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AN UPPER BOUND FOR THE X-RANKS OF POINTS OF $\mathbb{P}^n$ IN POSITIVE CHARACTERISTIC

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Abstract. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional variety. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ is the minimal cardinality of $S \subset X$ such that $P \in \langle S \rangle$. Here we study the pairs (X, P) such that $r_X(P) \geq n+2-m$, i.e. $r_X(P) = n+2-m$. These pairs exist only in positive characteristic, with X strange and P a strange point of X.

1. Introduction

Fix an integral and non-degenerate variety $X \subseteq \mathbb{P}^n$ defined over an algebraically closed field $\mathbb{K}$. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle \quad \rangle$ denote the linear span. Hence $r_X(P) = 1$ if and only if $P \in X$. Since X is non-degenerate, the X-ranks are defined and $r_X(P) \leq n+1$ for all $P \in \mathbb{P}^n$. For any integer $r > 0$ let $\sigma_r^0(X) \subseteq \mathbb{P}^n$ denote the the union all $(r-1)$-dimensional linear spaces spanned by r points of X. Let $\sigma_r(X)$ denote the closure of $\sigma_r^0(X)$ in $\mathbb{P}^n$ (sometimes called the $(r-1)$-secant variety of X). The border X-rank of a point $P \in \mathbb{P}^n$ is the minimal integer r such that $P \in \sigma_r(X)$. Each $\sigma_r(X)$ is irreducible. An easy estimate gives that either $\sigma_r(X) = \mathbb{P}^n$ or $\dim(\sigma_{r+1}(X)) > \dim(\sigma_r(X))$ ([1], 1.2). Hence $\sigma_x(X) = \mathbb{P}^n$, where $x := n - \dim(X)$. Moreover, either $\sigma_{r+1}(X) = \mathbb{P}^n$ or $\dim(\sigma_{r+1}(X)) \geq 2 + \dim(\sigma_r(X))$ ([1], Corollary 1.4). Even if $\sigma_x(X) = \mathbb{P}^n$ there may be points with X-rank $> x$. The main concern of this paper is to extend the basic estimate $r_X(P) \leq n - \dim(X)$ made in [15], Proposition 5.1, in characteristic zero to the case $p := \mathrm{char}(\mathbb{K}) > 0$, listing some exceptional pairs (X, P) for which $r_X(P) = n - \dim(X) + 1$ (e.g. take $(n, m, p) = (2, 1, 1)$, as X a smooth conic and as P its strange point ([10], Example IV.3.8.2); in this example every line through P intersects X in a unique point and hence we need 3 points of X to span a linear space containing P).

It is believed that the concept of X-rank may be useful for "real world applications". In the applications when X is a Veronese embedding of $\mathbb{P}^m$ the X-rank is also called the "structured rank" (this is related to the virtual array concept encountered in sensor array processing ([2], [8])). On this topic there was the 2008 AIM workshop Geometry and representation theory of tensors for computer science, statistics and other areas. In [15] a book in preparation is quoted ([14]). Up to now the applied part was toward engineering. All theory was done in characteristic

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zero. Our dream is to use these ideas together with specialists of computer algebra for real applications in coding theory. A preliminary step to fulfil this dream is to check the theory at least over an algebraically closed field with positive characteristic. Up to now the only general result on the X-rank (i.e. a result which does not use specific properties of very particular varieties X) is [15], Proposition 5.1. Hence its extension to positive characteristic seemed to be the first step needed to fulfil our dream. The aim of this paper is to prove that [15], Proposition 5.1, is not true in positive characteristic, but that it is "almost always true" and when it is not true it is "almost true" (it fails by $+1$). We also give a reasonable description of the projective varieties for which it is not true. The embedded variety $X \subseteq \mathbb{P}^n$ is said to be *strange* if there is $O \in \mathbb{P}^n$ such that $O \in T_Q X$ (the embedded tangent space in $\mathbb{P}^n$) for all $Q \in X_{reg}$ (or, equivalently, for a general $Q \in X$) ([4]). If X is strange, a point as above is called a *strange point* of X. The set of all strange points of X is either empty or a linear subspace of dimension at most $\dim(X)-1$ (unless $X = \mathbb{P}^n$). If $\mathrm{char}(\mathbb{K}) = 0$, then X is strange if and only if it is a cone and in this case the set of all strange points is its vertex (with the convention that a linear space is a cone with itself as its vertex). If X is strange with O as one of its strange points, but not a cone with vertex containing O, then $p := \mathrm{char}(\mathbb{K}) > 0$. If p is a large prime, then also $\deg(X)$ must be large (e.g. $\deg(X) \ge p(n-m)$) (see Proposition 3). We first prove the following result.

Theorem 1. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional variety. Fix $P \in \mathbb{P}^n$.*

(a) If P is not a strange point of X, then $r_X(P) \le n+1-m$.

(b) If P is a strange point of X, then $r_X(P) \le n+2-m$.

See Remark 4 for an example of an integral, non-degenerate and m-dimensional ($m \ge 2$) variety $X \subset \mathbb{P}^n$ with as strange points an $(m-1)$-dimensional linear space V and $r_X(P) = n-m+2$ for all $P \in V\backslash N$, where N is a hyperplane of V and $N \subset X$.

The proof of Theorem 1 is very elementary. To prove Theorem 1 we just follow the proof of [15], Proposition 5.1 (the case $\mathrm{char}(\mathbb{K}) = 0$ of Theorem 1), analysing the only missing piece in positive characteristic (a use of Bertini's theorem). In the one-dimensional case we are able to improve Theorem 1. A non-degenerate curve $X \subset \mathbb{P}^n$ is said to be *very strange* if its general hyperplane section is not in linearly general position ([18]). A very strange curve is strange ([18], Lemma 1.1).

Definition 1. Let $X \subset \mathbb{P}^n$, $n \ge 2$, be a non-degenerate strange curve and let O be its strange point. Let $\ell_O : \mathbb{P}^n\backslash\{O\} \to \mathbb{P}^{n-1}$ be the linear projection from O and $T \subset \mathbb{P}^{n-1}$ the closure of $\ell_O(Y\backslash\{O\})$. Thus T is non-degenerate and

$$\deg(X) = p^e s \cdot \deg(T) + \mu, \tag{1}$$

where μ is the multiplicity of X at O, while s and p^e are the separable and the inseparable degree of $\ell_O|X$, respectively ([4], Theorem 2.3). Now assume $n \ge 3$, $\mu = 0$ (i.e. $O \notin X$) and $s = 1$. We say that X is *flat* or *flat with respect to its strange point* O or a *flat strange curve* if for any $S \subset X$ such that $\sharp(S) \le n$ we have $\dim(\langle S\rangle) = \dim(\langle \ell_O(S)\rangle)$.

The proofs that $e > 0$ in the set-up of Definition 1 and that (1) holds are given in [4], §2 (see [4], eq. (2.1.1) and Theorem 2.3); the integer p^e is shown to be equal to the intersection multiplicity of $T_Q X$ with X at Q, where Q is a general point of X

(the so-called Generic Order of Contact Theorem proved in [9], 3.5, for embedded varieties with arbitrary dimension). See [12] for a very useful survey. For related details, see the proof of Proposition 3.

Notice that if $\mu = 0$, then (1) gives $\deg(X) \equiv 0 \mod p$.

Remark 1. Take the set-up of Definition 1.

(a) Since a strange curve (not a line) has a unique strange point, the point O is uniquely determined by X. Hence we do not need to specify it to check if a strange curve is flat or not.

(b) The assumption $(\mu, s) = (0, 1)$ implies that $\ell_O|X$ is generically injective. Flatness implies that $\ell_O|X$ is injective, but it is far stronger. We have $r_X(O) \geq 2$ if and only if $O \notin X$. We have $r_X(O) \geq 3$ if and only if $O \notin X$ and $\ell_O|X$ injective. If $\mu = 0$, then the flatness of a strange curve is equivalent to $r_X(O) = n+1$ (use that $r_X(P) \leq n+1$ for any $P \in \mathbb{P}^n$ and any non-degenerate reduced subset $X \subset \mathbb{P}^n$ and that for any finite $S \subset X$ we have $\dim(\langle \ell_O(S) \rangle) < \dim(\langle S \rangle)$ if and only if $O \in \langle S \rangle$).

(c) Part (b) shows that the "if" part of the following theorem is just the definition of flatness of a strange curve. It also gives the "only if" part if we first prove that X is a strange point of X with invariants $(\mu, s) = (0, 1)$.

Theorem 2. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate curve and $P \in \mathbb{P}^n$. We have $r_X(P) \geq n+1$ (i.e. $r_X(P) = n+1$) if and only if X is a flat strange curve and P is the strange point of X.*

V. Bayer and A. Hefez gave explicit equations for all plane strange curves in terms of the invariants μ, s and p^e introduced in Definition 1 ([4]). Later we extended the construction to strange varieties with a fixed strange point O, fix integers μ, s, p^e and a fixed image $T \subset \mathbb{P}^{n-1}$ with respect to the linear projection from O ([3]). All strange curves X such that $O \notin X$, $s = 1$ and $\ell_O(X)$ is a rational normal curve (where O is the strange point of X) are flat (Proposition 2). These curves are explicitely described by one equation in a Hirzebruch surface F_{n-1} ([3]). The other flat strange curves are very strange (Proposition 1) and we know only one example of these flat curves (see Example 1, i.e. [18], Example 1.2). See Remark 2 for another reason to say that the flat curves X with $\ell_O(X)$ a rational normal curve are "almost maximally linearly independent from the set-theoretic point of view".

The topic considered in [15] is very active (see also [7], [6], [5] and references therein). We stress that [15] and the other quoted papers are over $\mathbb{C}$: none of their statements and proofs is affected by the examples given here.

2. Proofs and related results

Proof of Theorem 1. If $P \in X$, then $r_X(P) = 1$. Hence to prove parts (a) and (b) we may assume $P \notin X$. First assume $m = 1$. Assume $r_X(P) \geq n+1$. Hence for a general hyperplane H containing P the set $(X \cap H)_{red}$ does not span H. Since X is connected, the cohomology exact sequence of the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{I}_X(1) \to \mathcal{I}_{X \cap H}(1) \to 0$$

gives that the scheme $X \cap H$ spans H. Thus $X \cap H$ is not reduced. Since $P \notin X$ and H is general among the hyperplanes containing P, $H \cap \mathrm{Sing}(X) = \emptyset$. Hence the non-reducedness of $X \cap H$ and the generality of H implies that X is a strange curve with P as its strange point. In the case $m = 1$ we have $r_X(P) \le n+1$ for all P, because X spans $\mathbb{P}^n$ proving parts (a) and (b) in the case $m = 1$.

Now assume $m \ge 2$ and that Theorem 1 is true for varieties of dimension $m-1$. Assume the existence of $P \in \mathbb{P}^n$ such that $r_X(P) \ge n+2-m$, but P is not a strange point of X. Fix a general hyperplane H containing P. Let $\ell_P : \mathbb{P}^n \backslash \{P\} \to \mathbb{P}^{n-1}$ be the linear projection from P. Since $P \notin X$, $\ell_P|X$ is a finite morphism. Bertini's theorem gives that $X \cap H$ is geometrically integral ([11], part 4) of Th. I.6.3). Fix a general $Q \in (X \cap H)_{reg}$. For general H we may take as Q a general point of X. Hence $P \notin T_QX$. Hence $P \notin (T_QX) \cap H = T_Q(X \cap H)$. Thus P is not a strange point of $X \cap H$. The inductive assumption gives $r_{X\cap H}(P) \le (n-1)-(m-1)+1 = n-m+1$. Since $r_X(P) \le r_{X\cap H}(P)$, we proved part (a) for all m, X, P.

Now assume that P is a strange point of X. Since we proved part (b) in the case $m = 1$, we may assume $m \ge 2$. Fix an integer $k \ge 3$ and a general $Q \in X_{reg}$. Let Y be the intersection of X with a general degree k hypersurface W such that $Q \in W$. The scheme $Y \backslash \{Q\}$ is geometrically integral by the characteristic free version of Bertini's theorem for very ample linear systems on non-complete varieties ([11], part 4) of Th. I.6.3). Since $k \ge 3$, it is easy to find W such that $Y = X \cap W$ is smooth at Q. Hence Y is geometrically integral and $Q \in Y_{reg}$. Since $k \ge 3$, we may find W as above such that $P \notin T_QW$. Hence $P \notin T_QW \cap T_QX = T_QY$. Hence P is not a strange point of Y. Part (a) applied to Y gives $r_X(P) \le r_Y(P) \le n-(m-1)+1$. □

Proof of Theorem 2. By part (c) of Remark 1 it is sufficient to prove the "only if" part. Fix X, P such that $r_X(P) \ge n+1$. The case $m = 1$ of Theorem 1 implies $r_X(P) = n+1$ and that P is a strange point of X. Call μ, s and p^e the invariants of X with respect to the linear projection ℓ_P from P. Since $r_X(P) \ge 2$, $P \notin X$, i.e. $\mu = 0$. Notice that $s = 1$ if and only if $\ell_P|X$ has separablle degree 1, i.e. it is generically injective. Since $r_X(P) \ge 3$, we have $\sharp((X \cap D)_{red}) \le 1$ for every line D such that $P \in D$. Thus $\ell_P|X$ in injective. Thus $s = 1$. As observed in part (c) of Remark 1 if $(\mu, s) = (0, 1)$ and P is the strange point of X, then the definition of flatness is equivalent to $r_X(P) \ge n+1$. □

Proposition 1. *Let $X \subset \mathbb{P}^n$, $n \ge 3$, be a non-degenerate and flat strange curve with O as its strange point. Then either X is very strange or $\ell_O(X)$ is a rational normal curve.*

Proof. Let O be the strange point of X. Set $d := \deg(\ell_O(X))$. If $d = n-1$, then $\ell_O(X)$ is a rational normal curve. Now assume $d \ge n$. By assumption $\mu = 0$ and $s = 1$. Fix a general $S \subset X$ such that $\sharp(S) = n-1$. Hence $\sharp(\ell_O(S)) = n-1$ and $\ell_O(S)$ spans a hyperplane of $\mathbb{P}^{n-1}$. Since $d \ge n$, there is $U \in \ell_O(X) \backslash \ell_O(S)$ such that $U \in \langle \ell_O(S) \rangle$. Fix $V \in X$ such that $\ell_O(V) = U$. Hence $\sharp(S \cup \{V\}) = n$. Since X is flat, $V \in \langle S \rangle$. Since this is true for a general $S \subset X$ such that $\sharp(S) = n-1$, X satisfies the definition of a very strange curve. □

Proposition 2. *Let $X \subset \mathbb{P}^n$, $n \ge 2$, be a non-degenerate and strange curve with O as its strange point and invariants $\mu = 0$ and $s = 1$, i.e. assume $O \notin X$ and that $\ell_O|X$ is generically injective. If either $n = 2$ or $\ell_O(X)$ is a rational normal curve of $\mathbb{P}^{n-1}$ (i.e. if $\deg(X) = (n-1)p^e$, where p^e is the inseparable degree of $\ell_O|X$), then X is flat.*

Proof. Fix $S \subset X$ such that $\sharp(S) \le n$. Let $u : C \to X$ be the normalization map. By assumption $\ell_O(X) \cong \mathbb{P}^1$ (even if $n = 2$). Since $\ell_O|X : X \to T \cong \mathbb{P}^1$ is purely inseparable, $C \cong \mathbb{P}^1$. Since $s = 1$, the morphism $\ell_O|X \circ u : \mathbb{P}^1 \to \mathbb{P}^1$ is purely inseparable. Hence it is injective. Thus the morphism $\ell_O|X$ is injective, not just generically injective. Hence $\sharp(\ell_O(S)) = \sharp(S) \le n$. Since any n points of a rational normal curve of $\mathbb{P}^{n-1}$ are linearly independent, we get $\dim(\langle \ell_O(S) \rangle) = \sharp(S) - 1$. $\square$

Remark 2. Take X as in Proposition 2. The proof of Proposition 2 gives that every $S \subset X$ such that $\sharp(S) \le n$ is linearly independent, i.e. X has no codimension 2 multisecant linear subspace from the set-theoretical point of view (but of course every tangent line of X at one of its smooth points contains a length p^e subscheme of X). We stress again that all curves X as in Proposition 2 are explicitely constructed in [3]. The rational normal curves of $\mathbb{P}^n$ are the only integral curves for which no hyperplane contains $n+1$ points of the curve, i.e. for which the reduction of every codimension 1 linear section is linearly independent.

Example 1. Here we check that the example of a very strange curve given in [18], Example 1.2, is a flat strange curve. Fix an integer $n \ge 3$, a prime p and a p-power q. Here $q = p^e$ is the inseparable degree of the linear projection from the strange point. Fix homogeneous coordinates $x_0, \dots, x_n$ of $\mathbb{P}^n$ and homogeneous coordinates $x_1, \dots, x_n$ of $\mathbb{P}^{n-1}$. Set $A := (0; \dots; 0; 1; 0)$ and $O := (1; 0; \dots; 0; 0)$. We recall that every point of the vertex of a cone T is a strange point of T. An integral hypersurface $\{f(x_0, \dots, x_n) = 0\}$ has O as one of its strange points if and only if in each monomial of f with a non-zero coefficient the variable x_0 appears with exponent divisible by p. Let X be the scheme with equations $x_0^q - x_1 x_n^{q-1}, x_1^q - x_2 x_n^{q-1}, \dots, x_{n-2}^q - x_{n-1} x_n^{q-1}$. The point O is a strange points of the $n-1$ hypersurfaces with these equations (the latter $n-2$ hypersurfaces are cones with vertex containing O). Set $X' := X \cap \{x_n \ne 0\}$. We have $(X \cap \{x_n = 0\})_{red} = \{A\}$. Since X is given by $n-1$ equations, each irreducible component of X_{red} has dimension at least 1. Hence A is in the closure of X'. Set $t := x_0/x_n$. The scheme $(X')_{red}$ has a rational parametrization

$$t \mapsto (t, t^q, t^{q^2}, \dots, t^{q^{n-1}}), \tag{2}$$

because in X' we have $x_i/x_n = (x_{i-1}/x_n)^q$ for every $i \in \{1, \dots, n-1\}$. Hence $(X')_{red}$ is integral, smooth, rational and its closure X_{red} in $\mathbb{P}^n$ has O as its strange point. Since $\deg(X_{red}) = q^{n-1}$ and X_{red} is set-theoretically the intersection of $n-1$ hypersurfaces of degree q, the algebraic set X_{red} is the complete intersection of these hypersurfaces, outside finitely many points. Hence the scheme X is a complete intersection and it is reduced outside finitely many points. Since X is a complete intersection, each local ring $\mathcal{O}_{X,Q}$, $Q \in X_{red}$, is Cohen-Macaulay. Hence X has no embedded component and it is generically reduced. Thus it is reduced. We have $O \notin X$. Set $Y := \ell_O(X) \subset \mathbb{P}^{n-1}$, $Y' := Y \cap \{x_n \ne 0\}$ and $A' := (0; \dots; 1; 0) = \ell_O(A) \in Y$. Since $\ell_O((t; t^q; \dots; t^{q^{n-1}}; 1)) = (t^q; \dots; t^{q^{n-1}}; 1)$ for all $t \in \mathbb{K}$, the curve Y' has a parametrization

$$z \mapsto (z, z^q, \dots, z^{q^{n-2}}), \tag{3}$$

where $z = t^q$. Hence $\ell_O|X' : X' \to Y'$ is injective and purely inseparable with inseparable degree q. Thus X has parameters $(\mu, s, p^e) = (0, 1, q)$. The parametrization (3) shows that Y' is smooth, that Y is strange with $O'' := (1; 0; \dots; 0; 0)$ as its strange point and that $Y \backslash Y' = \{A'\}$. Fix linearly independent $P_1, \dots, P_n \in X'$ and set $S := \{P_1, \dots, P_n\}$ and $M := \langle S \rangle$. The parametrization (2) shows that $(M \cap X')_{red} = \{P_1 + a_1(P_2 - P_1) + \cdots + a_{n-1}(P_n - P_1)\}$, where each a_i is an arbitrary element of $\mathbb{F}_q$. Since $\sharp((M \cap X')_{red}) = q^{n-1} = \deg(X)$, we get that this is a scheme-theoretic intersection and that $M \cap (X\setminus X') = \emptyset$. Since $M \cap X = (M \cap X')_{red}$ scheme-theoretically and $O \in T_{P_i}X$, we have $O \notin M$, i.e. $\dim(\ell_O(M)) = n-1$. Recall that $X\setminus X' = \{A\}$. Fix $S_1 \subset X$ such that $\sharp(S_1) = n$, $A \in S_1$ and S_1 is linearly independent. Let M_1 be the hyperplane spanned by S_1. Set $S_2 := S_1\setminus\{A\}$ and write $S_2 := \{P_1, \dots, P_{n-1}\}$. Set $Q_i := \ell_O(P_i)$, $1 \le i \le n-1$. We proved that $\sharp(\ell_O(S_2)) = n-1$, $\ell_O(S_2) \subset Y'$ and that $\ell_O(S_2)$ is linearly independent. Set $M_2 := \langle \ell_O(S_2)\rangle$. Since $A' = \ell_O(A)$, to conclude the proof of the flatness of X it is sufficient to prove $A' \notin M_2$. Let $E \subset \mathbb{P}^{n-1}$ the set $\{Q_1 + a_1(Q_2-Q_1) + \cdots + a_{n-2}(Q_{n-1}-Q_1)\}$, where each a_i is an arbitrary element of $\mathbb{F}_q$. Since $P_1 + a_1(P_2-P_1) + \cdots + a_{n-2}(P_{n-1}-P_1) \in X'$ for all $a_i \in \mathbb{F}_q$, we have $E \subseteq M_2 \cap \ell_O(X')$. Since $\ell_O|X'$ is injective, we have $\sharp(E) = q^{n-2} = \deg(Y)$. Thus $E = M_2 \cap Y$ and $(Y\setminus\ell_O(X')) \cap M_2 = \emptyset$. Since $\{A'\} = Y\setminus\ell_O(X')$, we get $A' \notin M_2$. Thus X is flat.

Remark 3. A theorem of Luiss' says that there is a unique smooth strange curve (if we exclude the lines): a smooth plane conic in characteristic 2 ([13], Proposition 3, or [10], Theorem IV.3.9). If $p = 2$ a smooth plane conic is obviously flat. This example shows that if $n = 2$ and $p = 2$ the ranks of the rational normal curves of $\mathbb{P}^n$ are not as in characteristic zero (see [7], [15], 4.1, or [5], 3.1). This phenomenon does not occur when $n = 3$. Let $C \subset \mathbb{P}^3$ be a rational normal curve. Let $TC := \cup_{Q\in C} T_QC \subset \mathbb{P}^3$ denote the tangent developable of C. If $P \in C$, then $r_C(P) = 1$. If $P \notin TC$, then $r_C(P) = 2$, because $\mathbb{P}^3$ is the secant variety of C ([1], Remark 1.6). Fix $P \in TC\setminus C$, say $P \in T_QC\setminus\{Q\}$ with $Q \in C$. Assume $r_C(P) = 2$ and take $P_1, P_2 \in C$ such that $P_1 \ne P_2$ and $P \in \langle\{P_1, P_2\}\rangle$. Since any length 3 scheme $Z \subset C$ spans a plane, $Q \notin \langle\{P_1, P_2\}\rangle$. Since $P \in T_QC \cap \langle\{P_1, P_2\}\rangle$, the linear space $M := \langle T_QC \cup \{P_1, P_2\}\rangle$ is a plane and $\text{length}(M \cap C) \ge 4$. Since $\deg(C) \ge 3$, we get a contradiction. Hence $r_C(P) \ge 3$. Since C is not strange, Theorem 1 gives $r_C(P) = 3$. Hence the stratification by ranks of C is the same as in characteristic zero.

Fix an integer $m \ge 2$. Here we construct m-dimensional examples of pairs (X, P) such that $r_X(P) = n+2-m$, i.e. such that the inequality in part (b) of Theorem 1 is an equality. Just taking cones we get an m-dimensional example from any one-dimensional example with the same codimension in an ambient projective space. This is the only example we know of pairs (X, P) with $m \ge 2$ and $r_X(P) = n+2-m$, i.e. a pair for which part (b) of Theorem 1 is sharp. Are there other examples?

Remark 4. Fix integers $n > m \ge 2$, an $(n-m+1)$-dimensional linear subspace M of $\mathbb{P}^n$ and an $(m-2)$-dimensional linear subspace N of $\mathbb{P}^n$ such that $M \cap N = \emptyset$, i.e. a complementary subspace. For any variety $Y \subset M$ let $C(N, Y) \subset \mathbb{P}^n$ denote the cone with vertex N and Y as its basis. Hence for each $O \in M$ the scheme $C(N, O)$ is an $(m-1)$-dimensional linear subspace of $\mathbb{P}^n$. We claim that $r_{C(N,Y)}(P) = r_Y(O)$ for every $P \in C(N, O)\setminus N$. Fix $P \in C(N, O)\setminus N$. Take an $(n-m+1)$-dimensional linear subspace M' of $\mathbb{P}^n$ such that $P \in M'$ and $N \cap M' = \emptyset$. The linear projection from N induces an isomorphism of pairs $(C(N, Y) \cap M', P) \cong (Y, O)$ as pairs of subvarieties, respectively of M' and of M. Thus $r_{C(N,Y)}(P) \le r_{C(N,Y)\cap M'}(P) = r_Y(O)$. To prove the reverse inequality we fix $P \in C(N, O)$ and $S \subset C(N, Y)$ computing $r_{C(N,Y)}(P)$. The image $S' \subset M$ of the linear projection of S from N is a set such that $\sharp(S') \le \sharp(S) = r_{C(N,Y)}(P)$. Since $O \in \langle S'\rangle$, we get $r_Y(O) \le \sharp(S') \le r_{C(N,Y)}(P)$. Taking as Y a flat curve with strange point O, $X = C(N, Y)$ and

$V = C(N, O)$ we get the existence (for all $n > m \geq 2$) of an integral, non-degenerate and m-dimensional variety $X \subset \mathbb{P}^n$ with as set of its strange points an $(m-1)$-dimensional linear space V and $r_X(P) = n - m + 2$ for all $P \in V \backslash N$, where N is an $(m-2)$-dimensional linear space and $N \subset X$.

Proposition 3. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional variety. Fix $O \in \mathbb{P}^n$ and assume that O is a strange point of X, but that X is not a cone with vertex containing O. Then* $\deg(X) \geq p \cdot (n-m)$.

Proof. Fix $A \in \mathbb{P}^n \backslash \{O\}$ and take any integral quasi-projective variety $E \subseteq \mathbb{P}^n \backslash \{O\}$ such that $A \in E_{reg}$. Set $x := \dim(E)$. The inclusion $j : E \subseteq \mathbb{P}^n$ induces an inclusion between the abstract tangent spaces $\Theta_{E,A}$ of E at A and the abstract tangent space $\Theta_{\mathbb{P}^n,A}$ of $\mathbb{P}^n$ at A. As usual in projective geometry we "complete" these vector spaces $\Theta_{E,A}$ and $\Theta_{\mathbb{P}^n,A}$ to projective spaces, respectively of dimension x and n, and call them T_AE and $T_A\mathbb{P}^n = \mathbb{P}^n$. Since $A \neq 0$, the submersion $\ell_O : \mathbb{P}^n \backslash \{O\} \to \mathbb{P}^{n-1}$ induces a linear surjective map of $\mathbb{K}$-vector spaces $\rho_O(A) : \Theta_{\mathbb{P}^n,A} \to \Theta_{\mathbb{P}^{n-1},\ell_O(A)}$. Since $\rho_O(A)$ is surjective, its kernel is one-dimensional. If we identify $\Theta_A\mathbb{P}^n$ with an affine n-dimensional open subset of $T_A\mathbb{P}^n = \mathbb{P}^n$, then the closure of this kernel is the line $\langle \{O, A\} \rangle$ (in the case $x = 1$, see [13], lines 3–4 of p. 215). Thus the differential of $\ell_O|E$ at A is injective if and only if $O \notin T_AE$. Thus the differential of $\ell_O|E$ at a general point of E is injective if and only if the closure $\overline{E} \subseteq \mathbb{P}^n$ of E is not strange with O as one of its strange points.

Let $T \subset \mathbb{P}^{n-1}$ denote the closure of $\ell_O(X \backslash \{O\})$. Since X is not a cone with vertex containing O, $\ell_O|X \backslash \{O\}$ is a generically finite morphism. Hence $\dim(T) = m$. Since T spans $\mathbb{P}^{n-1}$, we have $\deg(T) \geq n - m$. Since $\ell_O|X \backslash \{O\}$ is generically finite, the function field $K(X)$ of X is a finite extension of the function field $K(T)$. Since O is a strange point of X, this extension of fields is not separable (use the geometric interpretation of $\rho_O(A)$ just given and the differential criterion of separability, i.e. [17], Theorem 26.6, or [16], Th. 59 at p. 191, quoted in [10], Theorem II.8.6). Call p^e, $e \geq 1$, the inseparable degree of this extension of fields. A general fiber of $\ell_O|X \backslash \{O\}$ is a disjoint union of finitely many connected zero-dimensional schemes, each of them with degree p^e. Hence $\deg(X) \geq p^e \cdot \deg(T) \geq p(n-m)$. □

In the set-up of Proposition 3 if $O \in X$, then $\deg(X) > p \cdot (n-m)$. Proposition 3 is very weak, but we are unable to make a substantial improvement of it. In the case of a strange curve X the formula (1) relates $\deg(X)$ to other data. Nothing more can be said in the one-dimensional case. Indeed, the construction of [3] shows that we may take an arbitrary T spanning $\mathbb{P}^{n-1}$ and then find a solution X with arbitrary $e \geq 1$ and $\mu \geq 0$. Formula (1) is very useful to check if a curve X is strange. We observed after Definition 1 that if $\deg(X)/p \notin \mathbb{Z}$, then either X is not strange or its strange point belongs to X. If X is strange, we also see that the image curve T has much lower degree and hence it should be easier.

It seems to be very difficult to construct very strange curves. We know only the examples given in [18]. We expect that if they exist, then they have very large degree, at least p^{n-1} in $\mathbb{P}^n$.

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FIXED POINT PROPERTY FOR THE HYPERSPACES OF NON-METRIC CHAINABLE CONTINUA

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Abstract. The main purpose of this paper is to prove that some hyperspaces of a non-metric chainable continuum have the fixed point property.

1. Introduction

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space X is denoted by $\mathrm{w}(X)$.

A *generalized arc* is a Hausdorff continuum with exactly two non-separating points (end points) x, y. Each separable arc is homeomorphic to the closed interval $\mathbb{I} = [0, 1]$.

We say that a space X is *arcwise connected* if for every pair x, y of points of X there exists a generalized arc L with end points x, y.

An inverse system [3, pp. 135-142] is denoted by $\mathbf{X} = \{X_a, p_{ab}, A\}$. Suppose that we have two inverse systems $\mathbf{X} = \{X_a, p_{ab}, A\}$ and $\mathbf{Y} = \{Y_b, q_{bc}, B\}$. A *morphism of the system X into the system* $\mathbf{Y}$ [1, p. 15] is a family $\{\varphi, \{f_b : b \in B\}\}$ consisting of a nondecreasing function $\varphi : B \to A$ such that $\varphi(B)$ is cofinal in A, and of maps $f_b : X_{\varphi(b)} \to Y_b$ defined for all $b \in B$ such that the following

$$\begin{array}{ccc} X_{\varphi(b)} & \overset{p_{\varphi(b)\varphi(c)}}{\longleftarrow} & X_{\varphi(c)} \\ \downarrow f_b & & \downarrow f_c \\ Y_b & \overset{q_{bc}}{\longleftarrow} & Y_c \end{array} \tag{1.1}$$

diagram commutes. Any morphism $\{\varphi, \{f_b : b \in B\}\} : \mathbf{X} \to \mathbf{Y}$ induces a map, called the *limit map of the morphism*

$$\lim\{\varphi, \{f_b : b \in B\}\} : \lim \mathbf{X} \to \lim \mathbf{Y}$$

In the present paper we deal with the inverse systems defined on the same indexing set A. In this case, the map $\varphi : A \to A$ is taken to be the identity and we use the following notation $\{f_a : X_a \to Y_a; a \in A\} : \mathbf{X} \to \mathbf{Y}$.

The following result is well-known.

Theorem 1.1. [3, Exercise 2.5.D(b), p. 143]. *If for every $s \in S$ an inverse system $\mathbf{X}(s) = \{X_a(s), p_{ab}(s), A\}$ is given, then the family $\Pi\{\mathbf{X}(s) : s \in S\} = \{\Pi\{X_a(s) : s \in S\}, \Pi\{ p_{ab}(s) : s \in S\}, A\}$ is an inverse system and $\lim(\Pi\{\mathbf{X}(s) : s \in S\})$ is homeomorphic to $\Pi\{\lim \mathbf{X}(s) : s \in S\}$.*

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If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an inverse system, then we have inverse system $\mathbf{X} \times \mathbf{X} = \{X_a \times X_a, p_{ab} \times p_{ab}, A\}$. Let $X = \lim \mathbf{X}$. By Theorem 1.1 we infer that $X \times X$ is homeomorphic to the limit of inverse system $\mathbf{X} \times \mathbf{X}$.

We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *factorizing* [1, p. 17] if for each real-valued mapping $f : \lim \mathbf{X} \to \mathbb{R}$ there exist an $a \in A$ and a mapping $f_a : X_a \to \mathbb{R}$ such that $f = f_a p_a$.

An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be *σ-directed* if for each sequence $a_1, a_2, ..., a_k, ...$ of the members of A there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

Lemma 1.2. [1, Corollary 1.3.2, p. 18]. *If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a σ-directed inverse system of compact spaces with surjective bonding mappings, then it is factorizing.*

An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be *τ-continuous* [1, p. 19] if for each chain B in A with $\operatorname{card}(B) < \tau$ and $\sup B = b$, the diagonal product $\Delta \{p_{ab} : a \in B\}$ maps the space X_b homeomorphically into the space $\lim\{X_a, p_{ab}, B\}$.

An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be *τ-system* [1, p. 19] if:

a) $\tau \geq \mathrm{w}(X_a)$ for every $a \in A$,

b) The system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is τ-continuous,

c) The indexing set A is τ-complete.

If $\tau = \aleph_0$, then τ-system is called a σ-system. The following theorem is called the *Spectral Theorem* [1, p. 19].

Theorem 1.3. [1, Theorem 1.3.4, p. 19]. *If a τ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ with surjective limit projections is factorizing, then each map of its limit space into the limit space of another τ-system $\mathbf{Y} = \{Y_a, q_{ab}, A\}$ is induced by a morphism of cofinal and τ-closed subsystems. If two factorizing τ-systems with surjective limit projections and the same indexing set have homeomorphic limit spaces, then they contain isomorphic cofinal and τ-closed subsystems.*

Let us remark that the requirement of surjectivity of limit projections of systems in Theorem 1.3 is essential [1, p. 21].

In the sequel we will need the following theorem.

