi \circ \zeta \circ \pi^{(-1)}$. Indeed, the action of ζ preserves the fibration, i.e. $\pi \circ \zeta \circ \pi^{(-1)}$ is well defined, and ζ^2 acts as $(x, y, t) \mapsto (\zeta_3^2 x, y, t)$. In particular, this implies that the induced action on the base is at most of order 2 and if this induced action is trivial, then ζ restricts to an action of order 6 on each fiber.

An important property of automorphisms which preserve fibrations is given by the following statement:

Lemma 5.10. *If X is ζ-elliptic than $X^{[i]}$ does not contain curves of strictly positive genus.*

Proof. Assume that $X^{[1]}$ contains a curve C_0 which is not rational. Proposition 5.4 tells us that C_0 is a double section of the fibration and therefore, the action induced on the base is trivial. We are then in the situation where ζ induces an automorphism of order 6 on each fiber. Moreover, since ζ fixes at least two points per fiber, the points of intersection with C_0, it ought to be the identity: a contradiction. □

Remark. If X is ζ-elliptic, Lemma 5.10 tells us that the fixed locus of ζ contains no curves of positive genus. However, one could have an automorphism of order 6 commuting with an elliptic fibration and fixing a curve of genus one. Indeed, the proof of the Lemma does not exclude the fixed curve to be a fiber itself.

We can thus conclude

Conclusion 5.11. *The fixed locus of ζ consists either of a disjoint union of smooth rational curves and points or of a configuration containing possibly one elliptic curve. In Section 6.2, we show that the genus one situation is actually unique; all other cases, which correspond to X ζ-elliptic, are discussed in Section 7.*

6. The case where the fixed locus contains a genus 1 curve

Lemma 6.1. *If $g(C_0) = 1$ then $p_{\frac{1}{6}(3,4)} = l^{[1]} = 0$, $p_{\frac{1}{6}(2,5)} = 3$ and $l^{[2]} = 0$, $p_{\frac{1}{3}(2,2)} = 3$.*

Proof. If $l^{[1]}$ or $p_{\frac{1}{6}(3,4)}$ were to be strictly positive, Corollary 5.6 would imply that X is ζ-elliptic contradicting Lemma 5.10. Formula 5.1 gives us the value of $p_{\frac{1}{6}(2,5)}$. Similarly, the case $l^{[2]} > 0$ is excluded as we would reach a similar contradiction. Finally, the value for $p_{\frac{1}{3}(2,2)} = 3$ can be found in [AS08, Table 1]. □

A non-symplectic automorphim of order 6 which fixes a smooth elliptic curves fixes thus also three isolated points and nothing else. Actually,

Proposition 6.2. *If the fixed locus of a non-symplectic automorphism contains an elliptic curve C_0, then the action is defined uniquely i.e., the fixed loci of ζ, ζ^2 and ζ^3 are determined uniquely.*

- *The fixed locus of ζ and ζ^2 are identical : $X^{[1]}$ and $X^{[2]}$ consist of C_0 and three isolated points – as described in the previous Lemma.*
- *The fixed locus of ζ^3 is a superset of the previous fixed loci : it consists of C_0 and a second smooth elliptic curve C_1.*

Proof. Consider the elliptic fibration given by the linear system $|C_0|$. Since C_0 is in the fixed locus, the induced action on the base is of order 6, i.e. it is a cyclic action with two fixed points: the image of C_0 and some additional point Q. Since the Euler characteristic of a K3 surface is 24, the Euler characteristic of the fiber above Q is a multiple of 6. From Kodaira's classification of the possible singular fibers, the fiber above Q is of the type I_{6N} or I^*_{6N}. However, as will follow from Section 7.1, only in the case I_0 does ζ not fix any rational curves. The fiber above Q is thus smooth and the fixed loci of ζ and its powers are readily found. $\square$

Remark. An example of a K3 with a primitive non-symplectic automorphism of order 6 fixing an elliptic curve is given by the surface $y^2 = x^3 + (t^6 - 1)^2$, where the action is $\zeta : (x, y, t) \mapsto (x, y, \xi_6 t)$. The volume form $\omega = \frac{dx \wedge dt}{dy}$ gets mapped to $\zeta^* \omega = \xi_6 \omega$.

7. Elliptic case

In this Section we consider X to be ζ-elliptic. The induced automorphism, ψ, on $\mathbb{P}^1$, is either trivial or an involution. The two cases are analyzed respectively in Sections 7.2 and 7.3. Our discussion begins in Section 7.1 where we analyze how ζ acts on the fibers of π.

7.1. **Local analysis.** Let X be a K3 surface. The *Gram graph* of X is the incidence graph of the effective smooth rational curves on X. E.g., when the Picard lattice of X is isomorphic to $U \oplus E_8^2$ of S_X, then the Gram graph is as in figure 1. Let D be an effective divisor on X and ζ an automorphism of X. We call D *stable* if $\zeta(D) = D$, and we say that D is *fixed* if $\zeta|_D = \mathrm{id}$.

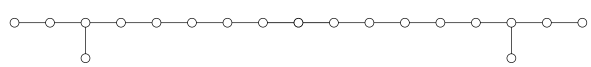

Figure 1. Gram graph of $U \oplus E_8^2$.

Lemma 7.1. *Consider a tree of rational curves on a surface X which are stable componentwise under the action of a primitive non-symplectic automorphism of order 6. Then, the points of intersection of the rational curves are fixed and the action at one fixed point determines the action on the whole tree.*

Proof. The key in this proof is to realize that the action of the automorphism on a given rational component and the action on a fixed point of this curve determine each other completely. Recall that an action of $\mathbb{C}$ will be of the form $z \mapsto \lambda z$, $\lambda \in \mathbb{C}^*$ under suitable coordinates. Now, λ is nothing but the eigenvalue associated to the fixed point of coordinate 0, or the inverse of the eigenvalue associated to the fixed point at infinity. Conversely, if one knows one eigenvalue of the automorphism localized at a point, then one knows the full action at that point. First, the eigendirections correspond to the components of the tree passing through the point Second, since the three types of points, $\frac{1}{6}(3,4)$, $\frac{1}{6}(2,5)$ and $\frac{1}{6}(1,0)$, all have distinct eigenvalues it is clear to which eigenvalue corresponds each direction. $\square$

Remark. It follows from the proof of the previous lemma that if we look at the types of points of intersection on a chain of smooth rational curves, these will embed in the following periodic sequence:

$$\ldots, \frac{1}{6}(2,5), \frac{1}{6}(3,4), \frac{1}{6}(3,4), \frac{1}{6}(2,5), \frac{1}{6}(1,0), \frac{1}{6}(1,0), \ldots$$

Example 7.2. *Consider a type IV^* configuration of rational curves which is stable under the action of ζ, a non-symplectic automorphism of order 6. Moreover, assume that it contains on one of the weight 1 curves, L, a point P of type $\frac{1}{6}(3,4)$, such that the eigendirection corresponding to the eigenvalue -1 is transversal to the L. Using Lemma 7.1 we can determine the action on the entire configuration. This action is illustrated in Figure 2.*

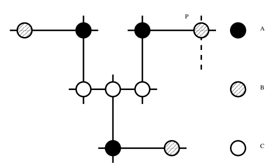

FIGURE 2. Study of the action on a type IV^* configuration.

From now on, we will focus only on fibers of type I_0, II, II^*, IV and IV^*. We will denote by ii the number of type II fibers, ii^* the number of type II^* fibers, etc. We focus our attention on these fibers because of the following theorem:

Proposition 7.3. [AS08, Proposition 4.2]*) Let X be ζ-elliptic, then the numbers (n,k) determine uniquely ii, ii^*, iv and iv^*. More precisely, π has*

1. *n type IV fibers if $k=2$.*
2. *$n-3$ type IV fibers and 1 type IV^* fiber if $k=3$.*
3. *$n-4$ type IV fibers and 1 type II^* fiber if $k=4$.*
4. *$n-7$ type IV fibers, 1 type IV^* fiber and and 1 type II^* fiber if $k=5$.*
5. *$n-8$ type IV and 2 type II^* fibers if $k=6$.*

Lemma 7.4. *Let X be ζ-elliptic, ψ trivial, and assume that X has a fiber of type II, IV, II^* or IV^*. When restricted to those fibers, ζ*

1. *fixes 1 point of type $\frac{1}{6}(3,4)$, namely the cuspidal point of the fiber. (Fiber of type II)*
2. *fixes 3 points of type $\frac{1}{6}(3,4)$, 4 points of type $\frac{1}{6}(2,5)$ and 1 rational curve. (Fiber of type II^*)*
3. *fixes 1 point of type $\frac{1}{6}(2,5)$, namely the common intersection point. (Fiber of type IV)*
4. *fixes 2 points of type $\frac{1}{6}(3,4)$ and 1 point of type $\frac{1}{6}(2,5)$. (Fiber of type IV^*)*

Moreover, π has also a section fixed by ζ and it is the only part of $X^{[1]}$ not completely included in the fibers.