Theorem 1.4. [7, Theorem 1.6, p. 402]. *If X is the Cartesian product $X = \prod\{X_s : s \in S\}$, where $\operatorname{card}(S) > \aleph_0$ and each X_s is compact, then there exists a σ-directed inverse system $\mathbf{X} = \{Y_a, P_{ab}, A\}$ of the countable products $Y_a = \prod\{X_\mu : \mu \in a\}$, $\operatorname{card}(a) = \aleph_0$, such that X is homeomorphic to $\lim \mathbf{X}$.*

2. Fixed point property for non-metric compact spaces

A *fixed point of a function* $f : X \to X$ is a point $p \in X$ such that $f(p) = p$. A space X is said to have the *fixed point property* provided that every surjective mapping $f : X \to X$ has a fixed point.

First Step in the proving fixed point property for hyperspaces of non-metric chainable continua is the following general Theorem for fixed point property for non-metric continua.

Theorem 2.1. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-system of compact spaces with the limit X and onto projections $p_a : X \to X_a$. Let $\{f_a : X_a \to X_a\} : \mathbf{X} \to \mathbf{X}$ be a morphism. Then the induced mapping $f = \lim \{f_a\} : X \to X$ has a fixed point if and only if each mapping $f_a : X_a \to X_a$, $a \in A$, has a fixed point.*

Proof. The if part. Let $F_a, a \in A$, be a set of fixed points of the mapping f_a.

Claim 1. *Every set F_a is closed.* This is a consequence of the following theorem [3, Theorem 1.5.4., p. 59]. *For any pair f,g of mappings of a space X into a Hausdorff space Y, the set*

$$\{x \in X : f(x) = g(x)\}$$

is closed in X.

It suffices to set $g(x) = x$ and $Y = X$.

Claim 2. *If $b \geq a$, then $p_{ab}(F_b) \subset F_a$.* Let x_b be any point of F_b. From the commutativity of the diagram (1.1) it follows $p_{ab}(f_b(x)) = f_a(p_{ab}(x))$. We have $p_{ab}(x) = f_a(p_{ab}(x))$ since $f_b(x) = x$. This mens that for the point $y = p_{ab}(x) \in X_a$ we have $y = f_a(y)$, i.e., $y \in F_a$. We infer that $p_{ab}(x) \in F_a$ and $p_{ab}(F_b) \subset F_a$.

Claim 3. $\mathbf{F} = \{F_a, p_{ab}|F_b, A\}$ *is an inverse system of compact spaces with the non-empty limit F.*

Claim 4. *The set $F \subset X$ is the set of fixed points of the mapping f.* Let $x \in F$ and let $x_a = p_a(x)$, $a \in A$. Now, $f_a(x_a) = x_a$ since $x_a \in F_a$. We infer that $f(x) = x$ since the morphism $\{f_a : a \in A\}$ induces f. The proof of the "if" part is complete.

The only if part. Suppose that the induced mapping f has a fixed point x. Let us prove that every mapping $f_a, a \in A$, has a fixed point. Now we have $f_a p_a(x) = p_a f(x)$. From $f(x) = (x)$ it follows $f_a p_a(x) = p_a(x)$. We infer that $p_a(x)$ is a fixed point for f_a. □

As an immediate consequence of this theorem and the Spectral theorem 1.3 we have the following result.

Theorem 2.2. *Let a non-metric continuum X be the inverse limit of an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a has the fixed point property and each bonding mapping p_{ab} is onto. Then X has the fixed point property.*

The following result is an application of Theorem 2.2.

Theorem 2.3. *Let S be an infinite set and $Q = \Pi\{X_s : s \in S\}$ Cartesian product of compact spaces. If each product $X_{s_1} \times X_{s_2} \times ... \times X_{s_n}$ of finitely many spaces X_s has the fixed point property, then Q has the fixed point property.*

Proof. We shall consider the following cases.

Case 1. $\mathrm{card}(S) = \aleph_0$. We may assume that $S = \mathbb{N}$. The proof is a straightforward modification of the proof of [9, Corollary 3.5.3, pp. 106-107]. Let $f : Q \to Q$ be continuous. For every $n \in \mathbb{N}$ define

$$K_n = \{x \in Q : (x_1, ..., x_n) = (f(x)_1, ..., f(x)_n)\}.$$

It is clear that for every n the set K_n is closed in Q and that $K_{n+1} \subset K_n$. For every $n \in \mathbb{N}$, let o_n be a given point of X_n and $p_n : Q \to X_1 \times ... \times X_n$ be the projection. Define continuous function $f_n : X_1 \times ... \times X_n \to X_1 \times ... \times X_n$ by

$$f_n(x_1, ..., x_n) = (p_n f)(x_1, ..., x_n, o_{n+1}, o_{n+2}, ...).$$

By assumption of Theorem f_n has the fixed point property, say $(x_1, ..., x_n)$. It follows that

$$(x_1, ..., x_n, o_{n+1}, o_{n+2}, ...) \in K_n.$$

We conclude that $\{K_n : n \in \mathbb{N}\}$ is a decreasing collection of nonempty closed subsets of Q. By compactness of Q we have that

$$K = \cap\{K_n : n \in \mathbb{N}\}$$

is nonempty. It is clear that every point in K is a fixed point of f.

Case 2 .card$(A) \geq \aleph_1$. By Theorem 1.4 there exists a σ-directed inverse system $\mathbf{X} = \{Y_a, P_{ab}, A\}$ of the countable products $Y_a = \prod\{X_\mu : \mu \in a\}$, card$(a) = \aleph_0$, such that Q is homeomorphic to $\lim \mathbf{X}$. By Case 1 each Y_a has the fixed point property. Finally, by Theorem 2.2 we infer that Q has the fixed point property. □

3. Fixed point property for the hyperspaces of non-metric chainable continua

In this Section we shall study the fixed point property of the hyperspaces of chainable continua.

A *chain* $\{U_1, ..., U_n\}$ is a finite collection of sets U_i such that $U_i \bigcap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$. A continuum X is said to be *chainable* or *arc-like* if each open covering of X can be refined by an open covering $u = \{U_1, ..., U_n\}$ such that $\{U_1, ..., U_n\}$ is a chain.

Second Step in the proving fixed point property for hyperspaces of non-metric chainable continua is the following general expanding Theorem for non-metric chainable continua into inverse σ-system.

Theorem 3.1. *If X is a chainable continuum, then there exists a σ-system $\mathbf{Q}_\sigma = \{Q_\Delta,\ p_{\Delta\Gamma},\ A_\sigma\}$ such that each Q_Δ is a metric chainable continuum, $p_{\Delta\Gamma}$ are surjections and X is homeomorphic with the inverse limit* $\lim \mathbf{Q}_\sigma$.

Proof. The proof is broken into several steps.

Step 1. *If X is a chainable continuum, then there exists a system $\mathbf{Q} = \{Q_a, q_{ab}, A\}$ such that each Q_a is a metric chainable continuum and X is homeomorphic with the inverse limit* $\lim \mathbf{Q}$. By [8, Theorem 2*] every chainable continuum X is homeomorphic with the inverse limit of an inverse system $\{Q_a, q_{ab}, A\}$ of metric chainable continua Q_a. One can assume that q_{ab} are onto mappings since a closed connected subset C of chainable continuum is chainable.

Step 2. *There exists a σ-system of chainable continua such that X is homeomorphic with its inverse limit.* The inverse system $\{Q_a, q_{ab}, A\}$ is not a σ-system. Now we shall prove that such inverse system exists. For each subset Δ_0 of $(A, \leq)$ we define sets Δ_n, $n = 0, 1, ...$, by the inductive rule $\Delta_{n+1} = \Delta_n \bigcup \{m(x, y) : x,\ y \in \Delta_n\}$, where $m(x, y)$ is a member of A such that $x, y \leq m(x, y)$. Let $\Delta = \bigcup\{\Delta_n : n \in \mathbb{N}\}$. It is clear that card$(\Delta)$ = card(Δ_0). Moreover, Δ is directed by $\leq$. For each directed set $(A, \leq)$ we define

$$A_\sigma = \{\Delta : \emptyset \neq \Delta \subset A,\ \text{card}(\Delta) \leq \aleph_0 \quad \text{and } \Delta \text{ is directed by } \leq\}.$$

Let us prove that A_σ is σ-directed and σ-complete. Let $\{\Delta^1,\ \Delta^2,\ ...,\ \Delta^n,\ ...\}$ be a countable subset of $A_\sigma = \{\Delta : \emptyset \neq \Delta \subset A,\ \text{card}(\Delta) \leq \aleph_0 \quad \text{and } \Delta \text{ is directed by } \leq\}$. Then $\Delta_0 = \cup\{\Delta^1,\ \Delta^2,\ ...,\ \Delta^n,\ ...\}$ is a countable subset of A_σ. Define sets Δ_n, $n = 0, 1, ...$, by the inductive rule $\Delta_{n+1} = \Delta_n \bigcup \{m(x, y) : x,\ y \in \Delta_n\}$, where $m(x,\ y)$ is a member of A such that $x, y \leq m(x, y)$. Let $\Delta = \bigcup\{\Delta_n : n \in \mathbb{N}\}$. It is clear that card$(\Delta)$ = card(Δ_0). This means that Δ is countable. Moreover $\Delta \supseteq \Delta^i$, $i \in \mathbb{N}$. Hence A_σ is σ-directed. Let us prove that A_σ is σ-complete. Let $\Delta^1 \subset \Delta^2 \subset ... \subset \Delta^n \subset ...$ be a countable chain in A_σ. Then $\Delta = \cup\{\Delta^i : i \in \mathbb{N}\}$ is countable and directed subset of A, i.e., $\Delta \in A_\sigma$. It is clear that $\Delta \supseteq \Delta^i$, $i \in \mathbb{N}$. Moreover, for each $\Gamma \in A_\sigma$ with property $\Gamma \supseteq \Delta^i$, $i \in \mathbb{N}$, we have $\Gamma \supseteq \Delta$. Hence $\Delta = \sup\{\Delta^i : i \in \mathbb{N}\}$. This means that A_σ is σ-complete.

If $\Delta \in A_\sigma$, let $\mathbf{Q}^\Delta = \{Q_b, q_{bb'}, \Delta\}$ and $Q_\Delta = \lim \mathbf{Q}^\Delta$. If $\Delta, \Gamma \in A_\sigma$ and $\Delta \subseteq \Gamma$, let $p_{\Delta\Gamma}$: $Q_\Gamma \to Q_\Delta$denote the map induced by the projections $q^\Gamma_\delta : Q_\Gamma \to Q_\delta$, $\delta \in \Delta$, of the inverse system $\mathbf{Q}^\Gamma$.

Now we shall prove that if $\mathbf{Q} = \{Q_a, q_{ab}, A\}$ is an inverse system, then $\mathbf{Q}_\sigma = \{Q_\Delta, p_{\Delta\Gamma}, A_\sigma\}$ is a σ-directed and σ-complete inverse system such that $\lim \mathbf{Q}$ and $\lim \mathbf{Q}_\sigma$ are homeomorphic. Each thread $x = (x_a : a \in A)$ induces the thread $(x_a : a \in \Delta)$ for each $\Delta \in A_\sigma$, i.e., the point $q_\Delta \in Q_\Delta$. This means that we have a mapping $H : \lim \mathbf{Q} \to \lim \mathbf{Q}_\sigma$ such that $H(x) = (q_\Delta : \Delta \in A_\sigma)$. It is obvious that H is continuous and 1-1. The mapping H is onto since the collections of the threads $\{q_\Delta : \Delta \in A_\sigma\}$ induces the thread in $\mathbf{Q}$. We infer that H is a homeomorphism since $\lim \mathbf{Q}$ is compact.

Finally, let us prove that every Q_Δ is chainable. We may assume that $\mathbf{Q}^\Delta = \{Q_b, q_{bb'}, \Delta\}$ is an inverse sequence since Δ is countable and $Q_\Delta = \lim \mathbf{Q}^\Delta$. Let $u = \{U_1, ..., U_n\}$ be an open covering of Q_Δ. There exists a $b \in \Delta$ and an open covering $u_b = \{U^b_1, ..., U^b_m\}$ of Q_b such that $\{q_b^{-1}(U^b_1), ..., q_b^{-1}(U^b_m)\}$ refines the covering $u = \{U_1, ..., U_n\}$. There is a chain $\{V^b_1, ..., V^b_p\}$ which refines u_b since Q_b is chainable. It is clear that $\{q_b^{-1}(V^b_1), ..., q_b^{-1}(V^b_p)\}$ is a chain which refines the covering u. Hence, Q_Δ is chainable.

Step 3. *One can assume that* $p_{\Delta\Gamma}$ *and* $p_\Delta : \lim Q_\sigma \to Q_\Delta$ *are onto mappings.* If $p_{\Delta\Gamma}$ and $p_\Delta : \lim Q_\sigma \to Q_\Delta$ are not onto mappings, then we shall use the inverse system $\mathbf{Q}^p_\sigma = \{p_\Delta(\lim Q_\sigma), p_{\Delta\Gamma}|p_\Delta(\lim Q_\sigma), A_\sigma\}$. Each $p_{\Delta\Gamma}|p_\Delta(\lim Q_\sigma)$ is chainable since a closed connected subset of chainable continuum is chainable.

The proof is completed since X is representable as the inverse limit of σ-system $\mathbf{Q}_\sigma = \{Q_\Delta, p_{\Delta\Gamma}, A_\sigma\}$ of metric chainable continua Q_Δ. □

Finally, we represent the various hyperspaces of a non-metric chainable continuum X as the inverse limits of hyperspaces of metric chainable continua.

Let X be a space. We define its hyperspaces as the following sets:

$2^X = \{F \subseteq X : F \text{ is closed and nonempty}\}$,
$\mathcal{C}(X) = \{F \in 2^X : F \text{ is connected}\}$,
$\mathcal{F}_n(X) = \{A \subset X : A \text{ is nonempty and } A \text{ has at most } n \text{ points}\}$.

For any finitely many subsets S_1, ..., S_n, let

$$\langle S_1, ..., S_n\rangle = \left\{F \in 2^X : F \subset \bigcup_{i=1}^n S_i, \text{ and } F \cap S_i \neq \emptyset, \text{ for each } i\right\}.$$

The topology on 2^X is the Vietoris topology, i.e., the topology with a base $\{< U_1, ..., U_n >: U_i$ is an open subset of X for each i and each $n < \infty$ }, and $C(X), X(n)$ are subspaces of 2^X. Moreover, $X(1)$ is homeomorphic to X.

The topology on 2^X is the Vietoris topology and $\mathcal{C}(X)$ and $\mathcal{F}_n(X)$ is a subspaces of 2^X.

Let X and Y be the spaces and let $f : X \to Y$ be a mapping. Define $2^f : 2^X \to 2^Y$ by $2^f(F) = f(F)$ for $F \in 2^X$. It is known that 2^f is continuous and $2^f(\mathcal{C}(X)) \subset \mathcal{C}(Y)$. Moreover, $2^f(\mathcal{F}_n(X)) \subset \mathcal{F}_n(Y)$. The restriction $2^f|\mathcal{C}(X)$ is denoted by $\mathcal{C}(f)$. Similarly, the restriction $2^f|\mathcal{F}_n(X)$ is denoted by $\mathcal{F}_n(f)$.

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces with the natural projections $p_a : \lim X \to X_a$, $a \in A$. Then $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$, $\mathcal{C}(\mathbf{X}) = \{\mathcal{C}(X_a), \mathcal{C}(p_{ab}), A\}$ and $\mathcal{F}_n(\mathbf{X}) = \{\mathcal{F}_n(X_a), \mathcal{F}_n(p_{ab}), A\}$ form inverse systems.

Lemma 3.2. *Let* $X = \lim \mathbf{X}$. *Then* $2^X = \lim 2^{\mathbf{X}}$, $\mathcal{C}(X) = \lim \mathcal{C}(\mathbf{X})$ *and* $\mathcal{F}_n(X) = \lim \mathcal{F}_n(\mathbf{X})$.

In [6, Corollary 5, p. 616] it is proved the following result.

Theorem 3.3. *The third symmetric product* $\mathcal{F}_3(X)$ *of a metric chainable continuum* X *has the fixed point property.*

The proof given there is purely metric. This means that it is reasonable to give the proof for non-metric chainable continua.

Theorem 3.4. *The third symmetric product* $\mathcal{F}_3(X)$ *of a chainable continuum* X (*metric or non-metric*) *has the fixed point property.*

Proof. If X is a metric chainable continuum, then apply Theorem 3.3. In order to complete the proof, we shall assume that X is non-metric chainable continuum. By Theorem 3.1 there exists a σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metric chainable continuum, p_{ab} are surjections *and* X is homeomorphic with the inverse limit $\lim \mathbf{X}$. Now we have a σ-system $\mathcal{F}_3(\mathbf{X}) = \{\mathcal{F}_3(X_a), \mathcal{F}_3(p_{ab}), A\}$ whose limit is homeomorphic to $\mathcal{F}_3(X)$. In order to apply Theorem 2.2 it suffices to prove that $\mathcal{F}_3(p_{ab})$ are surjections for every $a \leq b$. Let $\{x_1, x_2, x_3\} \in \mathcal{F}_3(X_a)$. The sets $p_{ab}^{-1}(x_1), p_{ab}^{-1}(x_2)$, $p_{ab}^{-1}(x_3)$ are non-empty since p_{ab} is onto. If $y_1 \in p_{ab}^{-1}(x_1), y_2 \in p_{ab}^{-1}(x_2)$ and $y_3 \in p_{ab}^{-1}(x_3)$, then $\{y_1, y_2, y_3\} \in \mathcal{F}_3(X_b)$ and $\mathcal{F}_3(p_{ab})(\{y_1, y_2, y_3\}) = \{x_1, x_2, x_3\} \in \mathcal{F}_3(X_a)$. Hence, $\mathcal{F}_3(p_{ab})$ is a surjection. By Theorem 3.3 each X_a has the fixed point property. Finally, by Theorem 2.2 we infer that $\mathcal{F}_3(X)$ has the fixed point property. $\square$

From the proof of the above theorem it is clear that is true the following theorem.

Theorem 3.5. *The nth-symmetric product* $\mathcal{F}_n(X)$ *of a chainable non-metric continuum* X *has the fixed point property if the nth-symmetric product of every chainable metric continuum has the fixed point property.*

From this theorem we shall give the following result.

Theorem 3.6. *If* X *is a non-metric chainable continuum, then* X *has the fixed point property.*

Proof. Now X is homeomorphic to $\mathcal{F}_1(X)$ which is homeomorphic to $\lim \mathcal{F}_1(\mathbf{X})$. From Theorem 3.5 it follows that $\lim \mathcal{F}_1(\mathbf{X})$ has the fixed point property since each metric chainable continuum $\mathcal{F}_1(Y)$ (homeomorphic to Y) has the fixed point property [4]. $\square$

Another hyperspace of a continuum is the hyperspace $\mathcal{C}(X) = \{F \in 2^X : F$ is connected$\}$. The following result is known.

Theorem 3.7. [11]. *If* Y *is a metric chainable continuum, then* $\mathcal{C}(Y)$ *has the fixed point property.*

For non-metric chainable continua we have the following result.

Theorem 3.8. *If* X *is a non-metric chainable continuum, then* $\mathcal{C}(X)$ *has the fixed point property.*

Proof. If X is a metric chainable continuum, then apply Theorem 3.7. In order to complete the proof, we shall assume that X is non-metric chainable continuum. By Theorem 3.1 there exists a σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metric chainable continuum, p_{ab} are surjections *and* X is homeomorphic with the inverse limit $\lim \mathbf{X}$. Now we have a σ-system $\mathcal{C}(\mathbf{X}) = \{\mathcal{C}(X_a), \mathcal{C}(p_{ab}), A\}$ whose limit is homeomorphic to $\mathcal{C}(X)$. In order to apply Theorem 2.2 it suffices to prove that $\mathcal{C}(p_{ab})$ are surjections for every $a \leq b$. Let $C \in \mathcal{C}(X_a)$. The set $p_{ab}^{-1}(C)$ contains a continuum D in Y such that $p_{ab}(D) = C$ ([10, Theorem 12.46, p. 262]). Hence, $\mathcal{C}(p_{ab})$ is a surjection. By Theorem 3.7 each X_a has the fixed point property. Finally, by Theorem 2.2 we infer that $\mathcal{C}(X)$ has the fixed point property. □

4. Fixed point property of the product of chainable continua

Dyer [2, Theorem 1, p. 663] showed the following result.

Theorem 4.1. *Suppose that M is the Cartesian product of n compact chainable metric continua $X_1, X_2, ..., X_n$ and f is a continuous mapping of M into itself. Then there is a point $x \in M$ such that $x = f(x)$.*

For $n = 2$ we have the following result.

Theorem 4.2. [5, p. 199, Exercise 22.26]. *If X and Y are metric chainable continua, then $X \times Y$ has the fixed point property.*

Dyer [2, Corollary, p.665] showed the following general result.

Theorem 4.3. *Cartesian product of the elements of any collection of chainable metric continua has the fixed point property.*

We will show that last Theorem 4.3 is true for non-metrizable chainable continua.

Theorem 4.4. *Cartesian product of the elements of any collection of chainable continua of the same weight has the fixed point property.*

Proof. If for every $s \in S$ we have a chainable non-metrizable continuum $X(s)$, then, for every $s \in S$, there exists an inverse system $\mathbf{X}(s) = \{X_a(s), p_{ab}(s), A(s)\}$ such that $X(s)$ is homeomorphic to $\lim \mathbf{X}(s)$ and every $X_a(s)$ is a metric chainable continuum (Theorem 3.1). If $\mathrm{w}(X(s_1)) = \mathrm{w}(X(s_2)), s_1, s_2 \in S$, then $A(s_1) = A(s_2)$ and we may suppose that $A(s) = A$ for every $s \in S$. By Theorem 1.1 the family $\Pi\{\mathbf{X}(s) : s \in S\} = \{\Pi\{X_a(s) : s \in S\}, \Pi\{ p_{ab}(s) : s \in S\}, A\}$ is an inverse system and $\lim(\Pi\{\mathbf{X}(s) : s \in S\})$ is homeomorphic to $\Pi\{\lim \mathbf{X}(s) : s \in S\}$. From Theorem 4.3 it follows that each $\Pi\{X_a(s) : s \in S\}$ has the fixed point property. Finally, from Theorem 2.2 it follows that $\Pi\{X(s) : s \in S\}$ has the fixed point property. □

QUESTION. Is it true that the assumption "*of the same weight*" in Theorem 4.4 can be omitted?

As an immediate application of Theorem 4.4 we give the following generalization of Brouwer Fixed-Point Theorem. Let L be a non-metric arc. The space X is said to be a *generalized n-cell* if it is homeomorphic to $L^n = L \times L \times ... \times L$ (n factors).

Theorem 4.5. *Every mapping $f : L^n \to L^n$ has a fixed point, i.e., L^n has the fixed point property.*

Theorem 4.5 implies the following result.

Theorem 4.6. *If $L_1, ..., L_n$ are arcs (metric or non-metric), then $L_1 \times L_2 \times ... \times L_n$ has the fixed point property.*

Proof. **Step 1**. *If M is a subarc of the arc L, then there exists a retraction $r : L \to M$.* Let a, b, c, d be end points of L and M such that $a \leq c < d \leq b$. We define $r : L \to M$ as follows:

$$r(x) = \begin{cases} c & \text{if} \quad a \leq x \leq c, \\ x & \text{if} \quad c \leq x \leq d, \\ d & \text{if} \quad d \leq x \leq b. \end{cases}$$

Step 2. *If $L_1, L_2, ..., L_n$ is a finite collection of arcs, then there an arc L such that $L_1, L_2, ..., L_n$ are subarc of L.* For each $i \in \{1, 2, ..., n\}$ let a_i, b_i be a pair of end points of L_i such that $a_i < b_i$. If we identify the pair of points $\{b_1, a_2\}, \{b_2, a_3\}, ..., \{b_{n-1}, a_n\}$ we obtain an arc L such that $L_i \subset L$ for each $i \in \{1, 2, ..., n\}$.

Step 3. *$L_1 \times L_2 \times ... \times L_n$ is a retract of L^n.* Let L and $L_1, L_2, ..., L_n$ be as in Step 2. Let $r_i : L \to L_i, i \in \{1, 2, ..., n\}$ be a retraction defined in Step 1. Let us prove that $r = r_1 \times r_2 \times ... \times r_n$ is a retraction of L^n onto $L_1, L_2, ..., L_n$. If $(y_1, y_2, ..., y_n) \in L^n$, then we have: $r_1 \times r_2 \times ... \times r_n(y_1, y_2, ..., y_n) = (r_1(y_1), r_2(y_2), ..., r_n(y_n)) \in L_1 \times L_2 \times ... \times L_n$ since $r_i(y_i) \in L_i$. If $(x_1, x_2, ..., x_n) \in L_1 \times L_2 \times ... \times L_n$, then $r_1 \times r_2 \times ... \times r_n(x_1, x_2, ..., x_n) = (r_1(x_1), r_2(x_2), ..., r_n(x_n)) = (x_1, x_2, ..., x_n) \in L_1 \times L_2 \times ... \times L_n$ since $r_i(x_i) \in x_i$.

Step 4. The product $L_1 \times L_2 \times ... \times L_n$ has the fixed point property since it is retract of the product L^n which has the fixed point property (Theorem 4.5). The proof is completed. □

Theorem 4.7. *If $L = \Pi\{L_s : s \in S\}$ is a Cartesian product of arcs L_s, then L has the fixed point property.*

Proof. Apply Theorems 4.6 and 2.3. □

For Cartesian product of two chainable continua the assumption concerning the weight in Theorem 4.4 can be omitted.

Theorem 4.8. *If X and Y are non-metrizable chainable continua, then $X \times Y$ has the fixed point property.*

Proof. First we shall prove that if X is any chainable continuum and if Y is a metric chainable continuum, then $X \times Y$ has the fixed point property. By Theorem 3.1 there exists a σ-directed inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ such that each X_a is a metric chainable continuum and X is homeomorphic to $\lim \mathbf{X}$. It is clear that $\mathbf{X}{\times}Y = \{X_a \times Y,\ p_{ab} \times id,\ A\}$ is a σ-directed inverse system whose limit is homeomorphic to $X \times Y$. Every $X_a \times Y$ has the fixed point property since it is the product of metric chainable continua (Theorem 4.2). Applying Theorem 2.2 we infer that $X \times Y$ has the fixed point property.

Suppose now that X and Y are non-metric chainable continua. Using again Theorem 3.1 we obtain a σ-directed inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ such that each X_a is a metric chainable continuum and X is homeomorphic to $\lim \mathbf{X}$. It is clear that $\mathbf{X}{\times}Y = \{X_a \times Y,\ p_{ab} \times id,\ A\}$ is a σ-representation of $X \times Y$. From the first part of this proof it follows that every $X_a \times Y$ has the fixed point property since it is the product of metric chainable continuum X_a and an chainable continuum Y. Applying Theorem 2.2 we infer that $X \times Y$ has the fixed point property. □

We close this section with the fixed point property for multifunctions on chainable continua.

A *multifunction*, $F : X \to Y$, from a space X to a space Y is a point-to-set correspondence such that, for each $x \in X$, $F(x)$ is a subset of Y. For any $y \in Y$, we write $F^{-1}(y) = \{x \in X : y \in F(x)\}$. If $A \subset X$ and $B \subset Y$, then $F(A) = \cup\{F(x) : x \in A\}$ and $F^{-1}(B) = \cup\{F^{-1}(y) : y \in B\}$.

A multifunction $F : X \to Y$ is said to be *continuous* if and only if: (i) $F(x)$ is closed for each $x \in X$, (ii) $F^{-1}(B)$ is closed for each closed set B in Y, (iii) $F^{-1}(V)$ is open for each open set V in Y.

A topological space X is said to have F.p.p (fixed point property for multi-valued functions) if for every multi-valued continuous function $F : X \to X$ there exists a point $x \in X$ such that $x \in F(x)$. It follows that X has F.p.p if for every single-valued continuous function $F : X \to 2^X$ there exists a point $x \in X$ such that $x \in F(x)$.

Theorem 4.9. [12]. *If X is any metric chainable continuum, then X has the F.p.p.*

Now we shall prove the following result.

Theorem 4.10. *Each chainable continuum X has the F.p.p.*

Proof. If an chainable continuum is metrizable, then it has F.p.p (Theorem 4.9). Suppose that chainable continuum X is non-metrizable. By virtue of Theorem 3.1 there exists a σ-system $\mathbf{X}_\sigma = \{X_\Delta, P_{\Delta\Gamma}, A_\sigma\}$ of metric chainable continua X_Δ and onto mappings $P_{\Delta\Gamma}$ such that X is homeomorphic to $\lim \mathbf{X}_\sigma$. Moreover, we have the inverse system $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$ whose limit is 2^X. Let $F : X \to 2^X$ be a continuous mapping. From Theorem 1.3 it follows that there exists a subset B cofinal in A such that for every $b \in B$ there exists a continuous mapping $F_b : X_b \to 2^{X_b}$ with the property that $\{F_b : b \in B\}$ is a morphism which induce F. Theorem 4.9 implies that the set $Y_b \subset X_b, b \in B$, of fixed points of F_b is non-empty. Let us prove that Y_b is a closed subset of X_b. We shall prove that $U_b = X_b \setminus Y_b$ is open. Let $x_b \in U_b$. This means that x_b and $F_b(x_b)$ are disjoint closed subset of X_b. By the normality of X_b there exists a pair of open sets U, V such that $x \in U$ and $Y_b \subset V$. From the continuity of F_b it follows that there exists an open set $W \subset U$ such that for every $x \in W$ we have $f(x) \subset V$. Hence, U_b is open and, consequently, Y_b is closed. Now, we shall prove that the collection $\{Y_b, p_{bc}|Y_c, B\}$ is an inverse system. To do this we have to prove that if $c > b$, then $p_{bc}(Y_c) \subset Y_b$. Let x_c be a point of Y_c. This means that $x_c \in f_c(x_c)$. Hence, $p_{bc}(x_c) \in p_{bc}(F_c(x_c)) = F_b p_{bc}(x_c)$. We conclude that the point $x_b = p_{bc}(x_c)$ has the property $x_b \in f_b(x_b)$, i.e., $x_b = p_{bc}(x_c) \in Y_b$. Finally, $p_{bc}(Y_c) \subset Y_b$. and $\{Y_b, p_{bc}|Y_c, B\}$ is an inverse system with non-empty limit. Let $Y = \lim\{Y_b, p_{bc}|Y_c, B\}$. In order to complete the proof we shall prove that for every $x \in Y$ we have $x \in F(x)$. Now we have $p_b(x) \in Y_b$, i.e., $p_b(x) \in F_b(p_b(x)) = p_b F(x)$, for every $b \in B$. It follows that $x \in F(x)$ since $x \notin F(x)$ implies that there is a $b \in B$ such that $p_b(x) \notin p_b F(x)$. We conclude that F has the fixed point property. $\square$

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ON A GENERALIZATION OF ATOMIC DECOMPOSITIONS

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Abstract. We generalize atomic decomposition for Banach spaces and called it T-atomic decomposition. A necessary condition for T-atomic decomposition is given. A characterization for a triangular atomic decomposition is also given. Finally, as an application of triangular atomic decompositions, we prove that if a Banach space has a triangular atomic decomposition, then it also has an approximative atomic decomposition, an atomic decomposition and a fusion Banach frame.

1. Introduction

Coifman and Weiss [3] introduced the notion of atomic decomposition for function spaces. Feichtinger and Gröchenig [5] extended the notion of atomic decomposition to Banach spaces. Frazier and Jawerth [6] had constructed wavelet atomic decompositions for Besov spaces which they called as ϕ-transform. Feichtinger [4] constructed Gabor atomic decompositions for the modulation spaces which are Banach spaces similar in many respects to Besov spaces, defined by smoothness and decay conditions. Atomic decompositions have played a key role in the development of wavelet theory and Gabor theory. Atomic decompositions and Banach frames were further studied in [1, 2, 8].

Motivated by Kozolov [10], we generalize atomic decompositions for Banach spaces. Infact, we introduce the notion of T-atomic decomposition for Banach spaces. Also, a necessary condition for T-atomic decomposition has been obtained. Further, a characterization for triangular atomic decomposition and a characterization for Banach frames have been obtained. Finally, as an application of triangular atomic decompositions, it has been proved that if a Banach space E has a triangular atomic decomposition, then E also has an approximative atomic decomposition, an atomic decomposition and a fusion Banach frame.

2. Preliminaries

Throughout this paper, E will denote a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), E^* the dual space of E, $L(E)$ the space of all linear operator on E, $[x_n]$ the closed linear span of $\{x_n\}$ in the norm topology of E, E_d an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$.

A sequence $\{x_n\}$ in E is said to be *complete* if $[x_n] = E$ and a sequence $\{f_n\}$ in E^* is said to be *total* over E if $\{x \in E : f_n(x) = 0, n \in \mathbb{N}\} = \{0\}$. A sequence of projections $\{v_n\}$ on E is *total* on E if $\{x \in E : v_n(x) = 0,\ n \in \mathbb{N}\} = \{0\}$.

Definition 2.1 ([5])**.** Let E be a Banach space and E_d be an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$. Let $\{x_n\} \subset E$ and $\{f_n\} \subset E^*$. Then, $(\{f_n\}, \{x_n\})$ is called an *atomic decomposition* for E with respect to E_d, if

(i) $\{f_n(x)\} \in E_d$, $x \in E$

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(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that

$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad x \in E$

(iii) $x = \sum_{i=1}^{\infty} f_i(x)x_i, \quad x \in E.$

The constants A and B, respectively, are called lower and upper atomic bounds of the atomic decomposition $(\{f_n\}, \{x_n\})$.

Definition 2.2 ([7])**.** Let E be a Banach space and E_d be an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$. Let $\{f_n\} \subset E^*$ and $S : E_d \to E$ be given. Then, $(\{f_n\}, S)$ is called a *Banach frame* for E with respect to E_d, if

(i) $\{f_n(x)\} \in E_d$, $x \in E$

(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad x \in E \tag{2.1}$$

(iii) S is a bounded linear operator such that

$S(\{f_n(x)\}) = x, \quad x \in E.$

The constants A and B, respectively, are called lower and upper frame bounds of the Banach frame $(\{f_n\}, S)$. The operator $S : E_d \to E$ is called the reconstruction operator (or, the pre-frame operator). The inequality (2.1) is called the Banach frame inequality.

A generalization of the concept of Banach frame namely, fusion Banach frame was introduced and studied in [9] and defined as follows:

Definition 2.3. Let E be a Banach space. Let $\{G_n\}$ be a sequence of subspaces of E and $\{v_n\}$ be a sequence of non-zero linear projections such that $v_n(E) = G_n$, $n \in \mathbb{N}$. Let $\mathcal{A}$ be a Banach space associated with E and $S : \mathcal{A} \to E$ be an operator. Then, $(\{G_n, v_n\}, S)$ is called a *frame of subspaces* (or, *fusion Banach frame*) for E with respect to $\mathcal{A}$, if

(i) $\{v_n(x)\} \in \mathcal{A}$, $x \in E$

(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that

$A\|x\|_E \leq \|\{v_n(x)\}\|_{\mathcal{A}} \leq B\|x\|_E, \quad x \in E$

(iii) S is a bounded linear operator such that

$S(\{v_n(x)\}) = x, \quad x \in E.$

The constants A and B, respectively, are called lower and upper frame bounds of the frame of subspaces $(\{G_n, v_n\}, S)$.