Proof. Since ψ is trivial, the fibers are preserved by ζ. Thus either ζ fixes the zero section σ_0, or there is another section σ_1 and ζ permutes the two. Assume σ_0 is not fixed. Pick a smooth fiber F of π. The automorphism ζ^2 is of order 3 on F and fixes the 2 points of intersection with the two sections σ_0 and σ_1. Therefore, there is a third fixed point. Since ζ permutes the first two, it fixes the third one. Since ψ is trivial, this point is of type $\frac{1}{6}(1,0)$ and there is a fixed section passing through that point.

Let us describe the action on the fibers explicitly.

1. (II) The point of intersection with the fixed section is not the node, as the section intersects the fiber with multiplicity 1, and is of type $\frac{1}{6}(0,1)$. On the other hand, the other fixed point, which ought to be the node, is of type $\frac{1}{6}(3,4)$.
2. (II^*) Since the Gram graph of this fiber has no non-trivial $\mathbb{Z}/2\mathbb{Z}$ automorphism, the curve of weight 6 is fixed. The remaining fixed points can be found using Lemma 7.1.
3. (IV) The section of the Weierstrass fibration is fixed and intersects the fiber at the curve of weight 1. Using Lemma 7.1 we see that there is a unique possible action, namely the one permuting the two other branches.
4. (IV^*) The action on the fiber follows from lemma 7.1 and is described in figure 3. The black dot corresponds to a point of type $\frac{1}{6}(2,5)$ and the two white dots to points of type $\frac{1}{6}(3,4)$.

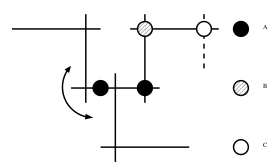

FIGURE 3. Action on a type IV^* fiber when ψ is trivial.

$\square$

Lemma 7.5. *Let X be ζ-elliptic and assume ψ is an involution. Let F be a fiber preserved by ζ, i.e. $\zeta(F) = F$. F is of type I_0, IV or IV^*. Moreover, ζ*

1. *fixes 3 points of type $\frac{1}{6}(3,4)$, when F is smooth.*
2. *fixes 1 point of type $\frac{1}{6}(3,4)$ and 1 point of type $\frac{1}{6}(2,5)$, when F is of type IV. This case is depicted in Example 7.2.*
3. *fixes 3 points of type $\frac{1}{6}(3,4)$, 3 points of type $\frac{1}{6}(2,5)$ and 1 rational curve, when F is of type IV^*.*

Moreover, every component of $X^{[1]}$ lies in one of those fibers.

Proof. Without loss of generality, we can assume that ψ is of the form $[x_0 : x_1] \mapsto [-x_0 : x_1]$, or $t \mapsto -t$. Since, the Weierstrass equation $y^2 = x^3 + p_{12}(t)$ is invariant under ψ, this implies that that the roots of p_{12} are double at 0 and ∞. The fibers which correspond to double roots are those of type I_0, IV and IV^*. Alternatively, one can perform a local analysis on the fibers, and see that these are the only possibilities. This analysis will also give us the exact nature of the fixed locus on each fiber. Take a fixed point P at the intersection of the section of π and a fiber F. Since ψ is an involution, the eigenvalue corresponding to the direction of the section is -1. Using Lemma 7.1 we can describe the local action in each case:

1. (I_0) There are 3 fixed points of type $\frac{1}{6}(3,4)$.
2. (IV) There is 1 fixed point of type $\frac{1}{6}(3,4)$, and one of type $\frac{1}{6}(2,5)$.
3. (IV^*) There are 3 points of type $\frac{1}{6}(3,4)$, 3 points of type $\frac{1}{6}(2,5)$ and 1 rational curve.

□

7.2. **Induced action on the base is trivial.**

Lemma 7.6. *Let X be ζ-elliptic, ψ trivial. Then $X^{[2]}$ determines $X^{[1]}$. The possibilities are listed in Table 1.*

Proof. From Proposition 7.3, we know that (n,k) determines the types of fibers of π. Since, Lemma 7.4 tells us that the action of ζ on each fiber is unique, it follows that $X^{[1]}$ is completely determined by $X^{[2]}$. □

Remark. Unfortunately, the converse is not true: the simple combinatorial data describing $X^{[1]}$ does not determine uniquely X or $X[2]$. See examples 4 and 10 in Table 1.

Finally, the existence of all the examples in Table 1 follows from Lemma 5.7.

7.3. **Induced action on the base is an involution.**

Lemma 7.7. *The fixed locus $X^{[1]}$ is contained in 2 fibers of π.*

Proof. Since ψ is an involution, it has two fixed points on $\mathbb{P}^1$, say 0 and ∞. Since all the fibers not above these points are permuted, $X^{[1]}$ is a subset of the fibers F_0 and F_∞ $(F_i = \pi^{(-1)}(i))$. □

Lemma 7.8. *Let X be a K3 surface and τ a non-symplectic automorphism of order 3 which is τ-elliptic. Call π the associated fibration. Assume that the multiset X_π of singular fibers of π can be decomposed $F \sqcup M$ with F a multiset of cardinality 2 whose elements come from $\{I_0, IV, IV^*\}$ and where each element of M has even multiplicity. Then there exists a pair (X', τ') consisting of a K3 surface and a non-symplectic automorphism of order 3 such that X' is τ'-elliptic, $X_\pi = X'_{\pi'}$, and $X^\tau = X'^{\tau'}$. Moreover, τ' factors as $\tau' = \zeta^2$ where ζ is a primitive non-symplectic automorphism of order 6 commuting with π'.*

Proof. This follows from the local analysis in Lemma 7.5, or from the fact that the only singular fibers corresponding to double roots of $p_{12}(t)$ are those of I_0, IV, and IV^*. □

Since the action on F_0 and F_∞ is determined by Lemma 7.5, we simply list all possibilities in Table 1.

8. Fixed Picard lattice.

A special case of the above classification consists of analysing only those automorphisms which fix the Picard group. Although this can be recovered from the previous sections, we will try to analyse the case separately to make the analogy with automorphisms of order 2 and 3 as studied by [Nik81] and [AS08].
Recall that for a K3 surface, X, the cohomology $H^2(X,\mathbb{Z})$ is a unimodular lattice of signature $(3,19)$ i.e., it is isomorphic to $U^3 \oplus E_8^2$. Also, it decomposes into the Picard lattice, S_X, and the transcendental lattice T_X:

$$H^2(X,\mathbb{Z}) \cong S_X \oplus T_X.$$

Given a lattice A, we will write $A^{\perp}$ for its orthogonal complement and A^* for its dual $\mathrm{Hom}(A,\mathbb{Z})$. We say that a lattice is *p-elementary* when $A^*/A = (\mathbb{Z}/p\mathbb{Z})^k$ for some $k \in \mathbb{N}$.

Lemma 8.1. *Let ζ be a primitive non-symplectic automorphism of order 6 of X which preserves the Picard lattice, then the Picard lattice $S_X = H^2(X,\mathbb{Z})^{\zeta}$ is a unimodular.*

Proof. Fix $p \in \{2,3\}$, and let $\zeta_p = (\zeta^*)^{6/p}$. The quotients S_X^*/S_X and $\left(S_X^{\perp}\right)^*/S_X^{\perp} = T_X^*/T_X$ are isomorphic. Hence, $p = 1 + \zeta_p^* + ... + (\zeta_p^*)^{p-1} = 0$ on T_X and $pT_X^* \subset T_X$. Since S_X is both 2 and 3 elementary, it is unimodular. □

Corollary 8.2. *The Picard lattice S_X is isomorphic to U, $U \oplus E_8$ or $U \oplus E_8^2$.*

Proof. By the Hodge index theorem, S_X is of signature $(1,*)$. By adjunction, the lattice is even. Using the classification of even unimodular lattices, e.g. in [Ser70], we get the desired result. □

Since in the three cases S_X decomposes as the direct sum of U with a negative definite lattice, it is easy to see that we fall everytime in the elliptic case. Moreover, using the Lefschetz topological formula or the fact that only the Picard lattice is fixed, one can see that there are no other irreducible fibers except for the given E_8 fibers generating part of the Picard lattice. Note that using lemma 7.1 one can see that in the case of rk $S_X > 2$ only the section and the rational lines of degree 3 are fixed.

Recall the following definition, due to Dolgachev [Dol96], of mirror pairs for K3 surfaces.

Definition 8.3. *The K3 surfaces (M,W) form a mirror pair whenever $S_M^{\perp} = S_W \oplus U$.*

When applied to the case of unimodular Picard lattices, we see that K3 surfaces form a pair when their Picard groups are respectively $U^i \oplus E_8^j$ and $U^{2-i} \oplus E_8^{2-j}$. I.e. the surfaces with Picard groups U and $U \oplus E_8^2$ are dual to one another while the surfaces with Picard group $U \oplus E_8$ are self-dual. This confirms the diagrams obtained for automorphisms of order 2 and 3 showing that mirror symmetry is a natural transformation preserving symmetries.

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