The following results are referred in this paper and are listed in the form of lemmas:

Lemma 2.4 ([12])**.** *If E is a Banach space and $\{f_n\} \subset E^*$ is total over E, then E is linearly isometric to the associated Banach space $E_d = \{\{f_n(x)\} : x \in E\}$, where the norm is given by $\|\{f_n(x)\}\|_{E_d} = \|x\|_E$, $x \in E$.*

Lemma 2.5 ([9])**.** *Let $\{G_n\}$ be a sequence of non-trival subspaces of E and $\{v_n\}$ be a sequence of non-zero linear projections with $v_n(E) = G_n$, $n \in \mathbb{N}$. If $\{v_n\}$ is total over E, then $\mathcal{A} = \{\{v_n(x)\} : x \in E\}$ is a Banach space with norm $\|\{v_n(x)\}\|_{\mathcal{A}} = \|x\|_E$, $x \in E$.*

3. Main Results

We begin with the following definition of T-atomic decomposition

Definition 3.1. Let E be a Banach space, E_d be an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$ and $T = (t_{nm})$ be a matrix of scalars such that

$$\sum_{j=1}^{\infty}|t_{nj}| \leq M < \infty, \qquad n = 1, 2, 3, \ldots \tag{3.1}$$

$$\lim_{n\to\infty} t_{nj} = 0, \qquad j = 1, 2, 3, \ldots \tag{3.2}$$

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} t_{nj} = 1. \tag{3.3}$$

Let $\{x_n\}$ be a sequence in E and $\{f_n\}$ be a sequence in E^*. Then, $(T, \{f_n\}, \{x_n\})$ is called a T-*atomic decomposition* for E with respect to E_d, if

(i) $\{f_n(x)\} \in E_d$, $x \in E$

(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that

$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad x \in E$

(iii) $\lim\limits_{n\to\infty} \sum\limits_{j=1}^{\infty} t_{nj}\left(\sum\limits_{i=1}^{j} f_i(x)x_i\right) = x$, $x \in E$.

In case, T is a triangular matrix, then $(T, \{f_n\}, \{x_n\})$ is said to be a *triangular atomic decomposition* for E with respect to E_d.

Regarding the existence of T-atomic decomposition, let E be a Banach space, $(\{f_n\}, \{x_n\})(\{f_n\} \subset E^*, \{x_n\} \subset E)$ be an atomic decomposition for E with respect to an associated Banach space E_d and $T = (t_{nm})$ be a matrix such that $t_{nn} = 1$, $n \in \mathbb{N}$ and $t_{nm} = 0$, $m \neq n$. Then $(T, \{f_n\}, \{x_n\})$ is a T-atomic decomposition for E with respect to E_d.

Also, one may observe that if E is a Banach space and $(T, \{f_n\}, \{x_n\})$ $(T = (t_{nm}), \{f_n\} \subset E^*, \{x_n\} \subset E)$ is a T-atomic decomposition for E with respect to E_d, then $\{x_n\}$ is complete in E and for each $n \in \mathbb{N}$, $\sigma_n : E \to E$ defined by

$$\sigma_n(x) = \sum_{j=1}^{\infty} t_{nj}\left(\sum_{i=1}^{j} f_i(x)x_i\right), \quad x \in E$$

is well defined bounded linear operator such that $\sup\limits_{1\leq n<\infty} \|\sigma_n\| < \infty$.

Conversely, we have the following example

Example 3.2. Let $E = c_0$, the space of all sequences convergent to 0 in $\mathbb{K}$. Let $T = (t_{nm})$ be a matrix such that $t_{nn} = 1$, $n \in \mathbb{N}$ and $t_{nm} = 0$, $n \neq m$. Let $\{e_n\}$ be the sequence of unit vectors in E and $\{f_n\}$ be a sequence in E^* defined by

$$f_n = (0, 0, \ldots, \underset{\substack{\uparrow \\ n^{\text{th}} \text{ position}}}{(-1)^n}, 0, 0, \ldots), \quad n \in \mathbb{N}.$$

Then, $\{e_n\}$ is complete in E and for each $n \in \mathbb{N}$, $\sigma_n : E \to E$ defined by

$$\sigma_n(x) = \sum_{j=1}^{\infty} t_{nj}\left(\sum_{i=1}^{j} f_i(x)e_i\right), \quad x \in E$$

is well defined bounded linear operator such that $\sup\limits_{1\leq n<\infty} \|\sigma_n\| < \infty$. But $\lim\limits_{n\to\infty} \sigma_n(x) \neq x$, for some $x \in E$. Infact, if we take $x = (1, 0, 0, \ldots) \in E$ then $\lim\limits_{n\to\infty} \sigma_n(x) \neq x$. Hence, $(T, \{f_n\}, \{e_n\})$ is not a T-atomic decomposition for E with respect to any associated Banach space E_d.

In the next result, we prove that for any matrix $T = (t_{nm})$ satisfying (3.1)-(3.3), every atomic decomposition for E is also a T-atomic decomposition for E.

Theorem 3.3. *Let $(\{f_n\}, \{x_n\})$ be an atomic decomposition for a Banach space E with respect to E_d. Then, for any matrix $T = (t_{nm})$ satisfying* (3.1)-(3.3), *$(T, \{f_n\}, \{x_n\})$ is a T-atomic decomposition for E with respect to E_d.*

Proof. Let c_E be the Banach space of all convergent sequences of elements of E with the norm $\|\{z_k\}\|_{c_E} = \sup\limits_{1\le k<\infty} \|z_k\|_E$. For each $n\in\mathbb{N}$, define $u_n : c_E \to E$ by

$$u_n(\{z_k\}) = \sum_{j=1}^{\infty} t_{nj} z_j, \quad \{z_k\} \in c_E.$$

Then, each u_n is well defined on c_E and

$$\begin{aligned} \|u_n\| &= \sup_{\{z_k\}\in c_E} \|u_n(\{z_k\})\| \\ &= \sum_{j=1}^{\infty} |t_{nj}| \le M, \quad n \in \mathbb{N}. \end{aligned}$$

Now, for any $\{x_1, x_2, \ldots, x_m, 0, 0, \ldots\} \in c_E$, we have

$$\lim_{n\to\infty} u_n(\{x_1, x_2, \ldots x_m, 0, 0, \ldots\}) \;=\; 0$$

and, for any $\{x, x, x, \ldots\} \in c_E$, we have

$$\lim_{n\to\infty} u_n(x, x, x, \ldots) \;=\; x, \quad x \in E.$$

Since, the set of all the elements of the form $\{x_1, x_2, \ldots, x_m, 0, 0, \ldots\}$ and $\{x, x, x, \ldots\}$, where $x_1, x_2, \ldots, x_m \in E$, $1 \le m < \infty$ and $x \in E$ is complete in c_E, we have

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} t_{nj} z_j = \lim_{n\to\infty} u_n(\{z_k\}) = \lim_{k\to\infty} z_k.$$

Define, $S_n(x) = \sum\limits_{i=1}^{n} f_i(x) x_i$, $n \in \mathbb{N}$ and $x \in E$. Then $\lim\limits_{n\to\infty} S_n(x) = x$, $x \in E$. Therefore, $\lim\limits_{n\to\infty} \sum\limits_{j=1}^{\infty} t_{nj} s_j(x) = x$, $x \in E$.

Hence, $(T, \{f_n\}, \{x_n\})$ is a T-atomic decomposition for E with respect to E_d. □

The converse of Theorem 3.3 may not be true as shown by the following example

Example 3.4. Let $E = \ell^2$. Define $\{x_n\} \subset E$ and $\{f_n\} \subset E^*$ by

$$\begin{aligned} x_n &= e_n - e_{n+1} \\ f_n(x) &= \langle e_1 + e_2 + \ldots + e_n, x\rangle, \quad x \in E,\ n = 1, 2, \ldots\ . \end{aligned}$$

Then, $(\{f_n\}, \{x_n\})$ is not an atomic decomposition for E with respect to any associated Banach space E_d. But, by Lemma 2.4, there exist an associated Banach space $E_{d_0} = \{\{f_n(x)\} : x \in E\}$ with the norm $\|\{f_n(x)\}\|_{E_{d_0}} = \|x\|_E$, $x \in E$ and a matrix $T = (t_{nm})$ given by $t_{nm} = \frac{1}{n}$, $m = 1, 2, \ldots, n$, $t_{nm} = 0$ for $m > n$ $(n = 1, 2, \ldots)$ such that $(T, \{f_n\}, \{x_n\})$ is a T-atomic decomposition for E with respect to E_{d_0}.

Indeed,

$$\begin{aligned}\sigma_n(x) &= \sum_{i=1}^{n}\frac{n-i+1}{n}f_i(x)x_i\\ &= \sum_{i=1}^{n}\frac{n-i+1}{n}\Big\langle\sum_{j=1}^{i}e_j,x\Big\rangle(e_i-e_{i+1})\\ &= \langle e_1,x\rangle e_1+\sum_{i=2}^{n}\left[\frac{n-i+1}{n}\Big\langle\sum_{j=1}^{i}e_j,x\Big\rangle e_i-\frac{n-i+2}{n}\Big\langle\sum_{j=1}^{i-1}e_j,x\Big\rangle e_i\right]\\ &\quad -\frac{1}{n}\Big\langle\sum_{j=1}^{n}e_j,x\Big\rangle e_{n+1}\end{aligned}$$

$$= \sum_{i=1}^{n}\frac{n-i+1}{n}\langle e_i,x\rangle e_i-\frac{1}{n}\sum_{i=2}^{n+1}\Big\langle\sum_{j=1}^{i-1}e_j,x\Big\rangle e_i,\quad x\in E, n=1,2,3,\dots\ .$$

Since, $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\langle e_i,x\rangle e_i=x$, $x\in E$, we have

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n-i+1}{n}\langle e_i,x\rangle e_i=x,\quad x\in E.$$

For each $n\in\mathbb{N}$, define $v_n:E\to E$ by

$$v_n(x)=\frac{1}{n}\sum_{i=1}^{n}\Big\langle\sum_{j=1}^{i}e_j,x\Big\rangle e_{i+1},\quad x\in E, n=1,2,\dots\ .$$

Then, each v_n is well defined bounded linear operator on E. Also, for each $n,k=1,2,3\dots$, we have

$$\|v_n(e_k)\|^2=\frac{1}{n^2}\sum_{i=1}^{n}\left|\Big\langle\sum_{j=1}^{i}e_j,e_k\Big\rangle\right|^2=\frac{n-k+1}{n^2}.$$

Therefore, $\lim\limits_{n\to\infty}\|v_n(e_k)\|^2=0$. Hence, $\lim\limits_{n\to\infty}v_n(x)=0$, $x\in\operatorname{span}\{x_i\}_{i=1}^{\infty}$. Also, since

$$\begin{aligned}\|v_n(x)\|^2 &= \frac{1}{n^2}\sum_{i=1}^{n}\left|\Big\langle\sum_{j=1}^{i}e_j,x\Big\rangle\right|^2\\ &\le \frac{1}{n^2}\sum_{i=1}^{n}\left\|\sum_{j=1}^{i}e_j\right\|^2\|x\|^2\\ &= \frac{n(n+1)}{2n^2}\|x\|^2\\ &\le \|x\|^2,\qquad x\in E,\ n=1,2,3,\dots\ ,\end{aligned}$$

we have, $\sup\limits_{1\le n<\infty}\|v_n\|<\infty$. Hence, $\lim\limits_{n\to\infty}\sigma_n(x)=x$, $x\in E$.

Next, we give a necessary condition for a T-atomic decomposition in a Banach space.

Theorem 3.5. *Let E be a Banach space and $T = (t_{nm})$ be a matrix satisfying* (3.1)-(3.3)*. If $(T, \{f_n\}, \{x_n\})(\{f_n\} \subset E^*,\ \{x_n\} \subset E)$ is a T-atomic decomposition for E with respect to E_d. Then for each $n, m \in \mathbb{N}$, there exists a linear operator $v_{nm} \in L(E)$ such that*

$$\lim_{n\to\infty}\lim_{m\to\infty} v_{nm}(x) = x, \quad x \in E.$$

Proof. For each $n, m = 1, 2, 3 \ldots$, define

$$v_{nm}(x) = \sum_{j=1}^{m} t_{nj}\left(\sum_{i=1}^{j} f_i(x)x_i\right), \quad x \in E.$$

Then, $v_{nm} \in L(E)$. Also

$$\lim_{m\to\infty} v_{nm}(x) = \sum_{j=1}^{\infty} t_{nj}\left(\sum_{i=1}^{j} f_i(x)x_i\right), \quad x \in E.$$

Since, $(T, \{f_n\}, \{x_n\})$ is a T-atomic decomposition for E, therefore

$$\lim_{n\to\infty}\sum_{j=1}^{\infty} t_{nj}\left(\sum_{i=1}^{j} f_i(x)x_i\right) = x, \quad x \in E. \qquad \square$$

Let E be a Banach space and $T = (t_{nm})$ be a triangular matrix satisfying (3.1)-(3.3). Let $\{x_n\}$ be any sequence in E and $\{f_n\}$ be any sequence in E^*. For each $n \in \mathbb{N}$, define

$$\begin{aligned}
\sigma_n(x) &= \sum_{j=1}^{n} t_{nj}\sum_{i=1}^{j} f_i(x)x_i, \quad x \in E,\ n = 1, 2, 3, \ldots, \\
E_0^{(T)} &= \{x \in E : \lim_{n\to\infty} \sigma_n(x) = x\} \text{ and} \\
E_1^{(T)} &= \{x \in E : \lim_{n\to\infty} \sigma_n(x) \text{ exists}\}.
\end{aligned}$$

The following result characterizes triangular atomic decompositions in terms of $\{\sigma_n\}$ and $E_0^{(T)}$

Theorem 3.6. *Let E be a Banach space and $T = (t_{nm})$ be a triangular matrix satisfying* (3.1)-(3.3)*. Let $\{f_n\} \subset E^*$ and $\{x_n\} \subset E$. Then there exists an associated Banach space E_{d_0} such that $(T, \{f_n\}, \{x_n\})$ is a triangular atomic decomposition for E with respect to E_{d_0} if and only if $\{\sigma_n\}$ is total on E and $E_0^{(T)} = E$.*

Proof. Assume that $E_0^{(T)} = E$ and $\{\sigma_n\}$ is total on E. Let $x \in E$ such that $f_n(x) = 0$ for all $n \in \mathbb{N}$. Then $\sigma_n(x) = 0$, $n \in \mathbb{N}$. So totality of $\{\sigma_n\}$ yields $x = 0$. Therefore, by Lemma 2.4, there exists an associated Banach space $E_{d_0} = \{\{f_n(x)\} : x \in E\}$ with the norm $\|\{f_n(x)\}\|_{E_{d_0}} = \|x\|_E$, $x \in E$. Also, by hypothesis, we have $\lim\limits_{n\to\infty}\sum\limits_{j=1}^{\infty} t_{nj}\Big(\sum\limits_{i=1}^{j} f_i(x)x_i\Big) = x$, $x \in E$. Hence, $(T, \{f_n\}, \{x_n\})$ is a triangular atomic decomposition for E with respect to E_{d_0}.

The converse part is straight forward. $\square$

We conclude this section with the following characterization of Banach frames in terms of $E_0^{(T)}$ and $E_1^{(T)}$

Theorem 3.7. *Let E be a Banach space and $T = (t_{nm})$ be a triangular matrix satisfying* (3.1)-(3.3)*. Let $\{x_n\} \subset E$ and $\{f_n\} \subset E^*$ such that $f_i(x_j) = \delta_{ij}$, $i, j \in \mathbb{N}$. Then there exist an associated Banach space E_d and a bounded linear operator*

$S : E_d \to E$ such that $(\{f_n\}, S)$ is Banach frame for E with respect to E_d if and only if $E_0^{(T)} = E_1^{(T)}$.

Proof. Let $(\{f_n\}, S)$ be a Banach frame for E. Then

$$\begin{aligned} f_m(\sigma_n(x)) &= f_m\bigg(\sum_{j=1}^{\infty} t_{nj}\bigg(\sum_{i=1}^{j} f_i(x)x_i\bigg)\bigg) \\ &= \bigg(\sum_{j=m}^{\infty} t_{nj}\bigg)f_m(x), \quad n, m = 1, 2, 3\ldots \text{ and } \ x \in E. \end{aligned}$$

Let $x \in E_1^{(T)}$. Then

$$\begin{aligned} f_m(x - \lim_{n\to\infty} \sigma_n(x)) &= f_m(x) - \lim_{n\to\infty}\bigg(\sum_{j=m}^{\infty} t_{nj}\bigg)f_m(x) \\ &= f_m(x)\bigg[1 - \lim_{n\to\infty}\sum_{j=1}^{\infty} t_{nj} + \lim_{n\to\infty}\sum_{j=1}^{m-1} t_{nj}\bigg] = 0 \end{aligned}$$

Therefore, by the frame inequality for the Banach frame $(\{f_n\}, S)$, we have $x \in E_0^{(T)}$.

Conversely, let $x \in E$ be such that $f_n(x) = 0$, $n = 1, 2, 3\ldots$. Then $\sigma_n(x) = 0$ for all $n \in \mathbb{N}$. Since, $E_0^{(T)} = E_1^{(T)}$, we have $x = 0$. Therefore, by Lemma 2.4, there exist associated Banach space $E_{d_0} = \{\{f_n(x)\} : x \in E\}$ with the norm $\|\{f_n(x)\}\|_{E_{d_0}} = \|x\|_E$, $x \in E$ and a bounded linear operator $S : E_{d_0} \to E$ defined by $S(\{f_n(x)\}) = x$, $x \in E$ such that $(\{f_n\}, S)$ is a Banach frame for E with respect to E_{d_0}. □

4. Applications

In this section, we give some applications of triangular atomic decompositions. First, we give the definition of approximative atomic decomposition introduced in [11].

Let E be a Banach space and let E_d be an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$. Let $\{x_n\} \subset E$ and $\{h_{n,i}\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}} \subset E^*$, where $\{m_n\}$ is an increasing sequence of positive integers. Then, $(\{h_{n,i}\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}}, \{x_n\})$ is called an approximative atomic decomposition for E with respect to E_d, if

(i) $\{h_{n,i}(x)\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}} \in E_d$, $x \in E$

(ii) there exist constants A and B with $0 < A \le B < \infty$ such that
$A\|x\|_E \le \|\{h_{n,i}(x)\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}}\|_{E_d} \le B\|x\|_E, \quad x \in E$

(iii) $x = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{m_n} h_{n,i}(x)x_i$, $x \in E$.

In the following result, we prove that if a Banach space has a triangular atomic decomposition, then it also has an approximative atomic decomposition

Theorem 4.1. *If a Banach space has a triangular atomic decomposition then it also has an approximative atomic decomposition.*

Proof. Let E be a Banach space having a triangular atomic decomposition $(T, \{f_n\}, \{x_n\})(T = (t_{nm}), \{f_n\} \subset E^*, \{x_n\} \subset E)$ with respect to E_d. Since, T is a triangular matrix, for each $n, m \in \mathbb{N}$, $m \ge n$,

$$\sum_{j=1}^{m} t_{nj}\bigg(\sum_{i=1}^{j} f_i(x)x_i\bigg) = \sum_{j=1}^{n} t_{nj}\bigg(\sum_{i=1}^{j} f_i(x)x_i\bigg), \quad x \in E.$$

For each $n \in \mathbb{N}$, define $\sigma_n : E \to E$ by

$$\sigma_n(x) = \sum_{j=1}^{n} t_{nj}\left(\sum_{i=1}^{j} f_i(x)x_i\right), \quad x \in E.$$

Then, each σ_n is well defined finite rank linear operator on E. Since, for each $n \in \mathbb{N}, \sigma_n(E)$ is finite dimensional. So, there exist a sequence $\{y_{n,i}\}_{i=m_{n-1}+1}^{m_n}$ in E and a total sequence $\{g_{n,i}\}_{i=m_{n-1}+1}^{m_n}$ in E^* such that

$$\sigma_n(x) = \sum_{i=m_{n-1}+1}^{m_n} g_{n,i}(x)y_{n,i}, \quad x \in E,\ n \in \mathbb{N},$$

where $\{m_n\}$ is an increasing sequence of positive integers with $m_0 = 0$. Define, $\{z_n\} \subset E$ and $\{h_{n,i}\}_{\substack{i=1,2,\ldots m_n \\ n\in\mathbb{N}}} \subset E^*$ by

$$\begin{aligned} z_i &= y_{n,i}, \quad i = m_{n-1}+1, \ldots, \quad m_n, \\ h_{n,i} &= \begin{cases} 0, & \text{if } \ i = 1, 2, \ldots, m_{n-1} \\ g_{n,i}, & \text{if } \ i = m_{n-1}+1, \ldots m_n, \end{cases} \qquad n \in \mathbb{N}\,. \end{aligned}$$

Then, for each $x \in E$,

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} h_{n,i}(x)z_i = \lim_{n\to\infty} \sigma_n(x) = x\,.$$

Let $x \in E$ be such that $h_{n,i}(x) = 0$, for all $i = 1, 2, \ldots, m_n,\ n \in \mathbb{N}$. Then $x = 0$. Therefore, by Lemma 2.4, there exists an associated Banach space $E_{d_0} = \{\{h_{n,i}(x)\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}} : x \in E\}$ with the norm given by $\|\{h_{n,i}(x)\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}}\|_{E_{d_0}} = \|x\|_E$, $x \in E$ such that, $(\{h_{n,i}\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}}, \{z_n\})$ is an approximative atomic decomposition for E with respect to E_{d_0}. □

Corollary 4.2. If a Banach space E has a triangular atomic decomposition, then it also has an atomic decomposition.

Proof. Follows in view of Theorem 4.1. □

Finally, we prove that, if for a suitably chosen triangular matrix T satisfying (3.1)-(3.3), E has a triangular atomic decomposition, then it also has a fusion Banach frame.

Theorem 4.3. *Let E be a Banach space and $T = (t_{nm})$ be a triangular matrix such that $t_{nm} \neq 0$, $n \geq m$. If $(T, \{f_n\}, \{x_n\})(\{f_n\} \subset E^*, \{x_n\} \subset E)$ is a triangular atomic decomposition for E, then E has a fusion Banach frame.*

Proof. By Theorem 4.1, E has approximative atomic decomposition. Let $\{x_n\} \subset E$ and $\{h_{n,i}\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}} \subset E^*$ be sequences such that $(\{h_{n,i}\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}}, \{x_n\})$ is an approximative atomic decomposition for E with respect to E_d, where $\{m_n\}$ is an increasing sequence of positive integers. For each $n \in \mathbb{N}$, define $u_n : E \to E$ by

$$u_n(x) = \sum_{i=1}^{m_n} h_{n,i}(x)x_i, \quad x \in E.$$

Then, each u_n is a well defined continuous linear operator on E with $\dim u_n(E) < \infty$ and $\lim_{n\to\infty} u_n(x) = x$, $x \in E$. Define $G_n = u_n(E)$, $n \in \mathbb{N}$. Then, each G_n is finite dimensional. Therefore, there exist a sequence $\{y_{n,i}\}_{i=1}^{m_n}$ in E and a total sequence $\{g_{n,i}\}_{i=1}^{m_n}$ in E^* such that

$$u_n(x) = \sum_{i=1}^{m_n} g_{n,i}(x) y_{n,i}, \quad x \in E \quad \text{and } n \in \mathbb{N}.$$

Now, for each $n \in \mathbb{N}$, define $v_n : E \to E$ by

$$v_n(x) = \sum_{i=1}^{m_n} g_{n,i}(x) y_{n,i}, \quad x \in E.$$

Then, each v_n is a projection on G_n such that $\{x \in E : v_n(x) = 0, n \in \mathbb{N}\}$=$\{0\}$. Therefore, by Lemma 2.5, there exist an associated Banach space $\mathcal{A} = \{v_n(x) : x \in E\}$ with the norm given by $\|\{v_n(x)\}\|_{\mathcal{A}} = \|x\|_E$, $x \in E$ and a bounded linear operator $S : \mathcal{A} \to E$ given by $S(\{v_n(x)\}) = x$, $x \in E$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to $\mathcal{A}$. □

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GENUS CALCULATIONS FOR TOWERS OF FUNCTION FIELDS ARISING FROM EQUATIONS OF C_{ab} CURVES

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ABSTRACT. We give a generalization of error-correcting code construction from C_{ab} curves by working with towers of algebraic function fields. The towers are constructed recursively, using defining equations of C_{ab} curves. In order to estimate the parameters of the corresponding one-point Goppa codes, one needs to calculate the genus. Instead of using the Hurwitz genus formula, for which one needs to know about ramification behavior, we use the Riemann-Roch theorem to get an upper bound for the genus by counting the number of Weierstrass gap numbers associated to a particular divisor. We provide a family of examples of towers which meet the bound.

1. INTRODUCTION

Let K be a perfect field, and let a and b be positive integers with $a > b$ and $\gcd(a, b) = 1$. Consider the polynomial $f \in K[x, y]$ given by

$$f(x, y) = \alpha_{a,0}x^a + \alpha_{0,b}y^b + \sum_{i,j} \alpha_{i,j}x^i y^j,$$

where $\alpha_{i,j} \in K$, $\alpha_{a,0}\alpha_{0,b} \neq 0$, and the summation is taken over non-negative integers i and j such that $aj + bi < ab$. The plane curve C defined by the equation $f = 0$ is called a C_{ab} curve. Such curves can be thought of as generalizations of the Weierstrass form of elliptic and hyperelliptic curves.

One has the following results (see Section 3.3 in [4], Proposition 4.6 in [5], or [3]). C is irreducible, with a single point at infinity. Let F/K be the associated function field, so $F = K(x, y)/(f)$. (We will use the function field notation of [8].) Let P_∞ be the unique place at infinity. If the affine points of C are non-singular, then the genus of C is $(a - 1)(b - 1)/2$, and the Riemann-Roch space $\mathcal{L}(mP_\infty)$ is generated by monomials of the form $x^i y^j$ where $aj + bi \leq m$. (In fact, a basis is $\{x^i y^j : 0 \leq i, 0 \leq j < b, aj + bi \leq m\}$.)

We will be concerned with the case where $K = \mathbb{F}_q$, the finite field with q elements. For the construction of geometric Goppa codes (as in [8], Chapter II), one uses a function field $F/\mathbb{F}_q$ of transcendence degree 1. For some divisor D, one needs to calculate a basis of functions for $\mathcal{L}(D)$ and then evaluate these functions at other rational places in the function field. A lower bound for the sum of the dimension and minimum distance of a Goppa code is known in terms of the genus of F. Since we can calculate a basis for $\mathcal{L}(mP_\infty)$ and know the genus of the function field of a non-singular C_{ab} curve, constructing a code with a C_{ab} curve essentially amounts to calculating rational places. Importantly, decoding methods exist for such curves, as is seen in [7].

In this paper, we present towers of function fields that give a generalization of the C_{ab} curve code constructions. These towers arise recursively from the defining equations of C_{ab} curves, which is explained in Section 2. Because of the way these towers are constructed, bases for the spaces $\mathcal{L}(mP_\infty)$ are easy to calculate. In Section 3, we explain why these bases consist of only monomials and then proceed to calculate an upper bound for the genus by counting the number of Weierstrass gap numbers. These results are summarized in Theorem 6. Examples of families of towers that achieve the bound are also given in Theorem 7.

2. Equations and valuations

Let C be a C_{ab} curve given by equation $f = 0$, for $f \in K[x, y]$. For each $n \geq 0$, let

$$C_n = \left\{(p_0, p_1, \dots, p_n) \in \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 : (p_{j-1}, p_j) \in C \text{ for } j = 1, \dots, n\right\}.$$

Consider the associated tower of function fields $\mathcal{F} = (F_0, F_1, \dots)$, where $F_0 = K(x_0)$ and $F_n = F_{n-1}(x_n)$ for $n \geq 1$. The x_k are related by the equation

$$f(x_{k-1}, x_k) = 0$$

for $k = 1, \dots, n$.

F_0 is a rational function field. The divisor associated to the function x_0 is $P_0 - P_\infty$, where

$$P_0 = \left\{\frac{x_0 \cdot f(x_0)}{g(x_0)} : f(x_0), g(x_0) \in K[x_0], x_0 \nmid g(x_0)\right\}$$

and

$$P_\infty = \left\{\frac{f(x_0)}{g(x_0)} : \deg g > \deg f\right\}.$$

To each place, we have an associated valuation. For the purposes of this paper, we are only interested in the valuation associated to P_∞, which is defined as

$$v_\infty\left(\frac{f(x)}{g(x)}\right) = \deg g - \deg f.$$

Proposition 1. *Let P_∞^n denote a place in $\mathbb{P}_{F_n}$ lying above P_∞, and let v_∞^n be the valuation associated to that place. Then P_∞^n is the only place in $\mathbb{P}_{F_n}$ above P_∞ and $v_\infty^n(x_n) = -a^n$.*

Proof For induction, suppose $n = 0$. The statement holds because we have $F_0 = K(x_0)$, so P_∞^0 is the unique place at infinity in the rational function field and $v_\infty^0(x_0) = -1$.

Now, suppose the statement is true for $n = k$. Since x_k and x_{k+1} satisfy the equation

$$\alpha_{a,0}x_k^a + \alpha_{0,b}x_{k+1}^b + \sum_{aj+bi<ab} \alpha_{i,j}x_k^i x_{k+1}^j = 0,$$

we have

$$\begin{aligned} av_\infty^{k+1}(x_k) &= v_\infty^{k+1}\left(\alpha_{a,0}x_k^a\right) \\ &= v_\infty^{k+1}\left(\alpha_{0,b}x_{k+1}^b + \sum_{aj+ib<ab} \alpha_{i,j}x_k^i x_{k+1}^j\right) \\ &\geq \min\left(\{bv_\infty^{k+1}(x_{k+1})\} \cup \{iv_\infty^{k+1}(x_k) + jv_\infty^{k+1}(x_{k+1}) : aj + ib < ab\}\right), \end{aligned}$$

by the triangle inequality. For the sake of contradiction, suppose

$$iv_\infty^{k+1}(x_k) + jv_\infty^{k+1}(x_{k+1}) \leq bv_\infty^{k+1}(x_{k+1})$$

for some non-negative i, j with $aj + bi < ab$ such that the left-hand side of the inequality is minimal. Then

$$\frac{v_\infty^{k+1}(x_k)}{v_\infty^{k+1}(x_{k+1})} \geq \frac{b-j}{i},$$

and since $aj + bi < ab$, we have $\frac{b-j}{i} > \frac{b}{a}$, so

$$\frac{v_\infty^{k+1}(x_k)}{v_\infty^{k+1}(x_{k+1})} > \frac{b}{a},$$

which means

$$\frac{a}{b} > \frac{v_\infty^{k+1}(x_{k+1})}{v_\infty^{k+1}(x_k)}.$$

On the other hand, if

$$iv_\infty^{k+1}(x_k) + jv_\infty^{k+1}(x_{k+1}) \leq bv_\infty^{k+1}(x_{k+1})$$

for i and j with $aj + bi < ab$ such that the left-hand side is minimal, then by the triangle inequality above,

$$av_\infty^{k+1}(x_k) \geq iv_\infty^{k+1}(x_k) + jv_\infty^{k+1}(x_{k+1}).$$

This implies

$$\frac{a-i}{j} \leq \frac{v_\infty^{k+1}(x_{k+1})}{v_\infty^{k+1}(x_k)}.$$

Since $aj + bi < ab$, we have $\frac{a}{b} < \frac{a-i}{j}$. So

$$\frac{a}{b} < \frac{v_\infty^{k+1}(x_{k+1})}{v_\infty^{k+1}(x_k)}.$$

We have a contradiction to our initial assumption. Therefore, we must have

$$bv_\infty^{k+1}(x_{k+1}) < iv_\infty^{k+1}(x_k) + jv_\infty^{k+1}(x_{k+1})$$

for all i, j with $aj + bi < ab$. Using the strict triangle inequality, we see that

$$av_\infty^{k+1}(x_k) = bv_\infty^{k+1}(x_{k+1}).$$

Since $\gcd(a, b) = 1$, b must divide $v_\infty^{k+1}(x_k)$. Let e denote the ramification degree of P_∞^{k+1} over P_∞^k, so

$$e = \frac{v_\infty^{k+1}(f)}{v_\infty^k(f)}$$

for any $f \in P_\infty^k$. Taking $f = x_k$, we have $ev_\infty^k(x_k) = v_\infty^{k+1}(x_k)$. By the inductive hypothesis, $v_\infty^k(x_k) = -a^k$, so b must divide e. However, since F_{k+1}/F_k is an extension of degree b, the ramification degree is at most b. Therefore, $e = b$, so P_∞^{k+1} is totally ramified over P_∞^k. Since ramification behaves well in towers, P_∞^{k+1} is totally ramified over P_∞^0, and thus unique.

From the formula for the ramification index,

$$\begin{aligned} v_\infty^{k+1}(x_k) &= ev_\infty^k(x_k) \\ &= b(-a^k). \end{aligned}$$

Then, since $av_\infty^{k+1}(x_k) = bv_\infty^{k+1}(x_{k+1})$, we have

$$v_\infty^{k+1}(x_{k+1}) = -a^{k+1},$$

as desired.

By induction, the statement is true for all non-negative integers n. □

Corollary 2. *For $n \geq k$,*

$$v_\infty^n(x_k) = -a^k b^{n-k}.$$

Proof By the formula for the ramification index, we have

$$e(P_\infty^n | P_\infty^k) = \frac{v_\infty^n(x_k)}{v_\infty^k(x_k)},$$

so

$$v_\infty^n(x_k) = -a^k b^{n-k}.$$

From the previous proposition, we know P_∞^n is totally ramified over P_∞^k for all $k < n$ because ramification works transitively in towers. Thus, $e(P_\infty^n | P_\infty^k) = [F_n : F_k] = b^{n-k}$. We also know $v_\infty^k(x_k) = -a^k$. The result follows immediately. □

3. Calculating the genera

In order to calculate the genera, we will count the number of Weierstrass gap numbers of P_∞^n in F_n/K. We do so by considering the pole orders of monomials. In the first subsection, which is motivated by Proposition 14 in [6], we show that the set of pole orders of polynomials is the same as the set of pole orders of monomials. In the second subsection, we show that under certain conditions, one obtains no new pole orders with rational functions. In the third subsection, assuming certain conditions, we calculate the number of Weierstrass gap numbers resulting from monomials, which gives us the genus of the function field. In the fourth subsection, we give examples of towers which satisfy the conditions and hence for which the genus formula applies.

3.1. Valuations of polynomials.

Let

$$I_n = (f(x_0, x_1), \ldots, f(x_{n-1}, x_n)) \subset F_n$$

be the ideal of the curve C_n, and let

$$\Gamma = K[x_0, \ldots, x_n]/I_n$$

be the coordinate ring of C_n. For notation, let $R_b = \{0, 1, \ldots, b-1\}$ be the set of residues mod b.

Claim 1. *Any polynomial $g(x_0, \ldots, x_n) \in \Gamma$ can be written as*

$$g(x_0, \ldots, x_n) = \sum_{\mathbf{e} \in \mathbb{Z}^{\geq 0} \times (R_b)^n} \lambda_{\mathbf{e}} x_0^{e_0} \ldots x_n^{e_n},$$

for $\mathbf{e} = (e_0, e_1, \ldots, e_n)$ and $\lambda_{\mathbf{e}} \in k$. In particular, for $i = 1, \ldots, n$, one has $0 \leq e_i < b$.

Proof Given the polynomial g, one can first reduce all powers of x_n to be less than b using the relation $f(x_{n-1}, x_n) = 0$. One can then reduce all powers of x_{n-1} to be less than b using the relation $f(x_{n-2}, x_{n-1}) = 0$. As this does not affect powers of x_n, one can continue on to reduce all powers of x_{n-2}, …, x_1, giving the resulting form. □

We will call a polynomial written in this form *b-reduced*.

Claim 2. *Let $g(x_0, \ldots, x_n) = x_0^{e_0} \ldots x_n^{e_n}$ and $h(x_0, \ldots, x_n) = x_0^{d_0} \ldots x_n^{d_n}$ be two monomials in Γ with $0 \leq e_i, d_i < b$ for $i = 1, \ldots, n$. Then $v_\infty^n(g) = v_\infty^n(h) \iff g = h$.*

Proof Since the pole order of x_i is $a^i b^{n-i}$, this means that

$$\sum_{i=0}^{n} e_i a^i b^{n-i} = \sum_{j=0}^{n} d_j a^j b^{n-j}.$$

Reducing modulo b,

$$e_n a^n \equiv d_n a^n \mod b.$$

Since $\gcd(a^n, b) = 1$ and $0 \leq e_n, d_n < b$, we have $e_n = d_n$. So

$$\sum_{i=0}^{n-1} e_i a^i b^{n-i} = \sum_{j=0}^{n-1} d_j a^j b^{n-j}.$$

Dividing through by b, we similarly obtain $e_{n-1} = d_{n-1}$, and so on to $e_1 = d_1$. Thus, $e_0 = d_0$, and so $g = h$. $\square$

Claim 3. *Let $g(x_0, \ldots, x_n) \in \Gamma$ be a polynomial. Then there exist constants $\lambda_{\mathbf{e}} \in k$ such that*

$$g(x_0, \ldots, x_n) = \sum_{\mathbf{e} \in \mathbb{Z}^{\geq 0} \times (R_b)^n} \lambda_{\mathbf{e}} x_0^{e_0} \ldots x_n^{e_n},$$

and

$$v_\infty^n(g) = \min \left\{ v_\infty^n(x_0^{e_0} \ldots x_n^{e_n}) : \lambda_{\mathbf{e}} \neq 0 \right\}.$$

Proof Since the pole orders at P_∞^n of any different b-reduced monomials are different, by the strong triangle inequality, the valuation at P_∞^n of a sum of b-reduced monomials is the minimum of the valuations of the monomials (i.e. there is no pole cancellation). $\square$

Therefore, to calculate the possible pole orders of all polynomials in Γ, it is enough to calculate the possible pole orders of monomials in Γ.

3.2. **Valuations of rational functions.** Before we can consider the possible pole orders of rational functions, we need the following proposition.

Proposition 3 (From [1], Chapter 2, Proposition 6)**.** *For $\bar{K}$ algebraically closed, let I be an ideal in $\bar{K}[x_1, \ldots, x_n]$. Suppose $V(I) = \{P_1, \ldots, P_m\}$ is a finite set of points in $\bar{K}^n$. For $\mathcal{O}_i = \mathcal{O}_{P_i}$, there is a natural isomorphism*

$$\bar{K}[x_1, \ldots, x_n]/I \longrightarrow \prod_{i=1}^{m} \mathcal{O}_i / I\mathcal{O}_i.$$

Theorem 4. *Let N be the semi-group of pole orders at P_∞^n in F_n generated by elements of Γ. Suppose, for some $r > 0$ and $r \notin N$, that there exists $\psi \in F_n$ with $(\psi)_\infty = rP_\infty^n$. Then, for $\psi = g/h$ with $g, h \in \Gamma$, there is a place in the support of $(h)_0$ corresponding to a singular point Q on C_n.*

Proof Fix some algebraic closure $\bar{K}$ of K. Let

$$I_n = (f(x_0, x_1), \ldots, f(x_{n-1}, x_n)) \subset \bar{K}[x_0, \ldots, x_n]$$

be an ideal. Let $\Gamma = \bar{K}[x_0, \ldots, x_n]/I_n$ be the polynomial ring of C_n.

Suppose there is an element $\psi \in F_n$ with pole only at Q_n. Then $\psi = g/h$ for polynomials $g, h \in \Gamma$. To prove the contrapositive, assume each zero P of h

corresponds to a non-singular point of C_n. Let the local ring of P be $\mathcal{O}_P \subset F_n$ with valuation v_P. Since g/h does not have a pole at P,

$$v_P(g) \geq v_P(h) > 0.$$

Since P is non-singular, $\mathcal{O}_P$ is a valuation ring, so there is a parameter $t_P \in \mathcal{O}_P$ such that $v_P(t_P) = 1$. Since each element $z \in \mathcal{O}_P$ can be written uniquely as $z = t^l u$ for $u \in \mathcal{O}_P^\times$ and $l = v_P(z)$, and since $v_P(g) \geq v_P(h)$, we have

$$g \in (h) \subset \mathcal{O}_P.$$

Thus, for the ideal $I = (h, I_n)$, we have

$$g = 0 \in \mathcal{O}_P / I\mathcal{O}_P$$

for each zero P of h. Since there are finitely many zeroes of h in C_n, $V(I)$ contains a finite number of points. By the isomorphism from the above proposition,

$$g = 0 \in \bar{K}[x_0, \ldots, x_n]/I = \Gamma/(h).$$

Hence, $g = \phi \cdot h \in \Gamma$ for some $\phi \in \Gamma$, so $\psi = g/h = \phi$, a polynomial. While ϕ may have coefficients in an extension of K, the valuation of ϕ is that of a polynomial, and thus in N. □

Thus, if C_n is non-singular, the set of pole orders of rational functions is the set of pole orders of monomials.

3.2.1. *A non-example.* We give an example of a singular C_{ab} curve with a rational function in the Riemann-Roch space.

Consider the curve $C \subset \overline{\mathbb{F}_q}^2$, for q odd, given by

$$C : y^2 - x^2(x-1) = 0.$$

In the associated function field $F/\mathbb{F}_q$, the functions x and y have poles of orders 2 and 3 at P_∞ and nowhere else. Let $P_{i,j}$ be the place corresponding to the point $(i, j) \in C$. Then the divisors associated to x and y are

$$\begin{aligned} (x) &= 2P_{0,0} - 2P_\infty \\ (y) &= 2P_{0,0} + P_{1,0} - 3P_\infty. \end{aligned}$$

Thus,

$$(y/x) = P_{1,0} - P_\infty,$$

so y/x has a pole that is not the pole of a monomial in x and y. There is one place in the support of $(x)_0$, which corresponds to the singular point $(0,0) \in C$, as is expected by the theorem. Note that $\mathcal{O}_{P_{0,0}}$ is not a valuation ring because there is no parameter. In particular, y is not in the ideal generated by x.

3.3. **Gap numbers of monomials.** We now calculate the number of Weierstrass gap numbers at P_∞^n by looking at the pole orders of monomials. We will assume that the curve C_n is non-singular. (If C_n is singular, we only obtain an upper bound for the genus by counting pole orders of monomials.)

For the function field F_n, the elements

$$1, x_0, x_1, \ldots, x_n$$

have poles at P_∞^n of orders

$$0, b^n, b^{n-1}a, \ldots, a^n,$$

respectively. The monomial $x_0^{e_0}x_1^{e_1}\dots x_n^{e_n}$ has a pole of order $e_0b^n+e_1b^{n-1}a+\cdots+e_na^n$. Thus, to calculate the genus, we need to calculate the number of positive integers α that do not have a solution in non-negative integers e_i to the equation

$$\alpha = e_0b^n + e_1b^{n-1}a + \cdots + e_na^n. \tag{1}$$

As we saw in Section 3.1, we can restrict to the case where $0 \le e_i < b$.

Proposition 5. *Let $g(F_n)$ denote the genus of the function field F_n. For $n \ge 0$,*

$$g(F_{n+1}) = bg(F_n) + \frac{(a^{n+1}-1)(b-1)}{2}.$$

Proof To prove this, we will use two methods of counting, show that the methods do not overlap, and then show that we have not missed anything.

Method 1: Let $A = \{\alpha_i : i = 1, 2, \dots, g(F_n)\}$ be the set of gap numbers of P_∞^n in F_n. We want to show that for any $\alpha \in A$, there is no monomial with a pole of order $b\alpha + la^{n+1}$ in F_{n+1} for $l \in R_b = \{0, \dots, b-1\}$.

Since α is a gap number of P_∞^n in F_n, this means that there is no solution in e_i to equation (1).

Suppose we have a monomial with pole order equal to $b\alpha + la^{n+1}$ in F_{n+1}. The variables in F_{n+1} are $1, x_0, \dots, x_{n+1}$, so this would mean we could find a solution to

$$e_0b^{n+1} + e_1ab^n + \cdots + e_na^nb + e_{n+1}a^{n+1} = b\alpha + la^{n+1}$$

with non-negative integers e_i. Then $e_{n+1} \equiv l \mod b$. Since e_{n+1} and l are in R_b, $e_{n+1} = l$, so

$$b\alpha = e_0b^{n+1} + e_1ab^n + \cdots + e_na^nb.$$

Dividing through by b, one has

$$\alpha = e_0b^n + e_1ab^{n-1} + \cdots + e_na^n.$$

Since $e_0, e_1, \dots, e_{n-1}$, and e_n are all non-negative integers, this contradicts the fact that α is a gap number of P_∞^n.

The result is that we have gap numbers $b\alpha + la^{n+1}$ of P_∞^{n+1} in F_{n+1} for $l = 0, \dots, b-1$ and for all $\alpha \in A$. This gives us the $bg(F_n)$ term in the genus formula.

Method 2: Since the only pole orders below a^{n+1} that are achievable with monomials in F_{n+1} are multiples of b, there is no monomial with pole order that is congruent to any of $1, 2, \dots, b-1 \mod b$ and less than a^{n+1}. Then, between a^{n+1} and $2a^{n+1}$, the only pole orders that are achievable are congruent to 0 or a^{n+1} mod b. Going on in this manner, for $l = 1, 2, \dots, b-1$, between la^{n+1} and $(l+1)a^{n+1}$, the only pole orders that are achievable are congruent to $0, a^{n+1}, 2a^{n+1}, \dots, la^{n+1}$ mod b. (Note that these are all distinct congruence classes because a^{n+1} is a unit mod b.)

Counting in this way gives that the number of gap numbers from 0 to la^{n+1} that are congruent to $la^{n+1} \mod b$ is $\lfloor \frac{la^n}{b} \rfloor$. Let p_{n+1} be the total number of these gap numbers in F_{n+1}. For the $b-1$ non-zero congruent classes modulo b, we get

$$p_{n+1} = \left\lfloor \frac{a^{n+1}}{b} \right\rfloor + \left\lfloor \frac{2a^{n+1}}{b} \right\rfloor + \cdots + \left\lfloor \frac{(b-1)a^{n+1}}{b} \right\rfloor$$

missing pole orders.

For any integer m, let $\overline{m} \in R_b$ be the residue of m modulo b. Since

$$\left\lfloor \frac{m}{b} \right\rfloor = \frac{m}{b} - \frac{\overline{m}}{b},$$

and since $\left\{\overline{a^{n+1}}, \overline{2a^{n+1}}, \ldots, \overline{(b-1)a^{n+1}}\right\} = \{1, 2, \ldots, b-1\}$, we get that

$$\begin{aligned} p_{n+1} &= \left(\frac{a^{n+1} + 2a^{n+1} + \cdots + (b-1)a^{n+1}}{b} - \frac{1 + 2 + \cdots + (b-1)}{b}\right) \\ &= \frac{(a^{n+1}-1)(b-1)}{2}. \end{aligned}$$

Overlap? Suppose there exists an element in F_{n+1} with pole order β that is counted by both methods. By the first method, $\beta = b\alpha + la^{n+1}$ for some $\alpha \in A$ and l such that $0 \leq l \leq b-1$. So $\beta \equiv la^{n+1} \mod b$. If β is counted by the second method, then since $\beta \equiv la^{n+1}$, β must be less than la^{n+1}. But $\beta = b\alpha + la^{n+1}$, so this is a contradiction. So

$$g(F_{n+1}) \geq bg(F_n) + \frac{(a^{n+1}-1)(b-1)}{2}.$$

Everything? Suppose there exists a gap number γ for which there is no monomial in F_{n+1} with pole order equal to γ. Then $\gamma \equiv la^{n+1} \mod b$ for some $l \in \{0, 1, \ldots, b-1\}$. Since $v_\infty^{n+1}(x_{n+1}^l) = -la^{n+1}$, we have $\gamma \neq la^{n+1}$. Thus, either $\gamma > la^{n+1}$ or $\gamma < la^{n+1}$.

If $\gamma > la^{n+1}$, then since $\gamma \equiv la^{n+1} \mod b$,

$$\gamma = mb + la^{n+1}$$

for some positive integer m. It follows that mb is a gap number of P_∞^{n+1}, because if it were the pole order of a function f, then the pole order of $f \cdot x_{n+1}^l$ is γ. Thus, m is a gap number of P_∞^n, and we counted all of these gap numbers in the first method.

Suppose $\gamma < la^{n+1}$. In the second method, we counted all integers that are congruent to la^{n+1} modulo b that are less than la^{n+1}, so γ must have been among them. ☐

We have our result.

Theorem 6. *Let q be a power of a prime. For $f(x,y) \in K[x,y]$, let $f = 0$ be the equation of a C_{ab} curve such that* $\gcd(a,b) = 1$. *For $n \geq 0$, let*

$$C_n = \left\{(p_0, \ldots, p_n) \in K^{n+1} : f(p_{i-1}, p_i) = 0 \text{ for } i = 1, \ldots, n\right\}.$$

Consider the tower of function fields $\mathcal{F} = (F_0, F_1, \ldots)$ where F_n is the function field associated to C_n. If C_n is non-singular, the genus of F_n is given by

$$g(F_n) = \frac{(b-1)a^{n+1} - (a-1)b^{n+1} + a - b}{2(a-b)},$$

and, for any positive integer m and P_∞ the place at infinity in F_n, a basis for $\mathcal{L}(mP_\infty)$ is given by

$$\left\{x_0^{e_0} x_1^{e_1} \ldots x_n^{e_n} : e_0 \geq 0, 0 \leq e_i < b \text{ for } i = 1, \ldots, n, \text{ and } \sum_{i=0}^{n} a^i b^{n-i} e_i \leq m\right\}.$$

Note that the above result is an upper bound. If C_n contains singular points, there will be fewer gap numbers, and so a lower genus.

3.4. **Examples.** Working in $\mathbb{F}_7$, let $f(x,y) = x^3 + y^2 - 3$, and let the curves C_n be defined as in Theorem 6. Note that 3 is neither a quadratic nor cubic residue in $\mathbb{F}_7$.

The Jacobian matrix of C_n is the $n \times (n+1)$ matrix J_n defined by

$$J_n = \begin{pmatrix} 3x_0^2 & 2x_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 3x_1^2 & 2x_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 3x_2^2 & 2x_3 & \dots & 0 & 0 \\ \vdots & & & & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & 2x_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \dots & 3x_{n-1}^2 & 2x_n \end{pmatrix}.$$

The affine curve C_n is nonsingular if the rank of J_n is n for all points on C_n. The rank drops precisely when we have a point that has two coordinates equal to zero. We aim to show that this can never happen by showing there can be no affine point of the form $(0, a_1, a_2, \dots, a_{n-1}, 0)$ on C_n.

From the equation

$$f(a_0, a_1) = a_0^3 + a_1^2 - 3 = 0,$$

if $a_0 = 0$, we have $a_1^2 = 3$. Since 3 is not a quadratic residue in $\mathbb{F}_7$, a_1 is not in $\mathbb{F}_7$, so $a_1 \in \mathbb{F}_{7^2} \setminus \mathbb{F}_7$. Then, from

$$f(a_1, a_2) = a_1^3 + a_2^2 - 3 = 0,$$

we get $a_2^2 = 3 - a_1^3$. Since $a_1 \in \mathbb{F}_{7^2} \setminus \mathbb{F}_7$, it follows that $3 - a_1^3$ is also in $\mathbb{F}_{7^2} \setminus \mathbb{F}_7$, and so $a_2 \in \mathbb{F}_{7^4} \setminus \mathbb{F}_7$. Similarly, it follows that each of $a_3, a_4, \dots, a_{n-1}$ is in $\mathbb{F}_{7^{2^i}} \setminus \mathbb{F}_7$ for some non-negative integer i.

From the equation

$$f(a_{n-1}, a_n) = a_{n-1}^3 + a_n^2 - 3 = 0,$$

if $a_n = 0$, then $a_{n-1}^3 = 3$. Since 3 is not a cubic residue in modulo 7, it follows that $a_{n-1} \in \mathbb{F}_{7^3}$. However, this contradicts our work above, which said that $a_{n-1} \in \mathbb{F}_{7^{2^i}}$. Hence, there can be no point on this curve with first and last coordinates equal to zero. As a result, there can be no point with two zero coordinates at all, which means that the Jacobian matrix has rank n, which means that the curve C_n is nonsingular. By Theorem 6, the genus of C_n and the corresponding function field F_n is

$$g(C_n) = \frac{3^{n+1} - 2^{n+2} + 1}{2}.$$

Working along these lines, we have the following result.

Theorem 7. *Let q be a prime power. Let $a, b \in \mathbb{N}$ with $\gcd(a,b) = 1$. If there exists $\alpha \in \mathbb{F}_q$ such that α is neither an ath or bth power in $\mathbb{F}_q$, then the function $f(x,y) = x^a + y^b - \alpha$ can be used to recursively define a nonsingular tower of curves.*

For the case where q is prime, we are guaranteed to have examples of this type when a and b both share factors with $q-1$, since the maps $x \mapsto x^a$ and $x \mapsto x^b$ are not one-to-one in this case. Since both maps are at least 2 to 1 and the element 1 is in the image of both maps, there must exist some $\alpha \in \mathbb{F}_q$ that is neither an ath or bth power.

In the case of $a = 2, b = 3$ and $q = 7$, the set of non-zero squares is $\{1, 2, 4\}$ and the non-zero cubes is $\{1, 6\}$. Hence, we can let α equal 3 or 5 and obtain a nonsingular tower.

It would be interesting to see if one can create examples for all combinations of a, b, and q.

4. Comments

For more flexibility, note that it is not required that one uses the same polynomial equation $f = 0$ for each level of the tower. Fixing a and b, if one has a sequence of polynomials $f_1, f_2, \dots$ for which $f_i = 0$ is the equation of a C_{ab} curve, then the genus formula given in Theorem 6 still holds.

In fact, consider a sequence $f_1, f_2, \dots,$ where $f_i = 0$ is the equation of a $C_{a_i b_i}$ curve, for $\gcd(a_i, b_i) = 1$ and $a_i > b_i$. As above, let $F_0 = K(x_0)$ and $F_{n+1} = F_n(x_{n+1})$ where $f_n(x_n, x_{n+1}) = 0$. Then, provided that $(a_i, b_j) = 1$ for all i, j, and that there are no singular points on the corresponding curve (except possibly at infinity), one has the following formula:

$$g(F_{n+1}) = b_{n+1} g(F_n) + \frac{(a_1 \cdots a_{n+1} - 1)(b_{n+1} - 1)}{2}.$$

A basis for $\mathcal{L}(mP_\infty)$ is given by

$$\left\{ x_0^{e_0} \dots x_n^{e_n} : \begin{array}{c} e_0 \geq 0, 0 \leq e_i < b_i \text{ for } i = 1, \dots, n, \\ \text{and } \sum_{i=0}^{n} a_1 \cdots a_i b_{i+1} \cdots b_n e_i \leq m \end{array} \right\}.$$

4.1. **Asymptotics.** We have seen that for the towers defined recursively by C_{ab} equations, the genus grows exponentially in a. The number of places of degree one will grow at most exponentially in b, which is strictly smaller than a. Thus, the ratio of number of places of degree one of F_n divided by the genus of F_n will tend to zero as $n \to \infty$. Therefore, these towers are not *asymptotically good*. (This also follows directly from [2], which states that for a recursively defined tower to be asymptotically good, the recursive equation must have balanced degrees.)

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A CONVOLUTION IDENTITY FOR EXCHANGEABLE RISKS

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Abstract. In this short note we provide an extension of a convolution identity for exchangeable dependent risks, which is motivated by an elegant proof of Panjer's algorithm derived in Mikosch (2006).

1. Introduction and Results

Let $X_i, 1 \le i \le n$ be independent and identically distributed random variables (risks), and denote $S_n = \sum_{i=1}^n X_i, n > 1$. In an insurance context S_n plays a crucial role for modelling aggregate portfolio losses where $X_i, 1 \le i \le n$ (assumed to be positive) model the losses of some insured portfolio over a specific time period. Since often the number of losses occurring in a particular time interval is itself random, say N, modeling of S_N is also of interest. In order to calculate the distribution function of S_N for particular choice of N, i.e., N being Binomial, Poisson or negative Binomial, the well-known Panjer's recursion algorithm is utilised; an elegant proof of that algorithm is given in Mikosch (2006).

If $X_1 \neq 0$ almost surely, following Mikosch (2006) we find the following equation for the conditional expectation

$$\boldsymbol{E}\left\{\frac{X_i}{S_n}\Big| S_n\right\} = \frac{1}{n}, \quad 1 \le i \le n \tag{1}$$

holds almost surely. Consequently, by removing the conditioning we are left with

$$\boldsymbol{E}\left\{\frac{X_i}{S_n}\right\} = \frac{1}{n}, \quad 1 \le i \le n.$$

Hence, for any $1 \le l < n$

$$\sum_{1 \le i \le l} \boldsymbol{E}\left\{\frac{X_i}{S_n}\right\} = \boldsymbol{E}\left\{\frac{S_l}{S_n}\right\} = \frac{l}{n},$$

or alternatively

$$\boldsymbol{E}\left\{\frac{S_l}{S_l + \sum_{l+1 \le i \le n} X_i}\right\} = \frac{l}{n}. \tag{2}$$

which is the main result of Theorem 2.1 in Mukhopadhyay (2010).

The independence assumption is in diverse applications not tenable. Therefore, it is of some interest to consider the validity of (1) for such instances.

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In the aforementioned paper several examples of dependent risks are provided extending thus the result (2). The reason why the dependence among the risks does not influence the expectation of the ratio S_l/S_n is the special dependence underlying those examples, namely the fact that the Gaussian random vectors with identical and equicorrelated components are exchangeable. We present below a more general result:

Lemma 1. *Let $X_i, 1 \le i \le n$ be exchangeable risks such that $X_i \neq 0$ almost surely. Then both* (1) *and* (2) *hold.*

Proof: The proof follows easily by the exchangeability assumption.

□

Remark: a) Exchangeability of $X_1, \ldots, X_n$ which means that the joint distribution function is invariant to permutation of the indices follows for instance if $X_i, 1 \le i \le n$ are conditionally independent.

b) The assumption that X_1 possesses a probability density function imposed in Theorem 2.1 of Mukhopadhyay (2010) is redundant.

2. Examples

In this section we give two examples of dependent risks $X_i, 1 \le i \le n$ which satisfy the convolution identity (2).

Example 1. Let $Z_1, \ldots, Z_n$ be standard Gaussian random variables with mean 0, variance 1, being further equicorrelated with correlation coefficient $\rho \in (-1, 1)$. Let R be a positive random variable independent of $Z_i, 1 \le i \le n$, and let $X_i = RZ_i, 1 \le i \le n$. The random vector $(X_1, \ldots, X_n)$ has a scale mixture Gaussian distribution. Since $X_i, 1 \le i \le n$ are exchangeable risks the result of Lemma 1 holds for this case.

Example 2. Consider a random vector $(Z_1, \ldots, Z_n)$ such that given $\Theta = \theta$ the conditional survival probability is specified by

$$\boldsymbol{P}\{Z_i > x_i, i = 1, \ldots, n | \Theta = \theta\} = \exp\Big(-\theta\lambda \sum_{1 \le i \le n} x_i\Big), \qquad x_i > 0, i \le n,$$

with some positive constant λ, and Θ being Gamma distributed with parameter $\alpha, 1$ (thus with mean α). Direct calculation shows that $(Z_1, \ldots, Z_n)$ has the multivariate Pareto distribution with survival function

$$\boldsymbol{P}\{Z_1 > x_1, \ldots, Z_d > x_n\} = \Big(1 + \lambda \sum_{1 \le i \le n} x_i\Big)^{-\alpha}, \quad x_i > 0, i \le n.$$

With R as in Example 1, applying Lemma 1 to $X_i = RZ_i, 1 \le i \le n$ we find that again (2) holds for this case.

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STABILITY OF CUBIC FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN $\mathcal{L}$-FUZZY NORMED SPACES

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Abstract. We determine some stability results concerning the cubic functional equation in non-Archimedean fuzzy normed spaces. Our result can be regarded as a generalization of the stability phenomenon in the framework of $\mathcal{L}$-fuzzy normed spaces.

1. Introduction

The stability of functional equations is an interesting area of research for mathematicians, but it can be also of importance to persons who work outside of the realm of pure mathematics.

It seems that the stability problem of functional equations had been first raised by Ulam [18]. Moreover the approximated mappings have been studied extensively in several papers. (See for instance [14], [6], [3], and [4]).

Fuzzy notion introduced firstly by Zadeh [19] that has been widely involved in different subjects of mathematics. Zadeh's definition of a fuzzy set characterized by a function from a nonempty set X to $[0, 1]$. Goguen in [5] generalized the notion of a fuzzy subset of X to that of an $\mathcal{L}$-fuzzy subset, namely a function from X to a lattice L.

Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in the field of science and engineering.

Later in 1984, Katsaras [9] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space.

With [10] and by modifying the definition of a fuzzy normed space in [2], Mirmostafaee and Moslehian in [12] introduced a notion of a non-Archimedean fuzzy normed space. Shekari et al. ([16]) considered the quadratic functional equation in $\mathcal{L}$-fuzzy normed space. Also Saadati and Park considered the $f(lx+y)+f(lx-y) = 2l^2 f(x) + 2f(y)$ and proved the Hyers-Ulam-Rasssias stability of this equation in $\mathcal{L}$-fuzzy normed spaces ([17]).

The stability problem for the cubic functional equation was proved by Jun and Kim [7] for mappings $f : X \to Y$, where X is a real normed space and Y is a Banach space. Later on, in ([8],[11],[13]) the problem of stability of some cubic equation were discussed.

Defining the class of approximate solutions of a given functional equation one can ask whether every mapping from this class can be somehow approximated

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by an exact solution of the considered equation in the non-Archimedean $\mathcal{L}$-fuzzy normed spaces. To answer this question, we establish a non-Archimedean $\mathcal{L}$-fuzzy Hyers-Ulam-Rassias stability of the cubic functional equation $f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)$. Then we define the non-Archimedean fuzzy continuity of the cubic mappings and we investigate the continuity of approximate cubic mappings.

2. **Preliminaries**

In this section, we provide a collection of definitions and related results which are essential and used in the next discussions.

Definition 2.1. Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $t, s \in \mathbb{R}$,
(N1) $N(x,c) = 0$ for $c \leq 0$;
(N2) $x = 0$ if and only if $N(x,c) = 1$ for all $c > 0$;
(N3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y, s+t) \geq min\{N(x,s), N(y,t)\}$;
(N5) $N(x,.)$ is a non-decreasing function on $\mathbb{R}$ and $lim_{t\to\infty} N(x,t) = 1$;
(N6) for $x \neq 0$, $N(x,.)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed linear space.

Definition 2.2. Let (X,N) be a fuzzy normed linear space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - lim_{n\to\infty} x_n = x$.

Definition 2.3. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is known that every convergent sequence in a fuzzy normed space is Cauchy and if each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and furthermore the fuzzy normed space is called a fuzzy Banach space.

Definition 2.4. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a t-norm if it satisfies the following conditions:
$(*1)$ $*$ is associative,
$(*2)$ $*$ is commutative,
$(*3)$ $a * 1 = a$ for all $a \in [0,1]$ and
$(*4)$ $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Definition 2.5. ([5]). Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice and let U be a non-empty set called the universe. An $\mathcal{L}$-fuzzy set in U is defined as a mapping $\mathcal{A} : U \to L$. For each u in U, $\mathcal{A}(u)$ represents the degree (in L) to which u is an element of $\mathcal{A}$.

Definition 2.6. ([1]). A t-norm on $\mathcal{L}$ is a mapping $*_L : L^2 \to L$ satisfying the following conditions:
(i) $(\forall x \in L)(x *_L 1_{\mathcal{L}} = x)$(: boundary condition);

(ii) $(\forall (x,y) \in L^2)(x *_L y = y *_L x)$(: commutativity);
(iii) $(\forall (x,y,z) \in L^3)(x *_L (y *_L z)) = ((x *_L y) *_L z)$(: associativity);
(iv) $(\forall (x,y,z,w) \in L^4)(x \leq_L x' \text{ and } y \leq_L y' \Rightarrow x *_L y \leq_L x' *_L y')$(: monotonicity).

A t-norm $*_L$ on $\mathcal{L}$ is said to be continuous if, for any $x, y \in \mathcal{L}$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converges to x and y, respectively, $lim_{n\to\infty}(x_n *_L y_n) = x *_L y$.

Definition 2.7. The triple $(V, \mathcal{P}, *_L)$ is said to be an $\mathcal{L}$-fuzzy normed space if V is vector space, $*_L$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{P}$ is an $\mathcal{L}$-fuzzy set on $V \times (0, \infty)$ satisfying the following conditions:
for all $x, y \in V$ and $t, s \in (0, \infty)$,
(a) $\mathcal{P}(x,t) >_L 0_{\mathcal{L}}$;
(b) $\mathcal{P}(x,t) = 1_{\mathcal{L}}$ if and only if $x = 0$;
(c) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
(d) $\mathcal{P}(x,t) *_L \mathcal{P}(y,s) \leq_L \mathcal{P}(x+y, t+s)$;
(e) $\mathcal{P}(x,t) : (0,\infty) \to L$ is continuous;
(f) $lim_{t\to 0}\mathcal{P}(x,t) = 0_{\mathcal{L}}$ and $lim_{t\to\infty}\mathcal{P}(x,t) = 1_{\mathcal{L}}$.
In this case, $\mathcal{P}$ is called an $\mathcal{L}$-fuzzy norm.

Definition 2.8. A negator on $\mathcal{L}$ is any decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$.

Definition 2.9. If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L$, then $\mathcal{N}$ is called an involutive negator.

In this paper, the involutive negator $\mathcal{N}$ is fixed.

Definition 2.10. A sequence (x_n) in an $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, *_L)$ is called a Cauchy sequence if, for each $\varepsilon \in L - \{0_{\mathcal{L}}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$, $\mathcal{P}(x_n - x_m, t) >_L \mathcal{N}(\varepsilon)$, where $\mathcal{N}$ is a negator on $\mathcal{L}$.

A sequence (x_n) is said to be convergent to $x \in V$ in the $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, *_L)$, if $\mathcal{P}(x_n - x, t) \to 1_{\mathcal{L}}$, whenever $n \to +\infty$ for all $t > 0$.

An $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, *_L)$ is said to be complete if and only if every Cauchy sequence in V is convergent.

Definition 2.11. Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $|.| : \mathbb{K} \to \mathbb{R}$ such that for any $a, b \in \mathbb{K}$ we have
(1) $|a| \geq 0$ and equality holds if and only if $a = 0$,
(2) $|ab| = |a||b|$,
(3) $|a + b| \leq max\{|a|, |b|\}$.

Note that $|n| \leq 1$ for each integer n. We always assume, in addition, that $|.|$ is non-tivial, i.e., there exists an $a_0 \in \mathbb{K}$ such that $|a_0| \neq 0, 1$.

Definition 2.12. A non-Archimedean $\mathcal{L}$-fuzzy normed space is a triple $(V, \mathcal{P}, *_L)$, where V is a vector space, $*_L$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{P}$ is an $\mathcal{L}$-fuzzy set on $V \times (0, +\infty)$ satisfying the following conditions:
for all $x, y \in V$ and $t, s \in (0, \infty)$,
(a) $\mathcal{P}(x,t) >_L 0_{\mathcal{L}}$;
(b) $\mathcal{P}(x,t) = 1_{\mathcal{L}}$ if and only if $x = 0$;
(c) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$;
(d) $\mathcal{P}(x,t) *_L \mathcal{P}(y,s) \leq_L \mathcal{P}(x+y, max\{t,s\})$;

(e) $\mathcal{P}(x,.) : (0,\infty) \to L$ is continuous;
(f) $lim_{t\to 0}\mathcal{P}(x,t) = 0_{\mathcal{L}}$ and $lim_{t\to\infty}\mathcal{P}(x,t) = 1_{\mathcal{L}}$.

3. Stability of cubic equation in $\mathcal{L}$-fuzzy normed spaces

Let $\mathbb{K}$ be a non-Archimedean field, X a vector space over $\mathbb{K}$ and $(Y,\mathcal{P},*_L)$ a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathbb{K}$.

In this section we investigate the cubic functional equation. We define an $\mathcal{L}$-fuzzy approximately cubic mapping. Let Ψ be an $\mathcal{L}$-fuzzy set on $X \times X \times [0,\infty)$ such that $\Psi(x,y,.)$ is nondecreasing,

$$\Psi(cx,cx,t) \geq_L \Psi(x,x,\tfrac{t}{|c|}),\ \forall x \in X,\ c \neq 0$$

and

$$lim_{t\to\infty}\Psi(x,y,t) = 1_{\mathcal{L}},\ \forall x,y \in X,\ t > 0.$$

Troughs this paper, we show the $a_1 *_L a_2 *_L ... *_L a_n$ by $\prod_{j=1}^{n} a_j$.

Definition 3.1. A mapping $f : X \to Y$ is said to be Ψ-approximately cubic if

$$\mathcal{P}(f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x),t) \geq_L \Psi(x,y,t),\ \forall x,y \in X, t > 0. \tag{3.1}$$

Theorem 3.2. *Let $\mathbb{K}$ be a non-Archimedean field, X a vector space over $\mathbb{K}$ and $(Y,\mathcal{P},*_L)$ a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathbb{K}$. Let $f : X \to Y$ be a Ψ-approximately cubic mapping. Suppose that $f(0) = 0$. If there exist an $\alpha \in \mathbb{R}$ $(\alpha > 0)$ and an integer k, $k \geq 2$ with $|2^k| < \alpha$ and $|2| \neq 0$ such that*

$$\Psi(2^{-k}x,2^{-k}y,t) \geq_L \Psi(x,y,\alpha t),\ \forall x \in X, t > 0, \tag{3.2}$$

and

$$lim_{n\to\infty}\prod_{j=n}^{\infty}\Psi(x,\tfrac{\alpha^j t}{|2|^{kj}}) = 1_{\mathcal{L}},\ \forall x \in X, t > 0,$$

then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mathcal{P}(f(x)-C(x),t) \geq_L \prod_{i=1}^{\infty}\mathcal{M}(x,\frac{\alpha^{i+1}t}{|2^{ki}|}),\ \forall x \in X, t > 0, \tag{3.3}$$

where

$$\mathcal{M}(x,t) = \prod_{i=1}^{\infty}\Psi(2^{i-1}x,0,t/4).$$

Proof. First, we show, by induction on j, that, for all $x \in X$, $t > 0$ and $j \geq 1$,

$$\mathcal{P}(f(2^j x)-4^j f(x),t) \geq_L \mathcal{M}_j(x,t). \tag{3.4}$$

Put $y = 0$ in (3.1). Then for all $x \in X$ and $t > 0$

$$\mathcal{P}(f(2x)-4f(x),t) \geq_L \Psi(x,0,t/4). \tag{3.5}$$

This proves (3.4) for $j = 1$. Let (3.4) holds for some $j > 1$. Replacing x by $2^j x$ in (3.5), we get

$$\mathcal{P}(f(2^{j+1}x)-4f(2^j x),t) \geq_L \Psi(2^j x,0,t/4).$$

Since $|4| \leq 1$, it follows that

$$\mathcal{P}(f(2^{j+1}x)-4^{j+1}f(x),t) \geq_L \mathcal{P}(f(2^{j+1}x)-4f(2^jx),t) *_L \mathcal{P}(4f(2^jx)-4^{j+1}f(x),t) = \mathcal{P}(f(2^{j+1}x)-4f(2^jx),t) *_L \mathcal{P}(f(2^jx)-4^jf(x),t/|4|) \geq_L \mathcal{P}(f(2^{j+1}x)-4f(2^jx),t) *_L \mathcal{P}(f(2^jx)-4^jf(x),t) \geq_L \mathcal{M}_j(x,t) *_L \Psi(2^jx,0,t/4) = \mathcal{M}_{j+1}(x,t).$$

Thus (3.4) holds for all $j \geq 1$. In particular, we have

$$\mathcal{P}(f(2^kx)-4^kf(x),t) \geq_L \mathcal{M}(x,t). \tag{3.6}$$

Replacing x by $2^{-(kn+k)}x$ in (3.6) and using the inequality (3.2), we obtain

$$\mathcal{P}(f(\tfrac{x}{2^{kn}})-4^kf(\tfrac{x}{2^{kn+k}}),t) \geq_L \mathcal{M}(\tfrac{x}{2^{kn+k}},t) \geq_L \mathcal{M}(x,\alpha^{n+1}t).$$

and so

$$\mathcal{P}(2^{2kn}f(\tfrac{x}{2^{kn}})-2^{2k(n+1)}f(\tfrac{x}{2^{k(n+1)}}),t) \geq_L \mathcal{M}(x,\tfrac{\alpha^{n+1}t}{2^{2kn}}) \geq_L \mathcal{M}(\tfrac{\alpha^{n+1}t}{2^{kn}})$$

Hence it follows that

$$\mathcal{P}(2^{2kn}f(\tfrac{x}{2^{kn}})-2^{2k(n+p)}f(\tfrac{x}{2^{k(n+p)}}),t) \geq_L \prod_{j=n}^{n+p}(\mathcal{P}(2^{2kj}f(\tfrac{x}{2^{kj}})-2^{2k(j+1)}f(\tfrac{x}{2^{k(j+1)}}),t) \geq_L \prod_{j=n}^{n+p}\mathcal{M}(x,\tfrac{\alpha^{j+1}t}{2^{kj}}).$$

Since $lim_{n\to\infty}\prod_{j=n}^{\infty}\mathcal{M}(x,\frac{\alpha^{j+1}t}{2^{kj}}) = 1_{\mathcal{L}}$ for all $x \in X$ and $t > 0$, $\{2^{2kn}f(\frac{x}{2^{kn}})\}$ is a Cauchy sequence in the non-Archimedean $\mathcal{L}$-fuzzy Banach space $(Y,\mathcal{P},*_L)$. Hence we can define a mapping $C : X \to Y$ such that

$$lim_{n\to\infty}\mathcal{P}(2^{2kn}f(\frac{x}{2^{kn}})-C(x),t) = 1_{\mathcal{L}}. \tag{3.7}$$

Next, for all $n \geq 1$, $x \in X$ and $t > 0$, we have

$$\mathcal{P}(f(x)-2^{2kn}f(\tfrac{x}{2^{kn}}),t) = \mathcal{P}(\textstyle\sum_{i=0}^{n-1}2^{2ki}f(\tfrac{x}{2^{ki}})-2^{2k(i+1)}f(\tfrac{x}{2^{k(i+1)}}),t) \geq_L \prod_{i=0}^{n-1}\mathcal{P}(2^{2ki}f(\tfrac{x}{2^{ki}})-2^{2k(i+1)}f(\tfrac{x}{2^{k(i+1)}}),t) \geq_L \prod_{i=0}^{n-1}\mathcal{M}(x,\tfrac{\alpha^{i+1}t}{|2^{ki}|})$$

and so

$$\mathcal{P}(f(x)-C(x),t) \geq_L \mathcal{P}(f(x)-2^{2kn}f(\frac{x}{2^{kn}}),t) *_L \mathcal{P}(2^{2kn}f(\frac{x}{2^{kn}})-C(x),t) \geq_L \prod_{i=0}^{n-1}\mathcal{M}(x,\tfrac{\alpha^{i+1}t}{|2^{ki}|}) *_L \mathcal{P}(2^{2kn}f(\tfrac{x}{2^{kn}})-C(x),t). \tag{3.8}$$

Taking the limit as $n \to \infty$ in (3.8), we obtain

$$\mathcal{P}(f(x)-C(x),t) \geq_L \prod_{i=1}^{\infty}\mathcal{M}(x,\frac{\alpha^{i+1}t}{|2^{ki}|}), \tag{3.9}$$

which proves (3.3). As $*_L$ is continuous, from a well known result in $\mathcal{L}$-fuzzy normed space(see[15], Chapter 12), it follows that

$$lim_{n\to\infty}\mathcal{P}(4^{kn}f(2^{-kn}(2x+y))+4^{kn}f(2^{-kn}(2x-y)-2.4^{kn}f(2^{-kn}(x+y))-2.4^{kn}f(2^{-kn}(x-y))-12.4^{kn}f(2^{-kn}x),t) = \mathcal{P}(C(2x+y)+C(2x-y)-2C(x+y)-2C(x-y)-12C(x),t)$$

for almost all $t > 0$.

On the other hand, replacing x, y by $2^{-kn}x, 2^{-kn}y$ in (3.1) and (3.2), we get

$$\mathcal{P}(4^{kn}f(2^{-kn}(2x+y))+4^{kn}f(2^{-kn}(2x-y)-2.4^{kn}f(2^{-kn}(x+y))-2.4^{kn}f(2^{-kn}(x-y))-12.4^{kn}f(2^{-kn}x),t)=\mathcal{P}(C(2x+y)+C(2x-y)-2C(x+y)-2C(x-y)-12C(x),t)\geq_L \Psi(2^{-kn}x,2^{-kn}y,\tfrac{t}{|2^{2kn}|}\geq_L \Psi(2^{-kn}x,2^{-kn}y,\tfrac{t}{|2^{kn}|}).$$

Since $lim_{n\to\infty}\Psi(x,y,\frac{\alpha^n t}{|2^{kn}|})=1_{\mathcal{L}}$, we infer that C is a cubic mapping.
For the uniqueness of C, let $C':X\to Y$ be another cubic mapping such that

$$\mathcal{P}(C'(x)-f(x),t)\geq_L \mathcal{M}(x,t).$$

Then we have, for all $x,y\in X$ and $t>0$,

$$\mathcal{P}(C(x)-C'(x),t)\geq_L \mathcal{P}(C(x)-2^{2kn}f(\tfrac{x}{2^{kn}}),t)*_L \mathcal{P}(2^{2kn}f(\tfrac{x}{2^{kn}})-C'(x),t).$$

Therefore from (3.7), we have $C=C'$. $\square$

Definition 3.3. Let $\mathbb{K}$ be a non-Archimedean field, X a normed space and $(Y,\mathcal{P},*_L)$ a non-Archimedean $\mathcal{L}$-fuzzy normed space over $\mathbb{K}$. Let $f:X\to Y$ be a Ψ-approximately cubic mapping. We say that $f:X\to Y$ is non-Archimedean $\mathcal{L}$-fuzzy continuous at a point $s_0\in X$ if for all $\varepsilon>0$ there exists $\delta>0$ such that for each s with $||s-s_0||<\delta$

$$lim_{t\to\infty}\mathcal{P}(T(s)-T(s_0),t\varepsilon)=1,$$

uniformly on X.

Theorem 3.4. *Let $\mathbb{K}$ be a non-Archimedean field, X a normed space and $(Y,\mathcal{P},*_L)$ a non-Archimedean $\mathcal{L}$-fuzzy normed space over $\mathbb{K}$. Let $f:X\to Y$ be a Ψ-approximately cubic mapping. If for some $x\in X$ and all $n\in\mathbb{N}$, the mapping $g:\mathbb{R}\to Y$ defined by $g(s)=2^{2kn}f(\frac{x}{2^{kn}})$ is non-Archimedean $\mathcal{L}$-fuzzy continuous. Then the mapping $s\mapsto C(sx)$ from $\mathbb{R}$ to Y is non-Archimedean $\mathcal{L}$-fuzzy continuous.*

Proof. Using Theorem (3.2) we deduce that, there exists a unique cubic mapping C such that

$$lim_{n\to\infty}\mathcal{P}(2^{2kn}f(\tfrac{x}{2^{kn}})-C(x),t)=1_{\mathcal{L}}.$$

By the non-Archimedean $\mathcal{L}$-fuzzy continuity of the mapping $t\mapsto 2^{2kn}f(\frac{tx}{2^{kn}})$, there exists δ such that for each s with $0<|s-s_0|<\delta$, we have

$$lim_{t\to\infty}\mathcal{P}(2^{2kn}f(\tfrac{sx}{2^{kn}})-2^{2kn}f(\tfrac{s_0x}{2^{kn}}),t\varepsilon)=1_L.$$

It follows that

$$lim_{t\to\infty}\mathcal{P}(C(sx)-C(s_0x),t\varepsilon)\geq_L lim_{t\to\infty}\mathcal{P}(C(sx)-2^{2kn}f(\tfrac{sx}{2kn}),t)*_L$$
$$\mathcal{P}(2^{2kn}f(\tfrac{sx}{2kn})-2^{2kn}f(\tfrac{s_0x}{2kn}),t)*_L \mathcal{P}(C(s_0x)-2^{2kn}f(\tfrac{s_0x}{2kn}),t)=1_L$$

for each s with $0<|s-s_0|<\delta$. Hence, the mapping $s\mapsto C(sx)$ is non-Archimedean $\mathcal{L}$-fuzzy continuous.

$\square$

Theorem 3.5. *Let $\mathbb{K}$ be a non-Archimedean field, X a normed space and $(Y,\mathcal{P},*_L)$ a non-Archimedean $\mathcal{L}$-fuzzy normed space over $\mathbb{K}$. Let $f:X\to Y$ be a Ψ-approximately cubic mapping. If for some $x\in X$ and all $n\in\mathbb{N}$, the mapping $g:\mathbb{R}\to Y$ defined by $g(s)=2^{2kn}f(\frac{x}{2^{kn}})$ is non-Archimedean $\mathcal{L}$-fuzzy continuous. Then $C(rx)=r^3C(x)$ for each $x\in X$ and $r\in\mathbb{R}$.*

Proof. For each $q \in \mathbb{Q}$, we have $C(qx) = q^3C(x)$. $\mathbb{Q}$ is a dense subset of $\mathbb{R}$. Fix $r \in \mathbb{R}$ and $t > 0$. Choose a rational sequence q_n such that $q_n \to r$. Then, there exists $\delta > 0$ such that

$$\mathcal{P}(C(rx) - r^3C(x), t) \geq_L$$
$$\mathcal{P}(C(rx) - C(q_nx), t) *_L \mathcal{P}(C(q_nx) - q_n^3C(x), t) *_L \mathcal{P}(q_n^3C(x) - r^3C(x), t).$$

By using the Theorem (3.4) for given $\varepsilon > 0$, we have

$$\mathcal{P}(C(rx) - r^3C(x), t) \geq_L 1 - \varepsilon *_L 1 *_L \mathcal{P}(q_n^3C(x) - r^3C(x), t).$$

By taking n tend to infinity,

$$\mathcal{P}(C(rx) - r^3C(x), t) \geq_L 1 - \varepsilon.$$

So the proof is complete. □

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SPECTRAL DECOMPOSITION OF A 4TH-ORDER COVARIANCE DOUGLAS TENSOR IN *P-FINSLER MANIFOLDS

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Abstract. In this note we wish to implement some very technical consequences due to professor(s) P.J. Basser and Sinisa Pajevic [5] to discuss the spectral decomposition of 4th-order Douglas tensor in *P-Finsler manifold. Actually, throughout the article, we have just reviewed the results of [5] for a particular 4th-order covariance Douglas tensor, most often enunciated in Finslerian geometry. Moreover, spectral decomposition techniques have been studied for isotropic Douglas tensor.

1. Introduction

A Finsler metric of a manifold or vector bundle is defined as a smooth assignment for each base point a norm on each fibre space, and thus the class of Finsler metrics contains Riemannian metrics as a special sub-class. For this reason, Finsler geometry is usually treated as a generalization of Riemannian geometry. In fact, there are many contributions to Finsler geometry which contain Riemannian geometry as a special case (see e.g., [4], [22], and references therein).

On the other hand, we can treat Finsler geometry as a special case of Riemannian geometry in the sense that Finsler geometry may be developed as differential geometry of fibred manifolds (e.g., [1]). In fact, if a Finsler metric in the usual sense is given on a vector bundle, then it induces a Riemannian inner product on the vertical subbundle of the total space, and thus, Finsler geometry is translated to the geometry of this Riemannian vector bundle. It is natural to question why we need Finsler geometry at all. To answer this question, we have few aspects of complex Finsler geometry to some subjects which are impossible to study via Hermitian geometry.
Let F be a Finsler metric on a holomorphic vector bundle $\pi : E \to M$ over a complex manifold M. The geometry of a Finsler bundle (E, F) is the study of

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the vertical bundle $V_E = \ker_\pi$ with a Hermitian metric g_{V_E} induced from the given Finsler metric. The main tool of the investigation in Finsler geometry is the Finsler connection. The connection is a unique one on the Hermitian bundle (V_E, g_{V_E}) satisfying some geometric conditions. Although it is natural to investigate (V_E, g_{V_E}) by using the Hermitian connection (V_E, g_{V_E}), it is convenient to use Finsler connection for investigating Finsler metrics. For example, the flatness of Hermitian connection (V_E, g_{V_E}) implies that the Finsler metric F is reduced to a flat Hermitian metric. However, if the Finsler connection is flat, then the metric F belongs to an important class, the so-called locally Minkowski metrics (we simply call these special metrics at Finsler metrics). If the Finsler connection is induced from a connection on E, then the metric F belongs to another important class, the so-called Berwald metrics (sometimes a Berwald metric is said to be modeled on a Minkowski space). In this sense, the big difference between Hermitian geometry and Finsler geometry is the connection used for the investigation of the bundle (V_E, g_{V_E}).

1.1. **Preliminaries** [6]**.** We consider an n-dimensional Finsler space

$$F^n = (M^n, L(x, y))$$

on a connected differentiable manifold M^n of dimension n. The fundamental function $L(x, y)$, a real valued function on the tangent bundle TM^n, is usually supposed to satisfy certain conditions from the geometrical standpoint, but only the homogeneity and the regularity are mainly important for our further considerations.

(1) $L(x, y)$ be positively homogeneous in y^i of degree one:
$L(x, py) = pL(x, y)$, for any $x \in M^n, y \in TM^n$ and $\forall p > 0$.

(2) $L(x, y)$ be regular:
$g_{ij} = \dot{\partial}_i \dot{\partial}_j F$ has non-zero $g = \det g_{ij}$,
where $F = L^2/2$ and $\dot{\partial}_i = \frac{\partial}{\partial y^i}$.

Let (g^{ij}) be the inverse of the metric (g_{ij}). We construct the following:

$$2\gamma^i_{jk} = g^{ir}(\partial_k g_{rj} + \partial_j g_{rk} - \partial_r g_{kj}),$$

$$2G^i(x, y) = g^{ij}\{(\dot{\partial}_j \partial_r F)y^r - \partial_j F\},$$

where $\partial_j = \frac{\partial}{\partial x^j}$. Then we have $\gamma^i_{jk}(x, y)y^j y^k = 2G^i(x, y)$.
We, now, consider a geodesic curve $C : x^i = x^i(t), (t_0 \leq t \leq t_1)$. on M^n whose arc length is defined by the integral $s = \int_{t_0}^{t_1} L(x, \dot{x})dt, \dot{x} \equiv \frac{dx^i}{dt}$. Then the extreme of this integral, called the geodesic, is given by the Euler differential equations $d(\dot{\partial}L)/dt - \partial_i L = 0$, which are written in the form:

$$\dot{x}^i[\ddot{x}^j + 2G(x, \dot{x})] - \dot{x}^j[\ddot{x}^i + 2G(x, \dot{x})] = 0. \tag{1.1}$$

The system of differential equations given by (1.1) can also be written as;

$$\frac{d^2x^i}{dt^2} = -2G^i(x, y),\ y^i = \frac{dx^i}{dt}, \tag{1.2}$$

where

$$G^i = \frac{1}{4} g^{ir} \left[y^s \left(\frac{\partial L^2_{(r)}}{\partial x^s} \right) - \frac{\partial L^2}{\partial x^r} \right], \tag{1.3}$$

and $g_{ij} = \frac{1}{2} L^2_{(i)(j)}$, $(i) = \dot{\partial} = \frac{\partial}{\partial y^i}$ and $(g^{ij}) = (g^{-1}_{ij})$.

Moreover, in order to introduce the geometrical quantities in F^n, we shall be concerned with a Finsler connection $F\Gamma = (F^i_{jk}(x,y), N^i_j(x,y), V^i_{jk}(x,y))$ on F^n. For a tensor field $F\Gamma$ gives rise to the h and v- covariant differentiations, we treat a tensor field $X^i(x,y)$ of type $(1,0)$ for brevity. Then we obtain two tensor fields as follows:

$$\nabla^h_j X^i = \delta_j X^i + X^r F^i_{rj}(x,y) \tag{1.4}$$

$$\nabla^v_j X^i = \dot{\partial}_j X^i + X^r V^i_{rj}(x,y), \tag{1.5}$$

where $\delta_j = \partial_j - N^r_j(x,y)\dot{\partial}_r$. The h and v-covariant derivatives $\nabla^h X$ and $\nabla^v X$ are tensor field of $(1,1)$-type.

The Berwarld connection coefficients $B\Gamma = (G^i_{ij}(x,y)), G^i_j(x,y), 0)$ can be derived from the function G^i as follows:

$$G^i_j = G^i_{(j)} \equiv \dot{\partial}_j G^i, \;\; G^i_{jk} = G^i_{j(k)} \equiv \dot{\partial} G^i_j. \tag{1.6}$$

The Berwald covariant derivative with respect to the Berwald connection can be written as;

$$T^i_{j;k} = \frac{\partial T^i_j}{\partial x^k} - T^i_{j(r)} G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}. \tag{1.7}$$

With the help of equation (1.7), we can obtain the commutation formulae, called Ricci identities:

$$X^i_{;j;k} - X^i_{;k;j} = X^r H^i_{rjk} - X^i_{;r} R^r_{jk}, \quad \dot{\partial}_k (X^i_{;j}) - (\dot{\partial}_k X^i)_{;j} = X^r G^i_{rjk}. \tag{1.8}$$

1.1.1. *Douglas Space, Douglas Tensor, Randers metric and *P Finsler Space.* In this subsection, we delineate a short introduction to the recent theory of Finsler manifolds.

We initiate with the equation (1.1) of geodesic of two dimensional Finsler space F^2. If we represent (x^1, x^2) by (x,y), assume x as the parameter t and use the mathematical terms $y' = \frac{dy}{dx}, y'' = \frac{dy'}{dx}$, then (1.1) $\forall i = 1, j = 2$ for F^2 can be written in the form:

$$y'' = f(x,y,y') = X_3 y'^3 + X_2 y'^2 + X_1 y' + X_0, \tag{1.9}$$

where $X_3 = G^1_{22}, X_2 = 2G^1_{12} - G^2_{22}, X_1 = G^1_{11} - 2G^2_{12}, X_0 = -G^2_{11}$ and $G^i_{jk} = G^i_{jk}(x,y,1,y')$ [23].

In case, if we are particularly concerned with a Riemannian space of dimension 2, then $G^i_{jk} = \gamma^i_{jk}$ are the usual Christoffel symbols, and hence $X's$ of (1.9) do not contain y' by definition. Consequently, $f(x,y,y')$ of those spaces is a polynomial in y' of degree at most three. Thus such a special property of $f(x, y, y')$ is equivalent to the fact that the expression $\dot{x}^1 G^2(x, \dot{x}) - \dot{x}^2 G^1(x, \dot{x})$ of equation (1.1) is a homogeneous polynomial in $\dot{x}^1, \dot{x}^2$ of degree three.

Now, we can extend the above fact to have the following definitions:

Definition 1.1. A Finsler space F^n is said to be of Douglas type or known as a Douglas space, if $D^{ij}(x, y) = G^i(x, y)y^j - G^j(x, y)y^i$ are homogeneous polynomials in y^i of degree three.

Proposition 1.1. *A Berwald space is said to be of Douglas type, if $G^i(x, y)$ of equation (1.1) are of the form $G^i_{jk}(x)y^j y^k/2$.*

Theorem 1.2. *A Finsler space F^2 of dimension 2 is said to be Douglas type, if and only if, in every local coordinate system (x, y) the differential equation $y'\prime = f(x, y, y')$ of geodesic is such that $f(x, y, y')$ is a polynomial of degree at most three.*

Now, let us consider the two Finsler spaces $F^n(M^n, L)$ and $\bar{F}^n(M^n, \bar{L})$ defined over a common underlying manifold M^n. A diffeomorphism $F^n \to \bar{F}^n$ is called geodesic if it maps an arbitrary geodesic of F^n to a geodesic of $\bar{F}^n$. In this case the change $L \to \bar{L}$ of the metric is called projective. It is also well known that the mapping $F^n \to \bar{F}^n$ is geodesic if and only if $\exists$ a scalar field $p(x, y)$ satisfying the following equation:

$$\bar{G}^i = G^i + p(x, y)y^i, \quad p \neq 0. \tag{1.10}$$

The projective factor $p(x, y)$ is positive homogeneous function of degree one in y. From equation (1.10), we obtain the following equations [6]:

$$\bar{G}^i_j = G^i_j + p\delta^i_j + p_j y^i, \quad p_j = p_{(j)} \tag{1.11}$$

$$\bar{G}^i_{jk} = G^i_{jk} + p_j\delta^i_k + p_k\delta^i_j + p_{jk}y^i, \quad p_{jk} = p_{j(k)}, \tag{1.12}$$

$$\bar{G}^i_{jkl} = G^i_{jkl} + p_{jk}\delta^i_l + p_{jl}\delta^i_k + p_{kl}\delta^i_j + p_{jkl}y^i, \; p_{jkl} = p_{jk(l)}. \tag{1.13}$$

If we substitute $p_{ij} = (\bar{G}_{ij} - G_{ij})/(n+1)$ and $p_{ijk} = (\bar{G}_{ij(k)} - G_{ij(k)})/(n+1)$ into equation (1.13), we obtain the so called Douglas tensor which is invariant under geodesic mappings, i.e.,

$$D^i_{jkl} = \frac{1}{(n+1)}\left[G^i_{jkl} - (y^i G_{jk(l)} + \delta^i_j G_{kl} + \delta^i_k G_{jl} + \delta^i_l G_{jk})\right], \tag{1.14}$$

which is invariant under geodesic mapping, that is,

$$D^i_{jkl} = \bar{D}^i_{jkl}. \tag{1.15}$$

We now consider the following notations and theorems for the Finsler space.

Definition 1.2. [2] In an n-dimensional differentiable space M^n, a Finsler metric $L(x, y) = \alpha(x, y) + \beta(x, y)$ is called Randeres metric, where $\alpha(x, y) = \sqrt{(a_{ij}(x)y^i y^j)}$ is a Riemannian metric in M^n and $\beta(x, y) = b_i(x)y^i$ is a differential 1-form in M^n. The Finsler space $F^n = (M^n, L)$ with Randers metric is called Randers space.

Definition 1.3. [2] The Finsler metric $L = \alpha^2/\beta$ is called Kropina metric and the Finsler space (M^n, L) equipped with Kropina metric is called Kropina space.

Definition 1.4. [2],[23] A Finsler space of dimension $n > 2$ is called C-reducible, if the tensor $C_{ijk} = \frac{1}{2} g_{ij,(k)}$ can be written in the form:

$$C_{ijk} = \frac{1}{n+1}(h_{ij}C_k + h_{ik}C_j + H_{jk}C_i,) \tag{1.16}$$

where $h_{ij} = g_{ij} - l_i l_j$ is the angular metric ensor and $l_i = L_{(i)}$.

Theorem 1.3. [23]*A Finsler space $F^n, n \geq 3$ is said to be C-reducible if and only if the metric is a Randers metric of a Kropian metric.*

Definition 1.5. [17, 18] A Finsler space F^n is called $^*P$-Finsler space, if the tensor $P_{ijk} = \frac{1}{2} g_{ij;k}$ can be written in the form:

$$P_{ijk} = \lambda(x, y) C_{ijk}. \tag{1.17}$$

Theorem 1.4. [17] *For $n > 3$ in a C-reducible $^*P$-Finsler space, $\lambda(x, y) = k(x)L(x, y)$ holds and $k(x)$ is only the function of position.*

1.1.2. *Spectral Decomposition of a 4th order covariance tensor.* Various techniques to characterize the variability of scalar and vector valued random variables have been evolved by many researchers. Specially, the Principal component analysis (PCA) for analyzing sample covariance matrices has been originally proposed by [29] and developed by [15]. Concerning to the same issue, many other methods, such as factor analysis [30] and independent component analysis (ICA)[24, 7, 16, 8] have been well established. However, yet now statistical framework for the variability of a tensor-valued random variables could be found. To overcome from the above problem, [5] have proposed a framework to delineate the covariance structure of random 2nd-order tensor variables. Further, expressions for the sample mean and covariance tensor associated with a 2nd-order tensor random variable have been derived and is shown that the covariance tensor is a 4th-order tensor, which can be decomposed as a linear combination of eigenvalues and the outer product of their corresponding eigentensors. Moreover, [5] have also proposed a new avenue to visualize angular or orientational feature of the 4th-order covariance tensor using the spectral decomposition framework.

1.2. Methodology and Theoretical Background of Spectral Analysis.

Here we discuss a brief digest over spectral analysis suggested by [5] and many others.

[5] mentioned that in order to perform spectral analysis on vector valued data, we first need to generate a sample covariance matrix S, and then expend it as a linear combination of eigenvalues and the outer product of their corresponding eigenvectors (for instance see [13]). It is also mentioned by [5] that however, there are many data types, such as 2nd-order and higher order tensors, for which the present approach is not suitable. Even, [9, 28] have shown that it is always possible

to express there higher order tensors as a linear combination of vectors, but recombining the tensor elements in this way may result vagueness (viz. failing to recognize diagonal and off-diagonal tensor elements). Moreover, it is also uneasy to perform affine transformations such as rotation, dilation or shear etc. in the condition when one have a higher order tensor as a linear combination of vectors. Representation of higher order tensor in vectorial form may also destroy the intrinsic geometric structure of original tensor data.

Keeping these issues in mind, [10] have proposed a normal distribution for 2nd and higher order tensor data, which generalizes the normal multivariate distribution. In order to preserve the original form of given tensor in this new distribution, [10] has replaced the familiar mean vector μ in the multivariate distribution, with a 3-dimensional 2nd-order mean tensor $\bar{D}$ and replaced the covariance matrix S, in the multivariate normal distribution, with a 3-dimensional 4th-order tensor Σ.

The normal distribution for 2nd-order tensor random variables. The exponent of a multi-varite normal probability density function $p(x)$ contains the quadratic form $(x-\mu)^T M^{-1}(x-\mu)$ of an n-dimensional normal random variable x, its man vector μ and the inverse of an $n \times n$ covariance matrix M is give as [25, 3]:

$$p(x) = \sqrt{\left[\frac{|M^{-1}|}{(2\pi)^n}\right]} e^{-(1/2)(x-\mu)^T M^{-1}(x-\mu)} = \sqrt{\left[\frac{|M^{-1}|}{(2\pi)^n}\right]} e^{-(1/2)(x_i-\mu_i)^T M_{ij}^{-1}(x_j-\mu_j)}, \tag{1.18}$$

where the Einstein summation convention have been used in the right most expression. It is also discussed by [25, 3] that the exponent $(x_i - \mu_i)^T M_{ij}^{-1}(x_j - \mu_j)$ is a scalar contraction of two n-dimensional second order covariance tensor, which in this context is a covariance matrix S, but it can be transformed as an n-dimensional 2nd-order tensor. Moreover, the interpretation of the random vector and covariance matrix as a tensor of 1st and 2nd-order respectively have been enhanced to have multi-variate normal distribution as a tensor variate normal distribution for a 2nd-order random tensor D,

$$\begin{aligned} p(D) &= \sqrt{\left[\frac{|\Sigma^{-1}|}{8\pi^6}\right]} e^{-(1/2)(D-\bar{D}):\Sigma^{-1}:(D-\bar{D})} \\ &= \sqrt{\left[\frac{|\Sigma^{-1}|}{8\pi^6}\right]} e^{-(1/2)(D_{ij}-\bar{D}_{ij}):\Sigma^{-1}_{ijmn}:(D_{mn}-\bar{D}_{mn})}. \end{aligned} \tag{1.19}$$

In the above expression, $\bar{D}$ is the mean tensor and $(D_{ij}-\bar{D}_{ij}) : \Sigma^{-1}_{ijmn} : (D_{mn}-\bar{D}_{mn})$ is a scalar contraction of the inverse of 3-dimensional 4th-order covariance tensor Σ_{ijmn} and two 3-dimensional 2nd-order tensors $(D_{ij} - \bar{D}_{ij})$ and $(D_{mn} - \bar{D}_{mn})$ [10]. Here in equation (1.19), the resulting exponent is a linear combination of quadratic functions formed from the products of elements $D, (D_{ij} - \bar{D}_{ij}), (D_{mn} - \bar{D}_{mn})$ weighted by the suitable coefficients, Σ^{-1}_{ijmn}. Also, in equation (1.19), the tensor "double dot product" operation as defined in [26] has been employed. In the expression $D : \Sigma^{-1} : D = D_{ij}\Sigma^{-1}_{ijkl}D_{kl}$, sums are taken for all indices i, j, k and l.

Moreover, the 4th-order covariance tensor Σ and its inverse Σ^{-1} are related to the symmetric 4th-order identity tensor Y as below:

$$\Sigma_{ijkl}\Sigma^{-1}_{klmn} = \Sigma^{-1}_{ijkl}\Sigma_{klmn} = Y_{ijmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}). \tag{1.20}$$

2. Spectral decomposition of a 4th order covariance Douglas tensor in *P-Finsler Manifolds

In order to discuss the spectral decomposition of a 4th-order covariance Douglas tensor, let us make the following useful assumptions to setup the relevant mathematical analysis.

Suppose F^n be a $^*P$-Finsler space satisfying the condition (1.17). Also, let $x_i \in M^n$ be the n-dimensional normal random variable in $^*P$-Finsler manifold and μ be the familiar mean vector, then for the hv-Ricci tensor (1.6), the probability density function $p(x)$ will have the form:

$$p(x) = \sqrt{\left[\frac{|G^{-1}|}{(2\pi)^n}\right]} e^{-(1/2)(x_i-\mu_i)^T G^{-1}_{ij}(x_j-\mu_j)}, \tag{2.1}$$

where G_{ij} is an n-dimensional 2nd-order covariance hv-Ricci tensor and is given as $G_{ij} = G_{i(j)} = \dot{\partial}_j G_i$.

Furthermore, if we consider the normal tensor variate probability function for G_{ij} as $P(G)$, then for the following 4th-order covariance Douglas tensor,

$$D_{ijkl} = \frac{1}{(n+1)}[G_{ijkl} - (y_i G_{jk(l)} + g_{ij}G_{kl} + g_{ik}G_{jl} + g_{il}G_{jk})], \tag{2.2}$$

the $P(G)$ will be given by

$$P(G) = \sqrt{\left[\frac{|D^{-1}|}{8\pi^6}\right]} e^{-(1/2)(G_{ij}-\bar{G}_{ij}):D^{-1}_{ijkl}:(G_{kl}-\bar{G}_{kl})}. \tag{2.3}$$

In the above expression $\bar{G}$ in the equation (2.3) is the mean hv-Ricci tensor of the hv-Ricci tensor G. Further, the 4th-order n-dimensional covariance Douglas tensor D_{ijkl} and its inverse D^{-1}_{ijkl} satisfy the identity:

$$D_{ijmn}D^{-1}_{klmn} = D^{-1}_{ijmn}D_{klmn} = Y_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{ij}\delta_{jk}). \tag{2.4}$$

The 4th-order covariance Douglas tensor D_{ijkl} may satisfy the following symmetry properties:

Since the hv-Ricci tensor G_{ij} is symmetric 2nd-order tensor, then D_{ijkl} given by (2.2) being the composition of G_{ij} and its partial derivatives should inherit symmetries such that its elements must remain same by exchanging particular pairs of indices.

For feasibility, if one sets the mean hv-Ricci tensor $\bar{G}_{ij}$ to zero. Then, since the product of two components of 2nd-order tensor commute in the scalar contraction (viz. $G_{ij}.D_{ijkl}.D_{kl}$), i.e., $G_{ij}G_{mn} = G_{mn}G_{ij}$, the corresponding coefficients of D_{ijkl}

must be indistinguishable and hence must be same, i.e., $D_{ijkl} = D_{klij}$.
From what has been discussed, it follows that symmetry of hv-Ricci tensor implies that $D_{ijkl} = D_{jikl}$ and $D_{ijkl} = D_{ijlk}$. Also, these symmetries of covariance Douglas tensor can reduce the possible number of independent components. In case of 3-dimension the symmetry can reduce the components of D_{ijkl} from 81 (i.e, 3^4) to 21 [19].

In concern with the symmetry properties, [5] has mentioned that actually the 21 independent components are also required to specify each element of the symmetric covariance matrix. Moreover, In case of three dimensional *P Finsler space, the 4th-order covariance Douglas tensor D_{ijkl} can be transformed to a 6-dimensional 2nd-order tensor which is a symmetric 6×6 covariance matrix having the same 21 independent components [28, 14, 32]. To perform such a transformation, we first write the scalar contraction $G_{ij}D^{-1}_{ijkl}G_{kl}$ in the form $\tilde{G}_r S_{rt} \tilde{G}_t$, where the hv-Ricci tensor G_{ij} is written as a 6-dimensional column vector $\tilde{G}$ and is expressed as $\tilde{G} = (G_{xx}, G_{yy}, G_{zz}, \sqrt{2}G_{xy}, \sqrt{2}G_{xz}, \sqrt{2}G_{yz})$. Here the factor $\sqrt{2}$ premultiplied with the off-diagonal elements of G_{ij} emphasize that the operation of matrix multiplication between $\tilde{G}$ and the 6-dimensional 2nd-order tensor S is isomorphic to the operation of tensor double product second order hv-Ricci tensor G_{ij} and 3-dimensional 4th-order covariance Douglas tensor D_{ijkl} [11, 12].

In order to have a conversion between 6-dimensional 2nd-order tensor and the 3-dimensional 4th-order covariance Douglas tensor, we have the following tensor representation as discussed by [5];
(2.5)
$$S = \begin{pmatrix} D_{xxxx} & D_{xxyy} & D_{xxzz} & \sqrt{2}D_{xxyy} & \sqrt{2}D_{xxzz} & \sqrt{2}D_{xxyz} \\ D_{xxyy} & D_{yyyy} & D_{yyzz} & \sqrt{2}D_{yyxy} & \sqrt{2}D_{yyxz} & \sqrt{2}D_{yyyz} \\ D_{xxzz} & D_{yyzz} & D_{zzzz} & \sqrt{2}D_{zzxy} & \sqrt{2}D_{zzxz} & \sqrt{2}D_{zzyz} \\ \sqrt{2}D_{xxxy} & \sqrt{2}D_{yyxy} & \sqrt{2}D_{zzxy} & 2D_{xyxy} & 2D_{xyxz} & 2D_{xyyz} \\ \sqrt{2}D_{xxxz} & \sqrt{2}D_{yyxz} & \sqrt{2}D_{zzxz} & 2D_{xyxz} & 2D_{xzxz} & 2D_{xzyz} \\ \sqrt{2}D_{xxyz} & \sqrt{2}D_{yyyz} & \sqrt{2}D_{zzyz} & 2D_{xyyz} & 2D_{xzyz} & 2D_{yzyz} \end{pmatrix}.$$

Again, according to [27, 31], the premultiplied factors 2 and $\sqrt{2}$ for different 3×3 blocks of the covariance 6×6 matrix S emphasize that this object transforms as a 6-dimensional second order tensor and the mapping between D_{ijkl} and S including corresponding multiplication operation is an isomorphism. This fact can be expressed by saying that with the column operation :, the set of 4th-order covariance Douglas tensor is isomorphic to the set of 2nd-order covariance tensors and matrix multiplication operation. Thus once we have the tensor representation of type (2.5), we can easily obtain S^{-1} from (2.5).

Let us now discuss about the latent-roots and latent-tensors of the 4th-order covariance Douglas tensor as below:

2.1. **Latent-roots and Latent-vectors of D_{ijkl}.** As it is evident from the foregoing discussion that we can represent the 2nd-order tensor in terms of a covariance matrix M and hence the latent-roots and the corresponding latent-vectors can be determined from the matrix M. Likewise, one can determine the eigenvalues (denoted by σ^2) and 2nd-order eigentensors E of a 4th-order covariance Douglas tensor [26, 20, 21]. The Fundamental expression is given as [9]:

$$D : E = \sigma^2 E, \tag{2.6}$$

where the tensor double dot product " : " has been employed to signify the tensor product operation.

Basically, if the two tensors, say U and V are of same order, then the tensor dot product for them will be given by,

$$U : V = \text{ Trace } (UV^T) = U_{ij}V_{kj}\delta_{ik} = U_{ij}V_{ij}. \tag{2.7}$$

Now, rearranging the terms for (2.6), we have

$$\left(D - \sigma^2 Y\right) : E = 0, \tag{2.8}$$

where Y is the 4th-order identity tensor defined by (2.4).

Now, likewise square matrices, the equation (2.8) has a non-trivial solution if and only if the Characteristic equation given by,

$$|D - \sigma^2 Y| = 0. \tag{2.9}$$

One can now perform the spectral decomposition, sometimes called eigentensor decomposition by developing the connection between the 4th-order covariance Douglas tensor D and the 6×6 matrix S as in equation (2.5)[9]. We now proceed to find the eigenvalues and eigenvectors of S with the fact that eigenvalues of D and S are same. for this purpose, we fabricate the 2nd-order eigentensor E of the 4th-order Douglas tensor D by considering the 6×1 eigenvectors of S using the following expression:

$$E^i = \begin{pmatrix} \epsilon^i_{xx} & \frac{1}{\sqrt{2}}\epsilon^i_{xy} & \frac{1}{\sqrt{2}}\epsilon^i_{xz} \\ \frac{1}{\sqrt{2}}\epsilon^i_{xy} & \epsilon^i_{yy} & \frac{1}{\sqrt{2}}\epsilon^i_{yz} \\ \frac{1}{\sqrt{2}}\epsilon^i_{xz} & \frac{1}{\sqrt{2}}\epsilon^i_{yz} & \epsilon^i_{zz} \end{pmatrix}, \tag{2.10}$$

where $\epsilon^i = (\epsilon^i_{xx}, \epsilon^i_{yy}, \epsilon^i_{zz}, \epsilon^i_{xy}, \epsilon^i_{yz}, \epsilon^i_{zx})^T$ is the i^{th} normalized eigenvector of S. Here the six 3×3 eigentensors represented by E^i are symmetric and mutually orthogonal and satisfy the following identity:

$$E^i : E^j = \delta^{3d}_{ij}. \tag{2.11}$$

here in equation (2.11), the symbol δ^{3d}_{ij} is the familiar Kronecker delta which sometimes known as 3-dimensional 2nd-order identity tensor. Moreover the superscript $3d$ placed on the Kronecker tensor is simply used to represent its dimensionality.

Also, the expression (2.11) is equivalent to the orthonormality condition for the six 6×1 eigenvectors of the corresponding covariance matrix S:

$$\epsilon^i.\epsilon^j = \delta_{ij}^{6d}, \tag{2.12}$$

where once again the symbol δ_{ij}^{6d} stands for the Kronecker tensor or the 6-dimensional 2nd-order identity tensor.

We, now, go through the spectral decomposition of the 4th-order covariance positive definite symmetric Douglas tensor D. This Douglas tensor can be decomposed into a linear combination of six positive definite latent-vectors denoted by σ_k^2 multiplied by the outer product of their corresponding six 2nd-order eigentensors, denoted by $E^k \otimes E^k$, that is to say,

$$D_{ijmn} = \sigma_k E_{ij}^k E_{mn}^k \sigma_k \text{ or } D = \sigma_k E^k \otimes E^k \sigma_k. \tag{2.13}$$

The above equation is usually refer to as spectral decomposition of a 4th-order covariance Douglas tensor. The Douglas tensor D being positive definite may possess six positive real latent-roots (even though some of them may be repeated)and the six relevant real valued 2nd-order latent-tensors. The above expression can precisely help in finding the inverse of the covariance Douglas tensor appeared in equation (2.3) as follows:

$$D_{ijmn}^{-1} = \sigma^{-1} E_{ij}^k E_{mn}^k \sigma_k^{-1} \text{ or } D^{-1} = \sigma_k^{-1} E^k \otimes E^k \sigma^{-1}. \tag{2.14}$$

For the detail theory of eigentensor decomposition,the readers should refer to [31]. Furthermore, the eigentensor decomposition can provide a lucid expression for the determinant of the 4th-order Douglas tensor $|D|$, i.e.,

$$|D| = \prod_{k=1}^{6} \sigma_k^2 \text{ and } |D^{-1}| = \prod_{k=1}^{6} \sigma_k^{-2}, \tag{2.15}$$

which can be applied in determining the normalization constant in equation (2.3). This factor can also delineate the multiplicative factor of $2^{3/2}$ between the normalization constants in equations (2.1) and (2.3). It is evident that this factor is simply the ratio of determinants of hv-ricci tensor $|G|$ and in equation (2.1) and $|D| = |S|$ in equation (2.3). Actually this factor is the Jacobian of transformation between G and D.

Let us now discuss a special case in which the 4th-order covariance Douglas tensor D_{ijkl} has the isotropic nature.

2.2. **Spectrally decomposed Douglas tensor bearing isotropic nature.** Here we describe the spectral decomposition of the 4th-order covariance Douglas tensor bearing isotropic nature. The detail study regarding this issue has been already done by [5, 10].Here, We precisely utilize the results of [5, 10] to meet our purpose. According to [5], isotropy of any tensorial quantity means the quantity has no orientation dependence. Informally, we can say that the tensorial quantity has an isotropic nature, if it is invariant from the aspect of its natural behavior, i.e, the nature which it bears must be invariant under any transformation like rotation, reflection, inversion etc. Now, as in our case, since the hv-Ricci tensor is symmetric, then the isotropy of Douglas tensor will have the form [19, 32, 31]:

$$D^{\text{iso}}_{ijkl} = \frac{\lambda_\alpha}{3}(\delta_{ij}\delta_{kl}) + \lambda_\beta \left(\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}\right). \tag{2.16}$$

Here, in the above expression, λ_α and λ_β are constants.
Now, according to [5], the spectrally decomposed 4th-order covariance isotropic Douglas tensor D^{iso}_{ijkl} will have the following latent-roots:

$$\sigma_1^2 = \lambda_\alpha; \sigma_2^2 = \sigma_3^2 = \sigma_4^2 = \sigma_5^2 = \sigma_6^2 = \lambda_\beta \tag{2.17}$$

and the relevant normalized latent-tensors are given as;

$$E^1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad E^2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.18}$$

$$E^3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad E^4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.19}$$

$$E^5 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad E^6 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.20}$$

In Finslerian geometry, these latent-tensors are able to describe the characteristics and geometric significance of the 4th-order covariance isotropic Douglas tensor. Now, with the 6th-order characteristic equation (2.9), there associate six coefficients each of which are scalar invariants of Douglas tensor D_{ijkl}. In fact, these coefficients and their functions are independent from the effect of change of coordinate system. Here we mention the six scalar invariants I_1, I_2, I_3, I_4, I_5 and I_6 which are already determined by [5].
These scalar invariants can be obtained by expanding equation (2.9)as below:

$$(\psi - \sigma_1^2)(\psi - \sigma_2^2)(\psi - \sigma_3^2)(\psi - \sigma_4^2)(\psi - \sigma_5^2)(\psi - \sigma_6^2) = 0, \tag{2.21}$$

where $\psi = \sigma^2$ is used for convenience. On collecting the like powers of σ and writing these coefficients in the form of latent-roots of Douglas tensor [5], we have

$$\psi^6 - I_1\psi^5 + I_2\psi^4 - I_3\psi^3 + I_4\psi^2 - I_5\psi + I_6 = 0. \tag{2.22}$$

Here, in the above expression, the following have been used [5]:

$$I_1 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2 + \sigma_6^2, \tag{2.23}$$

which stands for the trace of covariance matrix S.

$$\begin{aligned} I_2 = {} & \sigma_3^2\sigma_4^2 + \sigma_3^2\sigma_5^2 + \sigma_4^2\sigma_5^2 + \sigma_3^2\sigma_6^2 + \sigma_4^2\sigma_6^2 + \sigma_5^2\sigma_6^2 + \sigma_2^2\sigma_3^2 + \sigma_2^2\sigma_4^2 + \\ & + \sigma_2^2\sigma_5^2 + \sigma_2^2\sigma_6^2 + \sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_3^2 + \sigma_1^2\sigma_4^2 + \sigma_1^2\sigma_5^2 + \sigma_1^2\sigma_6^2, \end{aligned} \tag{2.24}$$

$$I_3=\sigma_3^2\sigma_4^2\sigma_5^2+\sigma_3^2\sigma_4^2\sigma_6^2+\sigma_3^2\sigma_5^2\sigma_6^2+\sigma_5^2\sigma_5^2\sigma_6^2+\sigma_2^2[\sigma_5^2\sigma_6^2+\sigma_4^2(\sigma_5^2+\sigma_6^2)+ +\sigma_3^2(\sigma_4^2+\sigma_5^2+\sigma_6^2)]+\sigma_1^2[\sigma_4^2\sigma_5^2+\sigma_4^2\sigma_6^2+\sigma_5^2\sigma_6^2+\sigma_3^2(\sigma_4^2+\sigma_5^2+\sigma_6^2)+\sigma_2^2(\sigma_3^2+\sigma_4^2+\sigma_5^2+\sigma_6^2)], \tag{2.25}$$

$$I_4=\sigma_3^2\sigma_4^2\sigma_6^2\sigma_6^2+\sigma_2^2[\sigma_4^2\sigma_5^2\sigma_6^2+\sigma_3^2(\sigma_5^2\sigma_6^2+\sigma_4^2(\sigma_5^2+\sigma_6^2))]+\sigma_1^2[\sigma_3^2\sigma_4^2\sigma_5^2+ +\sigma_3^2\sigma_4^2\sigma_6^2+\sigma_3^2\sigma_5^2\sigma_6^2+\sigma_4^2\sigma_5^2\sigma_6^2+\sigma_2^2(\sigma_5^2\sigma_6^2+\sigma_4^2(\sigma_6^2+\sigma_6^2)+\sigma_3^2(\sigma_4^2+\sigma_5^2+\sigma_6^2))], \tag{2.26}$$

$$I_5=\sigma_2^2\sigma_3^2\sigma_4^2\sigma_5^2\sigma_6^2+\sigma_1^2[\sigma_3^2\sigma_4^2\sigma_5^2\sigma_6^2+\sigma_2^2(\sigma_4^2\sigma_5^2\sigma_6^2+ +\sigma_3^2(\sigma_5^2\sigma_6^2+\sigma_4^2(\sigma_5^2+\sigma_6^2)))], \tag{2.27}$$

$$I_6=\sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2\sigma_5^2\sigma_6^2. \tag{2.28}$$

[5] mentioned that the invariant I_1 is the trace of covariance second order tensor S and I_6 is its determinant. Rest of the invariants stand for discrete combinations of eigenvalues of D_{ijkl}, which can be used to pursue distinguish features of Douglas space.

Concluding Remarks. The present paper is just a particularization of the fabulous article of [5]. The statistical methods have been employed to 4th-order covariance Douglas tensor in *P-Finsler manifold. Just the techniques provided by [5] have been applied and reviewed in the Finsler geometry. In a nutshell, in place of any general 4th-order covariance tensor, a particular Douglas tensor has been employed and then the spectral decomposition has been reviewed.

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TECHNIQUES FOR THE DECOMPOSITION OF CARTAN'S CURVATURE TENSOR IN COMPLEX FINSLER MANIFOLDS

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Abstract. A Finsler metric of a manifold or vector bundle is defined as a smooth assignment for each base point, a norm on each fiber space and thus the class of Finsler metrics contains Riemannian metrics as a special subclass. The geometry of complex Finsler manifold has been developed by [7]. In complex Finsler manifolds, the study of theory of curvatures has been an active field of research over past few decades. In the present article, our main purpose is to discuss some techniques of decomposition for the well known Cartan's first curvature tensor S^i_{jkh}. Moreover, we attempted to establish few significant results that may produce vital connections between complex Finsler and complex Einstein's manifolds. Also, by adopting the techniques of decomposition, various cases and conditions have been developed and their advantages in the study of theory of relativity & cosmology have been pursued.

1. Introduction

Since the explanation of various physical systems and Mathematical devices is much more concerned with the utilization of numerous algebraic quantities involved in the illustration of geometrical phenomenon and states in which they occur, it is mandatory to put forward the ideas of such basic algebraic quantities. Certain types of quantities are commonly identified as scalar, vector and tensors. Among these the tensors are quite crucial and tedious geometric structures as these are the only quantities involving three meaningful aspects altogether. That is, a tensor can suppose to be a tool having direction, magnitude and orientation dependency.

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Different types of tensors, according to their nature are applied to discuss different systems, their states and properties. But from geometric point of view, the geometry from its origin up to its recent extent is all about the curvature tensors.

Among all curvature tensor, yet known to us, Cartan's-curvature tensors are very surprising because of having special classes of differentiation obtained with the help of Cartan's postulates, which are most popularly used in Finsler geometry. Although, the Cartan's curvature tensor is capable to determine the properties of geometric configurations in Finsler manifolds, but sometimes it seems to be necessarily convenient to decompose it in order to study the basic aspects of the manifolds under consideration. Here are few basic concepts of complex Finsler manifolds and Cartan's-curvature tensor, which we seem necessary to study the proposed research topic.

The basic ideas of Finsler manifolds may be traced back to the famous lecture of Riemann "Über die hypotheses, welche der Geometrie zugrunde leigen". In this lecture of 1854, Riemann has discussed several new methods by means of which an n-dimensional manifold can be intimated with a special kind of distance or metric function. It is supposed that this special metric function should have three properties in common, namely; the function must be positive definite, its first order differential must be homogeneous and convex in latter. The distance function **ds** between two points whose coordinates are given by z^i & $z^i + dz^i$, in a complex Finsler manifold is defined by some function $F(z^i, dz^i)$, i.e.,

$$ds = F(z^i, dz^i) \ \ \forall i = 1, 2, \cdots, n.$$

If $\mathbb{D}$ be the domain (i.e., an open connected set) of an n-dimensional manifold V_n, which is completely covered by a coordinate system, such that any point of $\mathbb{D}$ is represented by a set of n independent variables $z^i (i = 1, 2, \cdots, n)$, then the set of points of $\mathbb{D}$, whose co-ordinates are expressible as the function of single parameter t is regarded as a curve of V_n. Thus, the equation

$$z^i = z^i(t), \tag{1}$$

represents a curve C in V_n. If the equations (1) are of class C^1, we shall regard the expression whose components are given by

$$\dot{z}^i = \frac{dz^i}{dt} \tag{2}$$

as the tangent vector to C.

Next, suppose that we are given a function $F(z^i, \dot{z}^i)$ of the line element $(z^i, \dot{z}^i)$ of the curve defined by C in $\mathbb{D}$, then we have the following conditions:

Condition (a): The function $F(z^i, \dot{z}^i)$ is positively homogeneous of degree one in the $\dot{z}^i$, i.e.,

$$F(z^i, k\dot{z}^i) = kF(z^i, \dot{z}^i) \text{ with } k > 0. \tag{3}$$

Condition (b): The function $F(z^i, \dot{z}^i)$ is positive if, not all $\dot{z}^i$ vanish simultaneously, i.e.,

$$F(z^i, \dot{z}^i) > 0 \text{ with } \sum_i (z^i)^2 \neq 0. \tag{4}$$

Condition (c): The function $F(z^i, \dot{z}^i)$ is convex in z^i. It follows from a well known theorem on complex function that $f(z^i)$ on n variables $u^1, u^2, \cdots, u^n$

of a domain $\mathbb{D}$ in the manifold is said to be convex, if it contains the whole segment of a straight line which connects any two of its points. The function $f(u^i)$ is said to be convex in $\mathbb{D}$, if it is defined in $\mathbb{D}$ and if the inequality

$$f\left(\frac{u^i_{(1)} + u^i_{(2)}}{2}\right) \leq \frac{1}{2}\left[f(u^i_{(1)}) + f(u^i_{(2)})\right] \tag{5}$$

is satisfied for all pairs of the points $u^i_{(1)}$ and $u^i_{(2)}$ of $\mathbb{D}$.

Eventually, if in the complex manifold C_n we introduce the fundamental function $F(z^i, \dot{z}^i)$ which is positively homogeneous of degree one with respect to the variables z^i and $\dot{z}^i$, then the function $F(z^i, \dot{z}^i) \geq 0$ is such that

$$F(z^i, \dot{z}^i) = |k| F(z^i, \dot{z}^i).$$

The arc length of the arc $z^i = z^i(t)$ for $t_1 \leq t \leq t_2$ is defined by;

$$s = \int_{t_1}^{t_2} F(z^i, \dot{z}^i). \tag{6}$$

Such manifold is called *complex Finsler manifold* [17] and is symbolized by F_n^C. Moreover, the function $F(z^i, \dot{z}^i)$ is assumed to be invariant under coordinate transformations.

Now, putting $F(z^i, \dot{z}^i) \stackrel{\text{def}}{=} F^2(z^i, \dot{z}^i)$ and $g_{ij}(z^i, \dot{z}^i) \stackrel{\text{def}}{=} \frac{\partial^2 F^2(z^i, \dot{z}^i)}{\partial \dot{z}^i \partial \dot{z}^j}$, we observe that g_{ij} is a symmetric covariant tensor and that

$$F^2(z^i, \dot{z}^i) = g_{ij}(z, \dot{z}) \dot{z}^i \dot{z}^j. \tag{7}$$

We shall assume that the fundamental tensor g_{ij} of the complex Finsler manifold has a rank n and we use g_{ij} and its associate g^{ij} to lower an raise the indices.

1.1. **Fundamental Postulates of E. Cartan**[8]**.** The theory of E-Cartan which treats the Finsler manifolds from an entirely different point of view has played the most prominent role in the development of Finsler Geometry. In this subsection, we shall take a brief look on Cartan's monograph in which he discussed his postulates, which he defined by means of special classes of covariant derivatives.

In order to be able to endow the Finsler manifold $F_n^{(C)}$ with a so-called "Euclidean connection", Cartan considered the manifold X_{2n-1} of the line elements $(z^i, \dot{z}^i)$ which is $(2n-1)$ dimensional, since only the ration of the $\dot{z}^i$ are necessary to define a direction in the tangent manifold $T_n(z^i)$. The coordinates are referring to the centre of the line element $(z^i, \dot{z}^i)$. All quantities such as tensors are to be defined by means of the functions of line elements.

In the manifold $F_n^{(C)}$, a metric is defined by means of a function $F(z^i, \dot{z}^i)$ satisfying the three conditions of Finsler manifold, but the manifold X_{2n-1} is said to be endowed with Euclidean connection if the following construction is imposed on X_{2n-1}.

I: A metric with symmetric components $g_{ij}(z^i, \dot{z}^i)$ is given such that the square of the distance between the centres z^i and $z^i + dz^i$ of the neighboring elements $(z^i, \dot{z}^i)$ and $(z^i + dz^i, \dot{z}^i) + d\dot{z}^i$ is given by the expression

$$g_{ij}(z, \dot{z}) dz^i dz^j. \tag{8}$$

Because, the dz^i form the components of a contravariant vector, it follows that the square of the length of an arbitrary contravariant vector will be defined by

$$g_{ij}(z,\dot{z})X^iX^j. \tag{9}$$

II: An analytic expression would came into existence, which would represent the variation of the vector X^i when its element of support $(z^i, \dot{z}^i)$ experiences an infinitesimal small change and becomes $(z^i + dz^i, \dot{z}^i + d\dot{z}^i)$. This variation of X^i will be represented by means of a covariant (or absolute) differential:

$$DX^i = dX^i + C^i_{kh}(z,\dot{z})X^k d\dot{z}^h + \Gamma^i_{kh}(z,\dot{z})X^k dz^h, \tag{10}$$

where the coefficients C^i_{kh} and Γ^i_{kh} are the functions of the element of support. Naturally, the first postulate can be applied to these by proceeding like below:
If a vector X^i is transposed from $(z^i, \dot{z}^i)$ to $(z^i + dz^i, \dot{z}^i + d\dot{z}^i)$ by parallel displacement, i.e., if the actual change dX^i in X^i is in accordance with the equation;

$$DX^i = 0 \text{ or } dX^i = -C^i_{kh}X^h d\dot{z}^i - \Gamma^i_{kh}X^k dz^h, \tag{11}$$

then the length of X^i as given equation (8) remains invariant.

III: The third postulate of E-Cartan contains the following four logics:

A: If the direction of a vector X^i coincides with that of its elements of support $(z^i, \dot{z}^i)$. its length is to be equal to $F(z^i, X^i)$.

B: Let X^i andY^i represent two vectors with a common element of support $(z^k, \dot{z}^k)$. When the latter undergoes an infinitesimal rotation about its own centre z^k and becomes $(z^k, \dot{z}^k + d\dot{z}^k)$, while the components X^i and Y^i remain invariant, then their corresponding covariant differentials (10) will be DX^i and DY^i and the following symmetric condition will hold good:

$$g_{ij}(z,\dot{z})X^iDY^j = g_{ij}(z,\dot{z})XD^iY^j. \tag{12}$$

C: If the direction of a vector with fixed components X^i coincides with that of its element of support, then its covariant differential given by equation (10) corresponding to an infinitesimal rotation of its element of support about its own centre vanishes identically.

D: When the displacement of a vector is such that the element of support is transported parallel to itself from z^k to $z^k + dz^k$, the coefficients like Γ^i_{kh} which appear in the covariant differential (10) will be symmetric in their lower indices h and k. In view of these conditions, we now conclude that following analytic aspect, which may be very useful to study the Cartan's curvature tensor and their covariant differentiation.

In view of equation (9), condition (A) obviously yields

$$F^2(z^i, \dot{z}^i) = g_{ij}\dot{z}^i\dot{z}^j. \tag{13}$$

Under the above conditions, [8] also gave a new form of equation (9) as below:

$$DX^i = dX^i + \Gamma^{*i}_{kj}X^k dz^j, \tag{14}$$

where he put the expression

$$\Gamma^{*i}_{kj} = \Gamma^i_{kj} - C^i_{kh}\Gamma^h_{rj}\dot{z}^r. \tag{15}$$

In view of the above expression, we now outline some properties of the covariant differentiation as discussed by [8]:

1.2. **Properties of covariant differentiation.** In view of the formula given by equation (10), we have the extended form of derivation of a tensor of any rank as below:

$$DT^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} = dT^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} + \sum_{\mu=1}^{r} T^{i_1 i_2 \cdots i_{\mu-1} i_{\mu+1} \cdots i_r}_{j_1 j_2 \cdots j_s} \times (C^{i_\mu}_{kh} d\dot{z}^h + \Gamma^{i_\mu}_{kh} dz^h) - \sum_{\theta=1}^{s} T^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_{\mu-1} j_{\mu+1} \cdots j_s} (C^k_{j_\mu h} d\dot{z}^h + \Gamma^k_{j_\mu h} dz^h). \tag{16}$$

In fact, there is no ambiguity that the tensor T of type (r, s) is a function of the element of support $(z, \dot{z})$. That is why the term dT involves the variation of the latter. Evidently it is quite clear that if we take this fact into account, we may easily observe that this differentiation technique obeys the usual laws of covariant differentiation viz. differential of the sum is equal to the sum of the differential and the product of ordinary differentiation.

[8] also considered the covariant differential of a contravariant vector $X^i = X^i(z, \dot{z})$ with respect to the unit vector $\dot{z}^h$ in the direction of element of support as follows:

$$DX^i = \left(F \frac{\partial X^i}{\partial \dot{z}^h} + A^i_{kh} X^k \right) Dl^h + X^i_{|h} dz^h, \tag{17}$$

where he used $X^i_{|h} = \frac{\partial X^i}{\partial z^h} - \frac{\partial X^i}{\partial \dot{z}^k} \frac{\partial G^k}{\partial \dot{z}^k} + \Gamma^{*i}_{kh} X^k$.

Now, in order to aim our purpose of study, we briefly discuss Cartan's curvature tensor as given by [8]. We take into account a commutation formula arising from the covariant derivatives as given in equation (17). Evidently, there exist two different processes of partial derivation, namely the process $X^i_{|h}$ which is defined in equation (17) and the process;

$$X^i|_h \equiv \left(F \frac{\partial X^i}{\partial \dot{z}^h} + A^i_{kh} X^k \right) \tag{18}$$

and hence in view of these differentiation processes, the equation (17) can be written as

$$DX^i = X^i|_h Dl^h + X^i_{|h} dz^h. \tag{19}$$

Further, we consider the commutation formula given in equation (18) corresponding to repeated application on indices as

$$X^i|_{hk} - X^i|_{kh} = F \left(F_{\dot{z}^k} \frac{\partial X^i}{\partial \dot{z}^h} - F_{\dot{z}^h} \frac{\partial X^i}{\partial \dot{z}^k} \right) + + X^r \left\{ F \left(\frac{\partial A^i_{rh}}{\partial \dot{z}^k} - \frac{\partial A^i_{rk}}{\partial \dot{z}^h} \right) + A^i_{km} A^m_{rh} - A^i_{mh} A^m_{rk} \right\}. \tag{20}$$

Using the result $F \left(\frac{\partial A^i_{rh}}{\partial \dot{z}^k} - \frac{\partial A^i_{rk}}{\partial \dot{z}^h} \right) = F_{\dot{z}^k} A^i_{rh} - F_{\dot{z}^h} A^i_{rk}$ in equation (20) and re minding equation (18), we have

$$X^i|_{hk} - X^i|_{kh} = \left\{ F_{\dot{z}^k} X^i|_h - F_{\dot{z}^h} X^i|_k \right\} + S^i_{jkh} X^j, \tag{21}$$

where we have written [5]:

$$S^i_{jkh} = A^i_{rk} A^r_{jh} - A^i_{rh} A^r_{jk}. \tag{22}$$

This tensor is called the *Cartan's first curvature tensor* and we shall consider this one for our decomposition studies. Here are few identities, which the Cartan's first curvature tensor satisfies:

$$S^i_{jkh} = -S^i_{jhk}, \tag{23}$$

i.e., Cartan's first curvature tensor is anti-symmetric in its last index pair.

If we lower the index of this curvature tensor by writing $S_{ijkh} = g_{rj} S^r_{ikh}$, we have

$$S_{ijkh} = -S_{jikh}. \tag{24}$$

2. Techniques for the Decomposition of Cartan's I-Curvature Tensor field

General 2^{nd} order tensors in the three dimensional manifolds contain nine independent components, but sometimes it is desirable to reduce the dimensionality of such tensor fields in a meaningful way as this process may let us know the physical state represented by a tensor. Various techniques for tensor decompositions are available to reduce the dimensionality or to transform the tensor in such a way that describes important aspects about those for which they are standing for. In order to decompose the Cartan's I-curvature tensor in complex Finsler manifolds, we shall make use of the following four techniques:

1: Technique of symmetric-antisymmetric Decomposition [22, 18, 3, 19, 1, 16, 11, 10]

2: Technique of eigenvector-eigenvalues Decomposition [22, 18, 13, 14, 23, 21, 15, 12]

3: Technique of isotropic-deviator Decomposition [22, 18, 24, 4]

4: Technique of singular value decomposition (SVD)[22, 18, 19, 20, 2, 1, 16, 6]

To use the aforementioned decomposition techniques, for the sake of feasibility, we first factorize the Cartan's I-curvature tensor in terms of the outer/open product of two second order tensors, each having 3^2-components in a three dimensional complex Finsler manifold as follows [3]:

$$S^i_{jkh} = G^i_j f_k. \tag{25}$$

Here, for the exhibition of physical significance due to such factorization, we would consider the first mixed tensor lying at the right side of equation (25) as an Einstein tensor and the second one covariant tensor as to describe the degree of curvature of a surface given by the function $F(z^i, \dot{z}^i)$. Also, a powerful reason behind this kind of assumption arises from one of the feature of Finsler manifold that the fundamental function intimated with $F_n^{(C)}$ is homogeneous in its first order differential.

It is remarkable that for the right hand side mixed tensor of rank 2, the property of symmetry and skew-symmetry is not an intrinsic one, as it is evident from the well known transformation law of mixed tensor's symmetry/anti-symmetry that the property of symmetry/anti-symmetry of a mixed tensor between a pair of dissimilar indices (one covariant and other contravariant) is not invariant under the transformation.

2.1. **Technique 1: Decomposition of S^i_{jkh} using technique first.** Let us decompose the two tensors of equation (25) one by one as follows:

Theorem 2.1. *Under the technique (1), the tensor G^i_j of Eq.(25) (which is preassumed to be an Einstein one) produces the following decomposition form:*

$$\|G^i_j\| = \frac{1}{2}[\|G^p_q\| + \|G^q_p\|]_{p,q=\alpha,\beta,\gamma} + \frac{1}{2}[\|G^p_q\| - \|G^q_p\|]_{p,q=\alpha,\beta,\gamma} \tag{26}$$

Proof. One can straightforwardly demonstrate this theorem by keeping in mind that "any tensor can be expressed as a sum of its symmetric and antisymmetric parts" [22, 18, 3, 19, 1, 16, 10, 11]. Thus under this assumption, the Einstein tensor G^i_j can be expressed as follows:

$$\underbrace{[\frac{1}{2}(G^i_j + G^j_i)]}_{Symmetric\,part} + \underbrace{[\frac{1}{2}(G^i_j - G^j_i)]}_{Antisymmetric\,part}\ , \tag{27}$$

where $G^i_j(z,\dot{z}) := R^i_j - \frac{R}{2}\delta^i_j$ is an Einstein tensor in terms of Ricci tensor and curvature scalar.
Also, the purely covariant form of this Einstein tensor can be found as

$$G_{ik} = g_{in}G^n_k = g_{in}(R^n_k - \frac{R}{2}\delta^n_k) = R_{ik} - \frac{R}{2}g_{ik}. \tag{28}$$

Now, Eq. (27) in matrix form can be written as

$$\|G^i_j\| = \frac{1}{2}\left\{\begin{pmatrix} G^\alpha_\alpha & G^\alpha_\beta & G^\alpha_\gamma \\ G^\beta_\alpha & G^\beta_\beta & G^\beta_\gamma \\ G^\gamma_\alpha & G^\gamma_\beta & G^\gamma_\gamma \end{pmatrix} + \begin{pmatrix} G^\alpha_\alpha & G^\alpha_\beta & G^\alpha_\gamma \\ G^\beta_\alpha & G^\beta_\beta & G^\beta_\gamma \\ G^\gamma_\alpha & G^\gamma_\beta & G^\gamma_\gamma \end{pmatrix}\right\} + \\ + \frac{1}{2}\left\{\begin{pmatrix} G^\alpha_\alpha & G^\alpha_\beta & G^\alpha_\gamma \\ G^\beta_\alpha & G^\beta_\beta & G^\beta_\gamma \\ G^\gamma_\alpha & G^\gamma_\beta & G^\gamma_\gamma \end{pmatrix} - \begin{pmatrix} G^\alpha_\alpha & G^\alpha_\beta & G^\alpha_\gamma \\ G^\beta_\alpha & G^\beta_\beta & G^\beta_\gamma \\ G^\gamma_\alpha & G^\gamma_\beta & G^\gamma_\gamma \end{pmatrix}\right\}, \tag{29}$$

where $G^\alpha_\alpha, G^\alpha_\beta, G^\alpha_\gamma \ldots$ ect. are the components of G^i_j in 3-dimensional complex Finsler manifold.
For the sake of convenience, we write the Eq. (29) as in the following notations:

$$\|G^i_j\| = \frac{1}{2}[\|G^p_q\| + \|G^q_p\|]_{p,q=\alpha,\beta,\gamma} + \frac{1}{2}[\|G^p_q\| - \|G^q_p\|]_{p,q=\alpha,\beta,\gamma}$$

□

Theorem 2.2. *Covariant differentiation of Eq.(2.2) yields an analytic expression which represents a relation between variation in Einstein tensor G^i_j and functions of element of support $C^i_{kh}(z,\dot{z})$ & $\Gamma^i_{jh}(z,\dot{z})$.*

Proof. Making use of the concept of covariant differentiation given by Eq. (16), we now differentiate Eq.(2.2) with respect to z^l as below:

$$D\|G^i_j\| = D[\frac{1}{2}(\text{Symmetric part})] + D[\frac{1}{2}(\text{Antisymmetric part})] \text{ of Eq. (29).}$$

Differentiation of Eq. (2.2) yields a lengthy but straightforward relation as below:

$$
\begin{aligned}
(30)\quad D\|G^i_j\| = \frac{1}{2}[\{d\|G^p_q\| + \|C^p_{rs}\|(z,\dot z)\|G^r_q\|d\dot z^s + \Gamma^p_{rs}(z,\dot z)\|G^r_q\|dz^s -\\
- \|C^r_{qs}\|(z,\dot z)\|G^p_r\|d\dot z^s - \Gamma^r_{qs}(z,\dot z)\|G^p_r\|dz^s\} + \{d\|G^q_p\| + \|C^q_{rs}\|(z,\dot z)\|G^r_p\|d\dot z^s +\\
+ \Gamma^q_{rs}(z,\dot z)\|G^r_p\|dz^s - \|C^r_{ps}\|(z,\dot z)\|G^q_r\|d\dot z^s - \Gamma^r_{ps}(z,\dot z)\|G^q_r\|dz^s\}] + \frac{1}{2}[\{d\|G^p_q\| +\\
+ \|C^p_{rs}\|(z,\dot z)\|G^r_q\|d\dot z^s + \Gamma^p_{rs}(z,\dot z)\|G^r_q\|dz^s - \|C^r_{qs}\|(z,\dot z)\|G^p_r\|d\dot z^s - \Gamma^r_{qs}(z,\dot z)\times\\
\|G^p_r\|dz^s\} - \{d\|G^q_p\| + \|C^q_{rs}\|(z,\dot z)\|G^r_p\|d\dot z^s + \Gamma^q_{rs}(z,\dot z)\|G^r_p\|dz^s - \|C^r_{ps}\|(z,\dot z)\times\\
\|G^q_r\|d\dot z^s - \Gamma^r_{ps}(z,\dot z)\|G^q_r\|dz^s\}].
\end{aligned}
$$

Most probably, the Eq. (30) predict an analytic expression which connects the variation of Einstein tensor with various components of tensorial and non-tensorial quantities such as $C^p_{rs}(z,\dot z)$ and $\Gamma^p_{rs}(z,\dot z)$ etc., which are themselves the functions of element of support $(z^i, \dot z^i)$.

This analytic expression may be of great geometrical as well as physical significance in the study of various properties of Einstein's manifolds when treated with Finsler Geometry. □

In order to study some possible/probable connections between Einstein and Finsler manifolds, we consider a special case where the covariant differentiation given by Eq. (30) vanishes.

Theorem 2.3. *The vanishing of the analytic expression (30) i.e., $D\|G^i_j\| = 0$ implies the existence of an Einstein's field equation of the form:*

$$\|R^i_j\|(z,\dot z) = \gamma[\|T^i_j\| + \frac{1}{2}\delta^i_j\|T\|](z,\dot z) = c \text{ (Stationary value)},$$

where T^i_j is the well known energy-momentum tensor and T is its trace.

Proof. Let us use the usual Einstein's tensor $G^i_j = R^i_j - \frac{R}{2}\delta^i_j$, where R^i_j being the Ricci tensor as well as R being the curvature scalar of complex Finsler manifold. If we assume the vanishing of an analytic expression (30), we have

$$D\|G^i_j\| = D\|R^i_j - \frac{R}{2}\delta^i_j\| = 0. \tag{31}$$

Now to derive the Einstein's field equations, we introduce a field tensor which is of the same rank and type with symmetric properties as the Ricci tensor R^i_j. Hence, in our case, we introduce an energy-momentum tensor T^i_j which is such that

$$D\|T^i_j\|(z,\dot z) = 0. \tag{32}$$

The equality of Eq. (31) and Eq. (32) implies

$$D\|G^i_j\| = D\|R^i_j - \frac{R}{2}\delta^i_j\| \propto D\|T^i_j\|, \tag{33}$$

or

$$D\|G^i_j\| = D\|R^i_j - \frac{R}{2}\delta^i_j\| = \gamma\|T^i_j\| = 0, \tag{34}$$

where γ is a constant of proportionality and in the study of theory of relativity, will be called cosmological constant. Thus, if we integrate Eq. (34) over the complex

Finsler manifold $F_n^{(C)}$ with respect to some coordinate z^l, we would have a well known Einstein's field equation in th complex Finsler manifold as below:

$$\int_{F_n^{(C)}} [D\|G_j^i\|(z,\dot{z})]dz^l = \int_{F_n^{(C)}} [D\|R_j^i - \frac{R}{2}\delta_j^i\|(z,\dot{z})]dz^l = \tag{35}$$

$$= \int_{F_n^{(C)}} [\gamma D\|T_j^i\|(z,\dot{z})]dz^l = \text{some constant of integration.}$$

The above expression is due to the well known Euler's condition $\int_{t_0}^{t_1} F(z^i,\dot{z}^i)dt =$ stationary, where $\dot{z}^i = \frac{dz^i}{dt}$. The expression (35) on simplification yields

$$\|G_j^i\|(z,\dot{z}) = \|R_j^i - \frac{R}{2}\delta_j^i\|(z,\dot{z}) = \gamma\|T_j^i\|(z,\dot{z}) = \tag{36}$$

$$= c \text{ (stationary value).}$$

Evidently, Eq. (36) stands for the Einstein's field equation in local component form. Moreover, if we contract Eq. (36) with respect to indices i and j, we have

$$\|G_i^i\|(z,\dot{z}) = \|R - \frac{R}{2}\|(z,\dot{z}) = \gamma\|T_i^i\|(z,\dot{z}) = c \text{ (stationary value).}$$

Thus with the help of expression mentioned just above, our field equation (36) implies

$$\|R_j^i\|(z,\dot{z}) = \gamma[\|T_j^i\| + \frac{1}{2}\delta_j^i\|T\|](z,\dot{z}) = c \text{ (stationary value),} \tag{37}$$

which is again the field equation used for analytical purposes. □

Definition 2.4. *There is a special case when the complex Finsler manifold is empty. Then in such case, the energy-momentum tensor must vanish, i.e.,* $T_j^i = 0$*. Hence from Eq. (37)* $\|R_j^i\|(z,\dot{z}) = 0$*. This condition gives rise to a special Finsler manifold which is Ricci flat and thereby called "Ricci flat complex Finsler manifold".*

In order to discuss the Eq. (25) completely, we now proceed to decompose the tensor f_{kh} using technique (1). Further, from the standpoint of physical significance, we assume that this second rank covariant tensor describes the degree of curvature of the Finsler surface given by the fundamental function $F(z^i,\dot{z}^i)$.

Theorem 2.5. *Under the technique (1), the splitting of tensor* f_{kh} *of Eq.(25) (which is preassumed to be a degree of curvature of Finsler surface) produces the following decomposition form:*

$$\|f_{kh}\| = \frac{1}{2}[\|f_{pq}\| + \|f_{qp}\|]_{p,q=\alpha,\beta,\gamma} + \frac{1}{2}[\|f_{pq}\| - \|f_{qp}\|]_{p,q=\alpha,\beta,\gamma} \tag{38}$$

Proof. We can straightforwardly proof this theorem by keeping in mind that "any tensor can be expressed as a sum of its symmetric and antisymmetric parts" [22, 18, 3, 19, 1, 16, 10, 11]. Thus under this assumption, the tensor f_{kh} can be expressed as follows:

$$\underbrace{[\frac{1}{2}(f_{kh} + f_{hk})]}_{Symmetric\ part} + \underbrace{[\frac{1}{2}(f_{kh} - f_{hk})]}_{Antisymmetric\ part}\ , \tag{39}$$

which in matrix form can be written as

$$
\begin{aligned}
(40)\quad \|f_{kh}\| = \frac{1}{2}\left\{\begin{pmatrix} f_{\alpha\alpha} & f_{\alpha\beta} & f_{\alpha\gamma} \\ f_{\beta\alpha} & f_{\beta\beta} & f_{\beta\gamma} \\ f_{\gamma_\alpha} & f_{\gamma\beta} & f_{\gamma\gamma} \end{pmatrix} + \begin{pmatrix} f_{\alpha\alpha} & f_{\alpha\beta} & f_{\alpha\gamma} \\ f_{\beta\alpha} & f_{\beta\beta} & f_{\beta\gamma} \\ f_{\gamma_\alpha} & f_{\gamma\beta} & f_{\gamma\gamma} \end{pmatrix}\right\} + \\
+ \frac{1}{2}\left\{\begin{pmatrix} f_{\alpha\alpha} & f_{\alpha\beta} & f_{\alpha\gamma} \\ f_{\beta\alpha} & f_{\beta\beta} & f_{\beta\gamma} \\ f_{\gamma_\alpha} & f_{\gamma\beta} & f_{\gamma\gamma} \end{pmatrix} - \begin{pmatrix} f_{\alpha\alpha} & f_{\alpha\beta} & f_{\alpha\gamma} \\ f_{\beta\alpha} & f_{\beta\beta} & f_{\beta\gamma} \\ f_{\gamma_\alpha} & f_{\gamma\beta} & f_{\gamma\gamma} \end{pmatrix}\right\},
\end{aligned}
$$

where $f_{\alpha\alpha}, f_{\alpha\beta}\ldots$ etc. are the components f_{kh} in 3-dimensional complex Finsler manifold.
Now, for the feasibility, we can write the Eq. (40) as in the following notations:

$$\|f_{kh}\| = \frac{1}{2}[\|f_{pq}\| + \|f_{qp}\|]_{p,q=\alpha,\beta,\gamma} + \frac{1}{2}[\|f_{pq}\| - \|f_{qp}\|]_{p,q=\alpha,\beta,\gamma}$$

□

Theorem 2.6. *Covariant differentiation of Eq.(2.14) yields an analytic expression which represents a relation between variation in degree of curvature f_{kh} of a Finsler surface $F(z^i, \dot{z}^i)$ and functions of element of support $C^i_{kh}(z,\dot{z})$ & $\Gamma^i_{jh}(z,\dot{z})$.*

Proof. We again make use of the concept of covariant differentiation given by (16) for the Eq. (38) as follows:
Differentiating Eq. (38) covariantly with respect to z^l we obtain

$$
\begin{aligned}
(41)\quad D\|f_{kh}\| = \frac{1}{2}[\{d\|f_{pq}\| - \|f_{rq}\|(\|C^r_{ps}\|(z,\dot{z})d\dot{z}^s + \Gamma^p_{rs}(z,\dot{z})dz^s) - \\
- \|f_{pr}\|(\|C^r_{qs}\|(z,\dot{z})d\dot{z}^s + \Gamma^r_{qs}(z,\dot{z})dz^s)\} + \{d\|f_{qp}\| - \|f_{rp}\|(\|C^r_{qs}\|(z,\dot{z})d\dot{z}^s + \\
+ \Gamma^r_{qs}(z,\dot{z})dz^s) - \|f_{qr}\|(\|C^r_{ps}\|(z,\dot{z})d\dot{z}^s + \Gamma^r_{ps}(z,\dot{z})dz^s)\}] + \frac{1}{2}[\{d\|f_{pq}\| - \|f_{rq}\times \\
\|(\|C^r_{ps}\|(z,\dot{z})d\dot{z}^s + \Gamma^p_{rs}(z,\dot{z})dz^s) - \\
- \|f_{pr}\|(\|C^r_{qs}\|(z,\dot{z})d\dot{z}^s + \Gamma^r_{qs}(z,\dot{z})dz^s)\} - \{d\|f_{qp}\| - \|f_{rp}\|(\|C^r_{qs}\|(z,\dot{z})d\dot{z}^s + \\
+ \Gamma^r_{qs}(z,\dot{z})dz^s) - \|f_{qr}\|(\|C^r_{ps}\|(z,\dot{z})d\dot{z}^s + \Gamma^r_{ps}(z,\dot{z})dz^s)\}]
\end{aligned}
$$

□

Theorem 2.7. *With the following trivial assumption*

- **A1:** *Introducing a fundamental tensor g_{ij} in Eq. (38) which is of same rank and type as f_{kh}, such that $Dg_{ij} = 0$.*
- **A2:** *To illustrate connection between the metric $ds^2 = F(z,\dot{z})$ and the Cartan's curvature tensor of the Finsler continuum, we assume $z_k = F(z,\dot{z})$ to be a two dimensional smooth curved surface in 3-dimensional complex Finsler manifold. Where the assumption of smoothness is in the sense that at each point, surface possesses a tangent plane.*

we can explore that the second rank covariant tensor f_{kh} describes theory of various curvatures, viz. principal curvatures of a Monge's surface and many more.

Proof. In view of the assumptions A1 and A2, the equation of the surface in complex Finsler manifold in Monge's form can be written as $z_k = F(z,\dot{z})$, where z_k being some complex coordinate. The function $F(z,\dot{z})$ is supposed to be differentiable as many times as desirable. Since the fundamental function $F(z,\dot{z})$ is a positive homogeneous function of degree one in $\dot{z}^i$, thereby using the well known Euler's theorem on homogeneous function, we have

$$F_{\dot{z}^i}(z,\dot{z}) = F(z,\dot{z}), \tag{42}$$

$$F_{\dot{z}^i\dot{z}^j}(z,\dot{z})\dot{z}^i = 0, \tag{43}$$

where the notations $F_{\dot{z}^i}$ and $F_{\dot{z}^i\dot{z}^j}$ are used to denote the derivatives of $F(z,\dot{z})$ with respect to $\dot{z}^i$ and $\dot{z}^i\dot{z}^j$ respectively.

Clearly, $[F(z,\dot{z})]_{(z,\dot{z}=0)}$ & $[F_{\dot{z}^i}(z,\dot{z})]_{(z,\dot{z}=0)} = 0$. Also, $[F_{\dot{z}^i\dot{z}^j}(z,\dot{z})]_{(z,\dot{z}=0)} = 0$. Thus by definition of fundamental function $F(z,\dot{z})$, we have

$$F(z^i,\dot{z}^i) = [g_{ij}(z^k)dz^i dz^j]^{1/2}, \tag{44}$$

where $g_{ij}(z^k)$ are the coefficients independent of z^i. This metric defined by Eq. (44) is the metric in Riemannian manifold. Moreover, since

$$F(z,\dot{z}) := \frac{1}{2}F^2(z,\dot{z}) \text{ and } g_{ij}(z,\dot{z}) = \frac{\partial^2 F^2(z,\dot{z})}{\partial\dot{z}^i\partial\dot{z}^j},$$

then from Eq. (7), the surface equation can be written as

$$F(z,\dot{z}) = \frac{1}{2}F^2(z,\dot{z}) = \frac{1}{2}g_{ij}(z,\dot{z})d\dot{z}^i d\dot{z}^j = z_k. \tag{45}$$

Now, by our assumption A1, we introduce f_{ij} in place of g_{ij} in Eq. (45), which yields

$$F(z,\dot{z}) = \frac{1}{2}f_{ij}(z,\dot{z})d\dot{z}^i d\dot{z}^j = z_k, \tag{46}$$

where this $f_{ij}(z,\dot{z})$ is equal to $\frac{\partial^2 F^2(z,\dot{z})}{\partial\dot{z}^i\partial\dot{z}^j}$ and hence able to determine the degree of surface at any point P of the manifold.

Further, as the normal section of the Monge's surface $z_k = F(z,\dot{z})$ of given complex Finsler manifold must have greatest and the least curvatures which will be called the principal curvatures. Thus to determine the principal curvatures say κ_a and κ_b, we should determine the eigenvalues (latent roots) of $\|f_{ij}\|(z,\dot{z})$. The latent root equation of the matrix $\|f_{ij}\|(z,\dot{z})$ is written as

$$|f_{ij} - \lambda\delta_{ij}| = 0, \tag{47}$$

where λ is called indeterminate.
Eq. (47) on expansion in usual way yields the following:

$$\begin{aligned}\lambda^3 &+ \lambda^2(f_{\alpha\alpha}f_{\beta\beta}f_{\gamma\gamma}) + \lambda(f_{\alpha\gamma}f_{\gamma\alpha} + f_{\alpha\beta}f_{\beta\gamma} + f_{\gamma\beta}f_{\beta\gamma} - f_{\beta\beta}f_{\gamma\gamma} - \\ &- f_{\alpha\alpha}f_{\gamma\gamma} - f_{\alpha\alpha}f_{\beta\beta}) + (f_{\alpha\alpha}f_{\beta\beta}f_{\gamma\gamma} - f_{\alpha\alpha}f_{\beta\gamma}f_{\gamma\beta} - f_{\alpha\beta}f_{\beta\gamma}f_{\gamma\gamma} + f_{\alpha\beta}f_{\gamma\alpha}f_{\beta\gamma} + \\ &+ f_{\alpha\gamma}f_{\beta\gamma}f_{\gamma\beta} - f_{\alpha\gamma}f_{\gamma\alpha}f_{\beta\beta}) = 0.\end{aligned} \tag{48}$$

But, as we have given the logic that for our Monge's surface, there may be only two principal curvatures κ_a and κ_b. Then if we vanish one of the index say γ, Eq. (48) will be reduced to the following form:

$$\lambda^2 - \lambda(f_{\alpha\alpha} + f_{\beta\beta}) + (f_{\alpha\alpha}f_{\beta\beta} - f_{\alpha\beta}f_{\beta\alpha}) = 0. \tag{49}$$

This expression produces the following facts:

$$\kappa_a + \kappa_b = (f_{\alpha\alpha} + f_{\beta\beta}) \text{ and } \kappa_a\kappa_b = (f_{\alpha\alpha}f_{\beta\beta} - f_{\alpha\beta}f_{\beta\alpha}) = |f_{ij}|. \tag{50}$$

If the coordinates $d\dot{z}^i$ and $d\dot{z}^j$ involved in the Eq. (46) are measured in the direction of the principal axis, we would have

$$\|f_{ij}\| = \begin{pmatrix} \kappa_a & 0 \\ 0 & \kappa_b \end{pmatrix},$$

and thus

$$z_k = F(z,\dot{z}) = \frac{1}{2}\kappa_a (d\ddot{z}^i)^2 + \frac{1}{2}\kappa_b (d\dot{z}^j)^2 + \cdots \tag{51}$$

Also, in a general direction, the Gaussian curvature would be weighted mean of the minimal and maximal values of κ_a and κ_b. Hence

$$[\mathbb{K}]_P = \kappa_a\kappa_b = |f_{ij}|(z,\dot{z}), \tag{52}$$

while the mean curvature

$$\mu = \frac{1}{2}(\kappa_a + \kappa_b). \tag{53}$$

Thus, we have shown that how a second rank covariant tensor describes the theory of various curvatures. □

We, now, proceed to discuss a special case which would provide a methodology to let us know that whether the underlying complex Finsler manifold is an Einstein one or not.

Theorem 2.8. *The constancy of Gaussian curvature tensor yields the necessary and sufficient condition for a complex Finsler manifold to be an Einstein one.*

Proof. As it is known to us that a manifold with constant Riemannian curvature tensor is an Einstein one. In the similar way, we now check for the constancy of Gaussian curvature Eq. (52) for Finsler manifold. If the Gaussian curvature of the surface $z_k = F(z,\dot{z})$ becomes constant at any point P of the Finsler manifold, we shall say that the Finsler manifold is an Einstein one. For this purpose, we use scalar representation of Gaussian curvature given by Eq. (52) as follows:
The Gaussian curvature of the Finsler manifold defined at any point with respect to a two directions $(z,\dot{z})$ is given by

$$R(z,\dot{z},Z) = \frac{[K_{ijkh}(z,\dot{z})\dot{z}^i\dot{z}^h Z^j Z^k]}{[g_{ih}(z,\dot{z})g_{jk}(z,\dot{z}) - g_{ij}(z,\dot{z})g_{kh}(z,\dot{z})]\dot{z}^i\dot{z}^h Z^j Z^k}. \tag{54}$$

But Eq. (52) also stands for the Gaussian curvature deduced from the Cartan's first curvature tensor, thereby from Eq. (52) and Eq. (54), we have

$$[\mathbb{K}]_P = [R(z,\dot{z},Z)]_P = |f_{ij}|(z,\dot{z}) = \frac{[K_{ijhk}(z,\dot{z})]}{[g_{ih}(z,\dot{z})g_{jk}(z,\dot{z}) - g_{ij}(z,\dot{z})g_{kh}(z,\dot{z})]} \tag{55a}$$

$$[\mathbb{K}]_p[g_{ih}(z,\dot{z})g_{jk}(z,\dot{z}) - g_{ij}(z,\dot{z})g_{kh}(z,\dot{z})] = k_{ijhk}(z,\dot{z}) \equiv |f_{ij}|(z,\dot{z}). \tag{55b}$$

Transvecting Eq. (55) with g^{hk} and summing over i and k from 1 to N, we obtain

$$[\mathbb{K}]_p[\delta^k_i g_{jk}(z,\dot{z}) - Ng_{ij}(z,\dot{z})] = g^{hk}(z,\dot{z})K_{ijhk}(z,\dot{z}) \equiv |f_{ij}|(z,\dot{z}),$$

or

$$[\mathbb{K}]_P g_{ij}(z,\dot{z})[1-N] = K_{ij}(z,\dot{z}) \equiv |f_{ij}|(z,\dot{z}). \tag{56}$$

Again transvecting with g^{ij} and summing over i and j from 1 to N, we get

$$[\mathbb{K}]_P g_{ij}(z,\dot{z})g^{ij}(z,\dot{z})[1-N] = K_{ij}(z,\dot{z})g^{ij}(z,\dot{z}) \equiv |f_{ij}|(z,\dot{z}),$$

which implies

$$N(1-N)[\mathbb{K}]_P = K(z,\dot{z}) \equiv |f_{ij}|(z,\dot{z}).$$

Substituting $[\mathbb{K}]_P = \frac{K(z,\dot{z})}{N(1-N)} \equiv |f_{ij}|(z,\dot{z})$ in Eq. (56), we obtain

$$|f_{ij}|(z,\dot{z}) \equiv K_{ij}(z,\dot{z}) = \frac{K(z,\dot{z})}{N} g_{ij}(z,\dot{z}), \tag{57}$$

which is the necessary and sufficient condition for a complex Finsler manifold to be an Einstein one. □

2.2. **Technique 2: Decomposition of Cartan's I-curvature Tensor Field by means of Eigenvalue-Eigenvector Method.** As from the standpoint of various mathematical and engineering applications, eigenvalue problems are among the most crucial problems in connection with matrices and tensors. Also, the study of such latentroot problems in quantum mechanics is highly insisted due to having *spectrum* (a set of eigenvalues) and the *spectral radii* (the largest of the absolute values of latentroots) of any tensor field. Further it is known that a set of special vectors and scalar values, customarily called eigenvectors and eigenvalues are associated with second rank tensors. Various analysis and visualization techniques use such sets of latentroots and latentvectors and are particularly crucial in the visualization of topological structures of and tensor field.

The eigenvectors of a tensor have the property that when the inner product of the original tensor and an eigenvector is taken, the consequence will be a vector which is a scalar multiple of the original eigenvector. That is if T is any tensor and X is its eigenvector then $TX = \lambda X$, where λ are the solutions of this equation and are the eigenvalues of T.

Here we again consider the open product given by Eq. (25) and use the technique (2)[22, 18, 13, 14, 23, 21, 15, 12] separately for each of the tensor residing to the right hand side in Eq. (25).

In case of 3-dimensional complex Finsler manifold, we have the Einstein's tensor $G^i_j(z,\dot{z})$ in matrix form as follows:

$$\|G^i_j\|(z,\dot{z}) = \begin{pmatrix} G^\alpha_\alpha & G^\alpha_\beta & G^\alpha_\gamma \\ G^\beta_\alpha & G^\beta_\beta & G^\beta_\gamma \\ G^\gamma_\alpha & G^\gamma_\beta & G^\gamma_\gamma \end{pmatrix}_{i,j=\alpha,\beta,\gamma} \tag{58}$$

Here, because of the manifold under consideration being complex Finsler, it is evident that each of the nine components of the above matrix will be the functions of so called element of support $(z,\dot{z})$ and hence will be the complex entries.

Now, by definition of latentroot-latentvector approach, we consider an eigenvector X_i having the components $X_\alpha, X_\beta, X_\gamma$ such that the characteristic equation for the given Einstein's tensor becomes

$$G^i_j X_i = \lambda X_j \text{ or,} \tag{59a}$$

$$(G^i_j - \lambda\delta^i_j) \tag{59b}$$

If we omit the null vector $X_i = 0$, the Eq. (59) implies

$$|G^i_j - \lambda\delta^i_j| = 0, \tag{60}$$

which for a 3-dimensional complex Finsler-Einstein's tensor expands to

$$D(\lambda) = \begin{pmatrix} G^{\alpha}_{\alpha} - \lambda & G^{\alpha}_{\beta} & G^{\alpha}_{\gamma} \\ G^{\beta}_{\alpha} & G^{\beta}_{\beta} - \lambda & G^{\beta}_{\gamma} \\ G^{\gamma}_{\alpha} & G^{\gamma}_{\beta} & G^{\gamma}_{\gamma} - \lambda \end{pmatrix} = 0, \tag{61}$$

where $D(\lambda)$ stands for the characteristic determinant.
Now, using Crammer's rule and simplifying the Eq. (61), we obtain

$$D(\lambda) = \lambda^3 - I^1\lambda^2 + I^2\lambda - I^3 = 0, \tag{62}$$

where I^1, I^2 and I^3 are the invariants defined by relations

$$I^1 = G^i_i = (G^{\alpha}_{\alpha} + G^{\beta}_{\beta} + G^{\gamma}_{\gamma}) \tag{63a}$$

$$I^2 = \frac{1}{2}(G^i_i G^j_j - G^i_j G^j_i) = \left(\begin{vmatrix} G^{\alpha}_{\alpha} & G^{\alpha}_{\beta} \\ G^{\beta}_{\alpha} & G^{\beta}_{\beta} \end{vmatrix} + \begin{vmatrix} G^{\alpha}_{\alpha} & G^{\alpha}_{\gamma} \\ G^{\gamma}_{\alpha} & G^{\gamma}_{\gamma} \end{vmatrix} + \begin{vmatrix} G^{\beta}_{\beta} & G^{\beta}_{\gamma} \\ G^{\gamma}_{\beta} & G^{\gamma}_{\gamma} \end{vmatrix} \right), \tag{63b}$$

$$I^3 = e_{ijk} G^i_i G^j_j G^k_k = \begin{vmatrix} G^{\alpha}_{\alpha} & G^{\alpha}_{\beta} & G^{\alpha}_{\gamma} \\ G^{\beta}_{\alpha} & G^{\beta}_{\beta} & G^{\beta}_{\gamma} \\ G^{\gamma}_{\alpha} & G^{\gamma}_{\beta} & G^{\gamma}_{\gamma} \end{vmatrix} \tag{63c}$$

In view of the above decomposition formulae, we now proceed to discuss an important theorem as below:

Theorem 2.9. *The characteristic equation (62) of Einstein's tensor G^i_j will have*

- **i:** *mixed type (i.e. purely real as well as complex) latentroots if the Einstein's tensor is self-conjugate*
- **ii:** *purely real latentroots if the Einstein's tensor is pure*
- **iii:** *purely complex latentroots if the Einstein's tensor is hybrid*

Proof. If we employ the basic feature of self-conjugacy of Einstein's tensor $G^i_j(z, \dot{z})$, i.e. the this tensor is self-conjugate if [9]:

$$G^i_j(z, \dot{z}) = \begin{pmatrix} G^{\mu}_{\nu} & G^{\bar{\mu}}_{\nu} \\ G^{\mu}_{\bar{\nu}} & G^{\bar{\mu}}_{\bar{\nu}} \end{pmatrix}, \tag{64}$$

which implies

$$\bar{G}^i_j(z, \dot{z}) = G^{\bar{i}}_{\bar{j}}(z, \dot{z}). \tag{65}$$

Then each components of the matrix given by Eq. (58) will satisfy the relation (65).

Applying the above methodology to our characteristic equation (62), we have

$$\begin{aligned} D(\lambda) = \lambda^3 - G^i_i(z, \dot{z})\lambda^2 + \frac{\lambda}{2}[G^i_i(z, \dot{z})G^j_j(z, \dot{z}) - G^i_j(z, \dot{z})G^j_i(z, \dot{z})] \\ - e_{ijk} G^i_i(z, \dot{z}) G^j_j(z, \dot{z}) G^k_k(z, \dot{z}) = 0, \end{aligned} \tag{66}$$

where we have substituted the values of invariants I^1, I^2 and I^3 from Eq. (63a), Eq. (63b) and Eq. (63c) respectively.

Applying Eq. (64) to Eq. (66), we obtain

$$D(\lambda)=\lambda^3-\lambda^2\begin{pmatrix} G^{\mu}_{\mu} & G^{\bar\mu}_{\mu}\\ G^{\mu}_{\bar\mu} & G^{\bar\mu}_{\bar\mu}\end{pmatrix}+\frac{\lambda}{2}[\begin{pmatrix} G^{\mu}_{\mu} & G^{\bar\mu}_{\mu}\\ G^{\mu}_{\bar\mu} & G^{\bar\mu}_{\bar\mu}\end{pmatrix}\begin{pmatrix} G^{\nu}_{\nu} & G^{\bar\nu}_{\nu}\\ G^{\nu}_{\bar\nu} & G^{\bar\nu}_{\bar\nu}\end{pmatrix}$$
$$-\begin{pmatrix} G^{\mu}_{\nu} & G^{\bar\mu}_{\nu}\\ G^{\mu}_{\bar\nu} & G^{\bar\mu}_{\bar\nu}\end{pmatrix}\begin{pmatrix} G^{\nu}_{\mu} & G^{\bar\nu}_{\mu}\\ G^{\nu}_{\bar\mu} & G^{\bar\nu}_{\bar\mu}\end{pmatrix}]-e_{ijk}(z,\dot z)[\begin{pmatrix} G^{\mu}_{\mu} & G^{\bar\mu}_{\mu}\\ G^{\mu}_{\bar\mu} & G^{\bar\mu}_{\bar\mu}\end{pmatrix}\begin{pmatrix} G^{\nu}_{\nu} & G^{\bar\nu}_{\nu}\\ G^{\nu}_{\bar\nu} & G^{\bar\nu}_{\bar\nu}\end{pmatrix}\times$$
$$\begin{pmatrix} G^{\xi}_{\xi} & G^{\bar\xi}_{\xi}\\ G^{\xi}_{\bar\xi} & G^{\bar\xi}_{\bar\xi}\end{pmatrix}]=0. \tag{67}$$

Simplifying Eq. (67) by keeping Eq. (65) in mind, we have

$$D(\lambda)=\lambda^3-\lambda^2(G^{\mu}_{\mu}\bar G^{\mu}_{\mu}-G^{\bar\mu}_{\mu}G^{\mu}_{\bar\mu})+\frac{\lambda}{2}[(G^{\mu}_{\mu}\bar G^{\mu}_{\mu}-G^{\bar\mu}_{\mu}G^{\mu}_{\bar\mu})$$
$$\times(G^{\nu}_{\nu}\bar G^{\nu}_{\nu}-G^{\bar\nu}_{\nu}G^{\mu}_{\bar\nu})-(G^{\mu}_{\nu}\bar G^{\mu}_{\nu}-G^{\bar\mu}_{\nu}G^{\mu}_{\bar\nu})(G^{\nu}_{\mu}\bar G^{\nu}_{\mu}-G^{\bar\nu}_{\mu}G^{\nu}_{\bar\mu})]-e_{ijk}(z,\dot z)\times$$
$$[(G^{\mu}_{\mu}\bar G^{\mu}_{\mu}-G^{\bar\mu}_{\mu}G^{\mu}_{\bar\mu})(G^{\nu}_{\nu}\bar G^{\nu}_{\nu}-G^{\bar\nu}_{\nu}G^{\nu}_{\bar\nu})(G^{\xi}_{\xi}\bar G^{\xi}_{\xi}-G^{\bar\xi}_{\xi}G^{\xi}_{\bar\xi})]=0. \tag{68}$$

From Eq. (68), one can immediately conclude that the eigenvalues and hence the corresponding eigenvectors of the Einstein's tensor $G^i_j(z,\dot z)$ will be of mixed type, i.e. some of them will be purely real and some will be complex.

We, now, consider the case (ii) when Einstein's tensor is *pure*, i.e. the tensor $G^i_j(z,\dot z)$ possesses the components of the form:

$$G^i_j(z,\dot z)=\begin{pmatrix} G^{\mu}_{\nu} & 0\\ 0 & G^{\bar\mu}_{\bar\nu}\end{pmatrix} \tag{69}$$

Then under this condition, the characteristic Eq. (66) for Einstein's tensor yields

$$D(\lambda)=\lambda^3-\lambda^2\begin{pmatrix} G^{\mu}_{\mu} & 0\\ 0 & G^{\bar\mu}_{\bar\mu}\end{pmatrix}+\frac{\lambda}{2}[\begin{pmatrix} G^{\mu}_{\mu} & 0\\ 0 & G^{\bar\mu}_{\bar\mu}\end{pmatrix}\begin{pmatrix} G^{\nu}_{\nu} & 0\\ 0 & G^{\bar\nu}_{\bar\nu}\end{pmatrix}$$
$$-\begin{pmatrix} G^{\mu}_{\nu} & 0\\ 0 & G^{\bar\mu}_{\bar\nu}\end{pmatrix}\begin{pmatrix} G^{\nu}_{\mu} & 0\\ 0 & G^{\bar\nu}_{\bar\mu}\end{pmatrix}]-e_{ijk}(z,\dot z)[\begin{pmatrix} G^{\mu}_{\mu} & 0\\ 0 & G^{\bar\mu}_{\bar\mu}\end{pmatrix}\begin{pmatrix} G^{\nu}_{\nu} & 0\\ 0 & G^{\bar\nu}_{\bar\nu}\end{pmatrix}\times$$
$$\begin{pmatrix} G^{\xi}_{\xi} & 0\\ 0 & G^{\bar\xi}_{\bar\xi}\end{pmatrix}]=0. \tag{70}$$

Again in view of the self-conjugacy condition (65), the Eq. (70) implies

$$D(\lambda)=\lambda^3-\lambda^2(G^{\mu}_{\mu}\bar G^{\mu}_{\mu})+\frac{\lambda}{2}[(G^{\mu}_{\mu}\bar G^{\mu}_{\mu})(G^{\nu}_{\nu}\bar G^{\nu}_{\nu})-(G^{\mu}_{\nu}\bar G^{\mu}_{\nu})\times$$
$$(G^{\nu}_{\mu}\bar G^{\nu}_{\mu})]-e_{ijk}(z,\dot z)[(G^{\mu}_{\mu}\bar G^{\mu}_{\mu})(G^{\nu}_{\nu}\bar G^{\nu}_{\nu})(G^{\xi}_{\xi}\bar G^{\xi}_{\xi})]=0, \tag{71}$$

which evidently shows that under purity condition of $G^i_j(z,\dot z)$, it's latentroots and hence the corresponding latentvectors will be purely real.

Taking account of the case (iii), when components of Einstein's tensor become hybrid, i.e. when

$$G^i_j(z,\dot z)=\begin{pmatrix} 0 & G^{\bar\mu}_{\nu}\\ G^{\mu}_{\bar\nu} & 0\end{pmatrix}. \tag{72}$$

Under the *hybrid* nature Eq. (72), the characteristic determinant Eq. (66) will take the form:

$$(73)\quad D(\lambda)=\lambda^3-\lambda^2\begin{pmatrix}0 & G^{\bar\mu}_{\mu}\\ G^{\mu}_{\bar\mu} & 0\end{pmatrix}+\frac{\lambda}{2}[\begin{pmatrix}0 & G^{\bar\mu}_{\mu}\\ G^{\mu}_{\bar\mu} & 0\end{pmatrix}\begin{pmatrix}0 & G^{\bar\nu}_{\nu}\\ G^{\nu}_{\bar\nu} & 0\end{pmatrix}$$
$$-\begin{pmatrix}0 & G^{\bar\mu}_{\nu}\\ G^{\mu}_{\bar\nu} & 0\end{pmatrix}\begin{pmatrix}0 & G^{\bar\nu}_{\mu}\\ G^{\nu}_{\bar\mu} & 0\end{pmatrix}]-e_{ijk}(z,\dot z)[\begin{pmatrix}0 & G^{\bar\mu}_{\mu}\\ G^{\mu}_{\bar\mu} & 0\end{pmatrix}\begin{pmatrix}0 & G^{\bar\nu}_{\nu}\\ G^{\nu}_{\bar\nu} & 0\end{pmatrix}\times$$
$$\begin{pmatrix}0 & G^{\bar\xi}_{\xi}\\ G^{\xi}_{\bar\xi} & 0\end{pmatrix}]=0.$$

In view of self-conjugacy, the Eq. (73) yields

$$(74)\quad D(\lambda)=\lambda^3-\lambda^2(G^{\bar\mu}_{\mu}G^{\mu}_{\bar\mu})+\frac{\lambda}{2}[(G^{\bar\mu}_{\mu}G^{\mu}_{\bar\mu})(G^{\bar\nu}_{\nu}G^{\mu}_{\bar\nu})-(G^{\bar\mu}_{\nu}G^{\mu}_{\bar\nu})\times$$
$$(G^{\bar\nu}_{\mu}G^{\nu}_{\bar\mu})]-e_{ijk}(z,\dot z)[(G^{\bar\mu}_{\mu}G^{\mu}_{\bar\mu})(G^{\bar\nu}_{\nu}G^{\nu}_{\bar\nu})(G^{\bar\xi}_{\xi}G^{\xi}_{\bar\xi})]=0.$$

From Eq. (74), its obvious that the latentroots as well as the corresponding latentvectors of Einstein's tensor will be purely complex. □

Now, by observing the proof of Theorem (2.9), we can easily estimate the ticklishness of the process of checking pure and hybrid nature of Einstein's tensor. The process of checking seems to be quite lengthy, because under this process one would need to check the pure and hybrid nature for each of the components of matrix given by Eq. (58). Therfore, for such a purpose, we shall utilize the two well known operators O^{sh}_{ir} and $*O^{sh}_{ir}$ defined by [9] (page 133). The O and $*O$ operators are defined as follows:

$$(75a)\qquad O^{sh}_{ir}=\frac{1}{2}(\delta^s_i\delta^h_r-F^s_iF^h_r),$$

$$(75b)\qquad *O^{sh}_{ir}=\frac{1}{2}(\delta^s_i\delta^h_r+F^s_iF^h_r).$$

Moreover the operators O and $*O$ satisfy the following relations:

(76a) $O+*O=A,$ where A being the identity operator,

(76b) $$O.O=O,$$

(76c) $$O.*O=O,$$

(76d) $$*O.O=O,$$

(76e) $$*O.*O=*O.$$

To omit the ambiguity regarding pure or hybrid nature of Einstein's tensor, we now use the above cited properties of operators O and $*O$.

Definition 2.10. *The Einstein,s tensor $G^i_j(z,\dot z)$ is declared to be pure or hybrid according to the following facts:*

1: *$G^i_j(z,\dot z)$ is pure if $OG^i_j(z,\dot z)=G^i_j(z,\dot z)$ or $*OG^i_j(z,\dot z)=0$,*
2: *$G^i_j(z,\dot z)$ is hybrid if $OG^i_j(z,\dot z)=0$ or $*OG^i_j(z,\dot z)=G^i_j(z,\dot z)$*

.

In view of the Definition (2.10), we now claim the following:

Proposition 2.11. *Einstein's tensor $G^i_j(z,\dot z)$ is hybrid in nature.*

Proof. The proof of the proposition follows directly from the Definition (2.10) and the relations (75a) and (75b).
Applying Eq. (75a) and Eq. (75b) to $G^i_j(z,\dot{z})$ one by one, we have

$$(77)\quad O^{si}_{jr}G^r_s(z,\dot{z}) = \frac{1}{2}(\delta^s_j\delta^i_r - F^s_jF^i_r)G^r_s(z,\dot{z}) = \frac{1}{2}(\delta^s_j\delta^i_rG^r_s - F^s_jF^i_rG^r_s)(z,\dot{z}) = \frac{1}{2}(G^i_j - G^i_j)(z,\dot{z}) = 0$$

and

$$(78)\quad *\,O^{si}_{jr}G^r_s(z,\dot{z}) = \frac{1}{2}(\delta^s_j\delta^i_r + F^s_jF^i_r)G^r_s(z,\dot{z}) = \frac{1}{2}(\delta^s_j\delta^i_rG^r_s + F^s_jF^i_rG^r_s)(z,\dot{z}) = \frac{1}{2}(G^i_j + G^i_j)(z,\dot{z}) = G^i_j(z,\dot{z}).$$

Now, comparing Eq. (77) and Eq. (78) with Definition (2.10), we conclude that Einstein's tensor is hybrid.

Moreover, we can predict the hybrid nature of Einstein's tensor by considering the work of [9], who has been verified that the fundamental metric tensor g_{ij} and the Ricci tensor R_{ij} both are hybrid in i and j. As the Einstein's tensor is the composition of Ricci tensor and metric tensor, so evidently it is hybrid and hence will have purely complex latentroots as well as latentvectors. □

Eventually, it remains to discuss eigenvalue-eigenvector decomposition for the second tensor $f_{kh}(z,\dot{z})$ (which is preassumed to be the degree of curvature) of Cartan's I-curvature tensor given by Eq. (25). Applying similar cases and conditions which have been applied for the Einstein's tensor in the preceeding sections, we can easily decompose $f_{kh}(z,\dot{z})$ and we can lucidly observe that the decomposition of this tensor has almost similar results as calculated for Einstein's tensor.

The eigenvalue-eigenvector decomposition of $f_{kh}(z,\dot{z})$ yields some great geometrical significances. Some of the significances are discussed as follows:

2.3. **Geometric configuration of $f_{kh}(z,\dot{z})$ and its latentroots-latentvectors.** Likewise the planar geometry of vectors, the second rank covariant, contravariant or mixed tensors have the natural geometries in the form of quadric surfaces. Generally the surface which is represented by general equation of second degree in x, y and z is called quadric surface or conicoid and is defined as

$$(79)\quad ax + by + cz + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0,$$

which can be reduced to any standard form like ellipsoid, hyperboloid of one sheet and two sheets and elliptic paraboloid etc.
In the similar way, we can write the quadratic surface of $f_{kh}(z,\dot{z})$ as

$$(80)\quad [r_k.f_{kh}.r_h] = 1, \quad \text{with} \quad \|f_{kh}(z,\dot{z})\| = \begin{pmatrix} f_{\alpha\alpha} & f_{\alpha\beta} & f_{\alpha\gamma} \\ f_{\beta\alpha} & f_{\beta\beta} & f_{\beta\gamma} \\ f_{\gamma\alpha} & f_{\gamma\beta} & f_{\gamma\gamma} \end{pmatrix}_{k,h=\alpha,\beta,\gamma}.$$

Expansion of the determinant of $f_{kh}(z,\dot{z})$ yields

$$(81)\quad f_{\alpha\alpha}z^2_\alpha + f_{\beta\beta}z^2_\beta + f_{\gamma\gamma}z^2_\gamma + (f_{\alpha\beta} + f_{\beta\alpha})z_\alpha z_\beta + (f_{\alpha\gamma} + f_{\gamma\alpha})z_\alpha z_\gamma + (f_{\beta\gamma} + f_{\gamma\beta})z_\beta z_\gamma = 1,$$

which are a subset of quadric surface and can be reduced to ellipsoid, real or imaginary elliptic cylinders, hyperboloid etc. If we consider only the symmetry of $f_{kh}(z,\dot{z})$, then $\exists$ a $1-1$ correspondence between $f_{kh}(z,\dot{z})$ and the quadric surface. Even for a general tensor, there is a symmetric plus a family of non-symmetric tensors which produce the same surface. This fact is due to the components $f_{\alpha\beta}+f_{\beta\alpha}$, $f_{\alpha\gamma}+f_{\gamma\alpha}$ and $f_{\beta\gamma}+f_{\gamma\beta}$ involved in Eq. (81).

If in any coordinate system $f_{kh}(z,\dot{z})$ becomes diagonalized, the Eq. (81) of quadric surface reduces to the form:

$$\lambda_1 z_\alpha^2+\lambda_2 z_\beta^2+\lambda_3 z_\gamma^2=1, \tag{82}$$

which is again a quadric surface having its geometric axes aligned with that specific coordinate system and of course the eigenvectors.

If all the latentroots are distinct, then clearly latentvectors will be orthogonal. If the latentroots are positive as well as distinct, the tensor $f_{kh}(z,\dot{z})$ is geometrically an ellipsoid with a circular cross section with two of its axes equal in length.

If all the eigenvalues are equal, the tensor $f_{kh}(z,\dot{z})$ is geometrically a sphere. Also, there is a relation among the invariants I^1, I^2 and I^3 of $f_{kh}(z,\dot{z})$ and its eigenvalues say λ_1,λ_2 and λ_3 given by

$$I^1=f_{kk}(z,\dot{z})=(f_{\alpha\alpha}+f_{\beta\beta}+f_{\gamma\gamma})(z,\dot{z})=\lambda_1+\lambda_2+\lambda_3, \tag{83a}$$

$$I^2=\frac{1}{2}(f_{kk}f_{hh}-f_{kh}f_{hk})=\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1, \tag{83b}$$

$$I^3=e_{khl}(f_{kk}f_{hh}f_{ll})=\lambda_1\lambda_2\lambda_3. \tag{83c}$$

These are called principal invariants of $f_{kh}(z,\dot{z})$ and geometrically concerned with the quadric surface given by Eq. (80).

3. Results and discussions

Here is the brief discussion over some vital results obtained from our article written in favor of decomposition techniques.

- Whatever be the rank of tensor, it can be firstly factorize into arbitrary number of tensors by means of open product so that each individual tensor could be decomposed with simplest decomposition techniques without seeking the higher order SVD techniques. Though the process may go on quite lengthy, but with this, checking actual geometric configurations of original tensor could seem rather convenient. As in our case the factorization of Cartan's first curvature tensor evolves two very surprising components namely Einstein's tensor and degree of curvature and both of them are widely used in the analysis of geometry of gravitation and differential geometry of curved surfaces.
- By adopting the process of covariant differentiation given by equation (16), for the symmetric and anti-symmetric part of Einstein's tensor, we have developed an expression, which most probably predicts some complicated relations among the components of Einstein's tensor and various tensorial and non-tensorial quantities as given by equation (30). Of course, this relation is vitally important as it describes direct or indirect correlation between components of complex Einstein's tensor and Christoffel's second kind bracket symbol, Ricci tensor, scalar curvature, Riemann metric tensor as well as a third order tensor $C^p_{rs}(z,\dot{z})$ etc.

- By introducing an energy-momentum tensor and after then Euler's well known stationary integrability condition, we have derived Einstein's field equation. Thus by first decomposition technique, we have shown that Cartan's curvature tensor is able producing Einstein's field equation and hence applicable to induce features of complex Finsler manifolds in Einstein's manifold.
- There is given a special case which evokes that if the energy momentum tensor and hence the components of Ricci's tensor vanishes the complex Finsler manifold reduces to a special manifold which should be called Ricci flat complex Finsler manifold.
- We exposed the second factorized part of Cartan's first curvature tensor in such a way that it describes the degree of curvature of smooth complex Finsler surface. The surface is assumed to be Monge's surface which will have greatest and least curvatures called principal curvatures. By decomposing second factorized part, we have calculated such principal curvatures for our Monge's surface.
- After calculating the latentroots of second factorized component of Cartan's first curvature tensor, we have calculated the Gaussian and mean curvatures of the Monge's surface. Thus we have shown that it is possible to derive the Gaussian, mean and principal curvatures from the given Cartan's curvature tensor.
- We have illustrated that if the Gaussian curvature tensor at any point of the Monge's surface becomes constant, then the complex Finsler manifold will turn into complex Einstein's manifold.
- In order to check the nature of latentroots/latentvector of Cartan's first curvature tensor using technique 2 we have introduced the case of self conjugacy, case of purity and hybridness of Einstein's tensor. Thus by merely checking the nature of latent roots/latent vectors, one can easily discuss the metric signatures and hence the complete characteristics of complex Finsler manifolds.
- For the feasibility and to avoid the heavy calculations, we have introduced the Yano's O and $*O$ operators and shown that what would be the effects of pure and hybrid nature of any tensor over its latentroots/latentvectors.
- We have given the geometric significance of our second factorized component of Cartans I-curvature tensor using technique 2. Here we have introduces an equation for quadric surface involving second factorized component and after checking its latent roots/latent vectors have discussed the geometric configurations generated by this quadric surface.

Remark 3.1. *The rest two techniques for decomposition will be studied in the next manuscript.*

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ON γ-SEMI-θ-CLOSED SETS AND LOCALLY γ-S-REGULAR SPACES

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Abstract. We continue studying the properties and characterizations of γ-semi-θ-closed sets, γ-s-closed relative to a space X and locally γ-s-regular spaces defined and discussed by B. Ahmad and S. Hussain in 2008 and S. Hussain and B. Ahmad in 2007 and 2009.

1. Introduction

In 1969, Signal and Arya [22] defined a new separation axiom called almost regularity which is weaker than regularity. It has been shown in [17], that for Hausdorff spaces, this axiom occupies a position between Urysohn's separation axiom and T_3-axiom. Maheswari and Prasad [18] have defined another axiom called s-regularity which is weaker than regularity (without T_2). In 1982, C. Dorsett [8] defined and investigated a new separation axiom called semi-regular space. It is shown that s-regularity is weaker than semi-regularity. In 1979, S. Kasahara [14] defined an operation α on topological spaces. B. Ahmad and M. Khan [7] defined and study locally s-regular spaces. It is interesting to mention that class of s-regular spaces is a proper subclass of locally s-regular spaces. In 1992 (1993) , B. Ahmad and F. U. Rehman [1] [21] introduced the notions of γ-interior, γ-boundary and γ-exterior points in topological spaces. B. Ahmad and S. Hussain further studied the properties of γ-operations in topological spaces in [2] [3]. Recently S. Hussain, B. Ahmad and T. Noiri [12] introduced and discussed γ-semi-open sets in topological spaces. Further B. Ahmad and S. Hussain [5] explored the characterizations of γ-semi-open (closed), γ-semi-closure (interior) and γ-semi-continuous functions. They defined and discussed γ-s-closed spaces and subspaces([4], [10]) using γ-semi-closure. It is

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known [10] that the concept of γ-s-closed space is a generalization of s-closed space [16]. They defined and discussed the characterizations and properties of γ-semi-θ-closed sets [10], γ-s-closed relative to a space X [4] and Locally γ-s-Regular spaces [4].

In this paper, we continue studying the properties and characterizations of γ-semi-θ-closed sets, γ-s-closed relative to a space X and locally γ-s-regular spaces defined and discussed in [4], [10], [11].

2. Preliminaries

Hereafter X will be represented as a topological space and we shall write a space in place of a topological space for our convenience.

Now we recall some notions defined in [5],[11], [12], [14] and [15]. In [14] an operation $\gamma : \tau \to$ P(X) is defined as a function from τ to the power set of X such that V $\subseteq V^\gamma$, for each V $\in \tau$, where V^γ denotes the value of γ at V. The operations defined by γ(G) = G, γ(G) = cl(G) and γ(G) = intcl(G) are examples of operation γ. Let A be a subset of space X . A point $x \in A$ is said to be γ-interior point [15] of A, if there exists an open nbd N of x such that $N^\gamma \subseteq$ A. We denote the set of all such points by int_γ(A). Thus

$$int_\gamma\ (\mathrm{A}) = \{x \in A : x \in N \in \tau \text{ and } N^\gamma \subseteq A\}.$$

Note that A is γ-open [15] iff A $=int_\gamma$(A). A is called γ- closed [21] iff $X - A$ is γ-open. A point x $\in$ X is called a γ-closure point [15] of A , if $U^\gamma \cap A \neq \phi$, for each open nbd U of x. The set of all γ-closure points of A is called γ-closure of A and is denoted by $cl_\gamma(A)$. A is called γ-closed, if $cl_\gamma(A) \subseteq$A. Note that cl_γ(A) is contained in every γ-closed superset of A. An operation γ on τ is said be regular [15], if for any open nbds U, V of $x \in X$, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$. An operation γ on τ is said to be open [15], if for every nbd U of each $x \in X$, there exists γ-open set B such that $x \in B$ and $U^\gamma \subseteq B$. A subset A of a space X is said to be a γ-semi-open set [12], if there exists a γ-open set O such that $O \subseteq A \subseteq cl_\gamma(O)$. The set of all γ-semi-open sets is denoted by $SO_\gamma(X)$. A is γ-semi-closed iff $X - A$ is γ-semi-open in X. Note that A is γ-semi-closed iff $int_\gamma(cl_\gamma(A)) \subseteq A$. The intersection of all γ-semi-closed sets containing A is called γ-semi-closure [5] of A and is denoted by $scl_\gamma(A)$. Note that A is γ-semi-closed iff $scl_\gamma(A) = A$. The union of γ-semi-open subsets of A is called γ-semi-interior [5] of A and is denoted by $sint_\gamma(A)$. A is γ-semi-regular [5], if it is both γ-semi-open and γ-semi-closed. The class of all γ-semi-regular sets of X is denoted by $SR_\gamma(A)$. Note that if γ is a regular operation, then the union of γ-semi-regular sets is γ-semi-regular. A space X is said to be γ-s-regular [11], if for any γ-semi-regular set A and $x \notin A$, there exist disjoint γ-open sets U and V such that $A \subseteq U$ and $x \in V$.

3. γ-semi-θ-closed Sets.

In [10] a point x of a space X is defined as the γ-semi-θ-closure point of a subset A of X, if $A \cap scl_\gamma(U) \neq \phi$, for every $U \in SO_\gamma(X)$ containing x. The set of all γ-semi-θ-closure points of A is called γ-semi-θ-closure of A and is denoted by $s_\gamma cl_\theta(A)$. A subset A is said to be γ-semi-θ-closed, if $A = s_\gamma cl_\theta(A)$. The complement of a γ-semi-θ-closed set is said to be γ-semi-θ-open.

Now we define:

Definition 3.1. *A function $f : X \to Y$ is said to be a γ-semi-θ-closed, if $f(K)$ is γ-semi-θ-closed in Y, for every γ-semi-θ-closed set K of X.*

Theorem 3.2. *A function $f : X \to Y$ is γ-semi-θ-closed iff $s_\gamma cl_\theta(f(A)) \subseteq f(s_\gamma cl_\theta(A))$, for every subset A of X, where γ is open.*

Proof. Necessity: let f be γ-semi-θ-closed and A any subset of X. Then $f(s_\gamma cl_\theta(A))$ is γ-semi-θ-closed. But $f(A) \subseteq f(s_\gamma cl_\theta(A))$ implies that $s_\gamma cl_\theta(f(A)) \subseteq f(s_\gamma cl_\theta(A))$ [10].

Sufficiency: let $s_\gamma cl_\theta(f(A)) \subseteq f(s_\gamma cl_\theta(A))$, for every subset A of X. Let B be a γ-semi-θ-closed set of X. Then $s_\gamma cl_\theta(f(B)) \subseteq f(s_\gamma cl_\theta(B)) = f(B)$. But $f(B) \subseteq s_\gamma cl_\theta(f(B))$. This proves that $s_\gamma cl_\theta(f(B)) = f(B)$. This gives that $f(B)$ is γ-semi-θ-closed. Hence the proof. □

Theorem 3.3. *Let γ be an open operation. Then the following are equivalent for a function $f : X \to Y$:*

(1) f is γ-semi-θ-closed.

(2) $s_\gamma cl_\theta(f(A)) \subseteq f(s_\gamma cl_\theta(A))$, for every subset A of X.

(3) For every subset B of Y and every γ-semi-θ-open set U of X containing $f^{-1}(B)$, there exists a γ-semi-θ-open set V of Y containing B such that $f^{-1}(V) \subseteq U$.

(4) For every point $y \in Y$ and every γ-semi-θ-open set U of X containing $f^{-1}(y)$, there exists a γ-semi-θ-open set V of Y containing y such that $f^{-1}(V) \subseteq U$.

Proof. The proof is similar to that of Theorem 5.2 [6] and is thus omitted. □

Recall [Lemma 1 [4]] that a subset A of a space X is γ-s-closed relative to X iff every cover of A by γ-semi-θ-open sets of X has a finite subcover.

Theorem 3.4. *Let $f : X \to Y$ be a γ-semi-θ-closed function such that $f^{-1}(y)$ is γ-s-closed relative to X, for each point y of Y. If K is γ-s-closed relative to Y, then $f^{-1}(K)$ is γ-s-closed relative to X, where γ is a regular operation.*

Proof. Let $\{U_\alpha : \alpha \in I\}$ be any cover of $f^{-1}(K)$ by γ-semi-θ-open sets of X. For each $y \in K$, $f^{-1}(y)$ is γ-s-closed relative to X and by Lemma 1 [4], there exists a finite subset $I(y)$ of I such that $f^{-1}(y) \subseteq \cup\{U_\alpha : \alpha \in I(y)\}$. Let $U(y) = \cup\{U_\alpha : \alpha \in I(y)\}$, then U(y) is γ-semi-θ-open in X [4]. Since f is γ-semi-θ-closed, by Theorem 3.4, there exists a γ-semi-θ-open set V(y) containing y such that $f^{-1}(V(y)) \subseteq U(y)$, since $\{V(y) : y \in K\}$ is a γ-semi-θ-open cover of K. By Lemma 1 [4], there exists a finite subset K_0 of K such that $K \subseteq \cup\{V(y) : y \in K_0\}$. Therefore, we obtain

$$f^{-1}(K) \subseteq \cup\{f^{-1}(V(y)) : y \in K_0\} \subseteq \cup\{U_\alpha(y) : \alpha \in I(y), y \in K_0\}.$$

This shows that $f^{-1}(K)$ is γ-s-closed relative to X. This completes the proof. □

Corollary 1. *Let $f : X \to Y$ be a γ-semi-θ-closed surjection such that $f^{-1}(y)$ is γ-s-closed relative to X for each point $y \in Y$. If Y is γ-s-closed, then X is γ-s-closed, where γ is a regular operation.*

In [4], we defined that a filterbase $\Im$ on X is said to γ-SR-converge to $x \in X$, if for each $V \in SR_\gamma(X)$, there exists $F \in \Im$ such that $F \subseteq V$.

Definition 3.5. *A point $x \in X$ is called γ-semi-θ-adherent point of a filterbase $\Im$ in X, if $x \in [s_\gamma ad]_\theta(\Im) = \bigcap\{s_\gamma cl_\theta(F) : F \in \Im\}$.*

Definition 3.6. *A filterbase $\Im$ is said to be γ-semi-θ-directed towards $S \subseteq X$, if every filterbase subordinate to $\Im$ has a γ-semi-θ-adherent point in S.*

Definition 3.7. *A function $f : X \to Y$ is said to be γ-semi-θ-perfect, if for every filterbase $\Im$ in f(X) γ-SR-converges to $y \in Y$, $f^{-1}(\Im)$ is γ-semi-θ-directed towards $f^{-1}(y)$.*

Theorem 3.8. *Every γ-semi-θ-perfect function is γ-semi-θ-closed, where γ is open.*

Proof. Let $f : X \to Y$ be a γ-semi-θ-perfect function and A any subset of X. Let $y \in s_\gamma cl_\theta(f(A))$. Then there exists a filterbase $\Im$ on $f(A)$ which γ-SR-convergrs to y. Put $\xi = \{f^{-1}(F) \cap A : F \in \Im\}$. Then ξ is a γ-filterbase on X subordinate to the filterbase $f^{-1}(\Im)$. Since $f^{-1}(\Im)$ is γ-semi-θ-directed towards $f^{-1}(y)$, we have $f^{-1}(y) \cap [s_\gamma ad]_\theta(\xi) \neq \phi$. Therefore, we obtain $y \in f(s_\gamma cl_\theta A)$. By Theorem 3.2, f is γ-semi-θ-closed. Hence the proof. □

Theorem 3.9. *A function $f : X \to Y$ is γ-semi-θ-perfect iff $[s_\gamma ad]_\theta f(\Im) \subset f([s_\gamma ad]_\theta(\Im))$, for every filterbase $\Im$ in X.*

Proof. Necessity: suppose that $f : X \to Y$ is γ-semi-θ-perfect. Let $\Im$ be a filterbase in X and $y \in [s_\gamma ad]_\theta f(\Im)$. Then there exists a filterbase ξ in $f(X)$ which is subordinate to $f(\Im)$ and γ-SR-converges to y. Put $H = \{f^{-1}(G) \cap F : F \in \Im, G \in \xi\}$. Then H is filterbase in X subordinate to $f^{-1}(\xi)$. Since f is γ-semi-θ-perfect, $f^{-1}(\xi)$ is γ-semi-θ-directed towards $f^{-1}(y)$. Therefore we have $f^{-1}(y) \cap [s_\gamma ad]_\theta(H) \neq \phi$ and hence $y \in f([s_\gamma ad]_\theta(\Im))$. This proves that $[s_\gamma ad]_\theta f(\Im) \subset f([s_\gamma ad]_\theta \Im)$. This proves necessity.

Sufficiency: suppose that $[s_\gamma ad]_\theta f(\Im) \subset f([s_\gamma ad]_\theta(\Im))$, for every filterbase $\Im$ in X. We prove that f is γ-semi-θ-perfect. Assume contrary that f is not γ-semi-θ-perfect, then there exists a filterbase $\Im$ in $f(X)$ such that $\Im$ γ-SR-converges to a point $y \in Y$. But $f^{-1}(\Im)$ is not γ-semi-θ-directed towards $f^{-1}(y)$. Thus there exists a filterbase ξ in X which is subordinate to $f^{-1}(\Im)$ and $f^{-1}(y) \cap [s_\gamma ad]_\theta(\xi) = \phi$. Therefore we have $y \notin [s_\gamma ad]_\theta f(\xi)$ and hence $y \notin s_\gamma cl_\theta(f(G_1))$, for some $G_1 \in \xi$. Then there exists a γ-semi-open set V containing y such that $scl_\gamma(V) \cap f(G_1) = \phi$. Since $\Im$ γ-SR-converges to y and ξ is subordinate to $f^{-1}(\Im)$, there exists a $G_2 \in \xi$ such that $f(G_2) \subset scl_\gamma(V)$. Consequently, we obtain $G_1 \cap G_2 = \phi$. This contradicts that ξ is a filterbase. This proves that f is γ-semi-θ-perfect. Hence the proof. □

4. γ-s-closed Relative to a Space.

In [10] a subset A of a space X is defined to be γ-s-closed relative to X, if for every cover of A by γ-semi-open sets of X, there exists a finite subset I_0 of I such that $A \subseteq \bigcup_{\alpha \in I_0} scl_\gamma(V_\alpha)$.

Theorem 4.1. *A subset K of a space X is γ-s-closed relative to X iff $K \cap [s_\gamma ad]_\theta(\Im) \neq \phi$, for every filterbase $\Im$ in K.*

Proof. Necessity: suppose that K is γ-s-closed relative to X. Assume that $\Im$ is a filterbase in K such that $K \cap [s_\gamma ad]_\theta(\Im) = \phi$. Then for each $x \in K$, there exists a γ-semi-open set U_x containing x and $F_x \in \Im$ such that $F_x \cap scl_\gamma(U_x) = \phi$. Since K is γ-s-closed relative to X, there exists a finite number of points $x_1, x_2, ..., x_n$ in K such that $K \subset \bigcup\{scl_\gamma(U_{x_j}) : j = 1, 2, ..., n\}$. Put $F = \bigcap\{F_{x_j} : j = 1, 2, ..., n\}$, then we obtain $F \cap K = \phi$. This contradicts that $\Im$ is a filterbase in K. Therefore $K \cap [s_\gamma ad]_\theta(\Im) \neq \phi$. This completes necessity.

Sufficiency: suppose that $K\cap[s_\gamma ad]_\theta(\Im) \neq \phi$, for every filterbase $\Im$ in K. Assume that K is not γ-s-closed relative to X. Then there exists a cover $\{U_\alpha : \alpha \in I\}$ of K be γ-semi-open sets of X such that $K \nsubseteq \bigcup\{scl_\gamma(U_\alpha) : \alpha \in I_o\}$ for every $I_0 \in J(I)$, where $J(I)$ denotes the family of all finite subsets of I. Put $F_{I_o} = \bigcap\{K - scl_\gamma(U_\alpha) : \alpha \in I_o\}$, for each $I_o \in J(I)$. Then $\Im = \{F_{I_o} : I_o \in J(I)\}$ is a filterbase in K and $K \cap [s_\gamma ad]_\theta(\Im) = \phi$. This is a contradiction. Therefore, K is γ-s-closed relative to X. This completes the proof. □

Theorem 4.2. *If $f : X \to Y$ is a γ-semi-θ-perfect function, then $f^{-1}(K)$ is γ-s-closed relative to X, for every $K \subseteq Y$, γ-s-closed relative to Y. Where γ is open.*

Proof. Let $f : X \to Y$ be a γ-semi-θ-perfect function and $K \subset Y$ γ-s-closed relative to Y. We prove that $f^{-1}(K)$ is γ-s-closed relative to X. Let $\Im$ be a filterbase in X. Then $\xi = \{f(F)\cap K : F \in \Im\}$ is a filterbase in K subordinate to the filterbase $f(\Im)$. By Theorem 4.1, $K\cap[s_\gamma ad]_\theta(\xi) \neq \phi$ and hence we obtain $K\cap[s_\gamma ad]_\theta f(\Im) \neq \phi$. Since f is γ-semi-θ-perfect, therefore by Theorem 3.8, we obtain $K \cap f([s_\gamma ad]_\theta(\Im)) \neq \phi$. This gives $f^{-1}(K) \cap [s_\gamma ad]_\theta(\Im) \neq \phi$. Hence by Theorem 4.1, we conclude that $f^{-1}(K)$ is γ-s-closed relative to X. Hence the proof. □

Theorem 4.3. *Let γ be an open operation. A function $f : X \to Y$ is γ-semi-θ-perfect iff*

(1) f is γ-semi-θ-closed, and

(2) $f^{-1}(y)$ is γ-s-closed relative to X, for each $y \in Y$.

Proof. Necessity: let $f : X \to Y$ be a γ-semi-θ-perfect function. Then (1) follows from Theorem 3.8 and (2) follow from Theorem 4.2, because each singleton set in Y is γ-s-closed relative to Y [10].

Sufficiency: we suppose on the contrary that conditions (1) and (2) hold but f is not γ-semi-θ-perfect. Then there exists a filterbase $\Im$ in $f(X)$ such that $\Im$ γ-SR-converges to a point $y \in Y$. But $f^{-1}(\Im)$ is not γ-semi-θ-directed towards $f^{-1}(y)$. Thus there exists a filterbase ξ in X which is subordinate to $f^{-1}(\Im)$ and $f^{-1}(y) \cap [s_\gamma ad]_\theta(\xi) = \phi$. Therefore we have $y \notin [s_\gamma ad]_\theta f(\xi)$ and hence $y \notin s_\gamma cl_\theta(f(G_1))$, for some $G_1 \in \xi$. Then there exists a γ-semi-open set V containing y such that $scl_\gamma(V) \cap f(G_1) = \phi$. Since $\Im$ γ-SR-converges to y and ξ is subordinate to $f^{-1}(\Im)$, therefore, there exists a $G_2 \in \xi$ such that $f(G_2) \subset scl_\gamma(V)$. Consequently, we obtain $G_1 \cap G_2 = \phi$. This contradicts that ξ is filterbase. This proves that f is γ-semi-θ-perfect. This completes the proof. □

5. Locally γ-s-Regular Spaces .

In [10] a subset A of a space X is called γ-regular-open, if $A = int_\gamma(cl_\gamma(A))$. The set of all γ-regular-open sets in X is denoted by $RO_\gamma(X, \tau)$. Note that $RO_\gamma(X, \tau) \subseteq \tau_\gamma \subseteq \tau$ [10].

A space X is a γ-extremally disconnected space [10], if $cl_\gamma(U)$ is a γ-open set, for every γ-open set U in X. A space X is said to be locally γ-s-regular [11], if for each point of X has a γ-regular-open nbd which is γ-s-regular subspace of X.

The following Theorem shows that locally γ-s-regularity is a γ-regular-open hereditary property:

Theorem 5.1. *Every γ-regular-open subspace of a locally γ-s-regular space is locally γ-s-regular, where γ is a regular operation.*

Proof. Let Y be a γ-regular-open subspace of a locally γ-s-regular space X and $x \in Y$. Since X is locally γ-s-regular, therefore $x \in X$ has a γ-regular-open set V containing x which is γ-s-regular subspace of X. Since $int_{\gamma_Y}((cl_{\gamma_Y})(A)) = Y \cap int_{\gamma_X}((cl_{\gamma_X})(A))$ [2], for any γ-open set Y of X and any set A of Y. Therefore, there exists a γ-regular-open set U of Y such that $U = Y \cap V$ and $x \in U$. Now we show that U is γ-s-regular subspace of Y. Since V is γ-s-regular, by Theorem 3.4 [11], for γ-semi-regular set V_1 of V containing x, there exists γ-open set W such that $x \in W \subset cl_{\gamma_V}(W) \subset V_1$. Since $V \in RO_\gamma(X)$ and $W \subseteq V$, therefore we have $x \in W \subseteq cl_{\gamma_X}(W) \cap V \subseteq V_1$ [11]. But $cl_{\gamma_X}(W) \subset cl_{\gamma_X}(V_1) = V_1$. This gives $x \in W \subset cl_{\gamma_X}(W) \subset V_1$. Thus we have $x \in W \cap U \subset cl_{\gamma_V}(W) \cap U \subset V_1 \cap U$. Since Y is γ-semi-closed in X [5], therefore we obtain $cl_{\gamma_X}(W \cap U) \subset cl_{\gamma_X}(W) \cap cl_{\gamma_X}(U) = cl_{\gamma_X}(W) \cap U$ or $cl_{\gamma_X}(W \cap U) \subset cl_{\gamma_X}(W) \cap U$. This gives that $x \in W \cap U \subset cl_{\gamma_X}(W \cap U) \subseteq cl_{\gamma_X}(W) \cap U \subseteq V_1 \cap U$ or $x \in W \cap U \subset cl_{\gamma_U}(W \cap U) \subset V_1 \cap U$, where $W \cap U$ is γ-open and $V_1 \cap U$ is γ-semi-regular in U, since γ is regular. Thus, by Theorem 3.4 [11], U is γ-s-regular. This proves that Y is γ-s-regular. This completes the proof. □

Corollary 2. *Every γ-clopen subspace of a locally γ-s-regular space is locally γ-s-regular, where γ is a regular operation.*

Corollary 3. *Every γ-open subspace of a locally γ-s-regular space is locally γ-s-regular, if X is γ-extremally disconnected, where γ is a regular operation.*

Next we characterize locally γ-s-regular spaces as:

Theorem 5.2. *In a space X, the following are equivalent:*

(1) X is locally γ-s-regular.

(2) Every point $x \in X$ has a γ-regular-open set containing x which is a γ-s-reguar subspace of X.

(3) Every point $x \in X$ has a γ-open nbd U of x such that $int_{\gamma_X}(cl_{\gamma_X}(U))$ is a γ-s-regular subspace of X.

(4) Every point $x \in X$ has a γ-open set U containing x such that $scl_{\gamma_X}(U)$ is a γ-s-regular subspace of X.

(5) Every point $x \in X$ has a γ-open nbd U such that $s_\gamma cl_\theta(U)$ is a γ-s-regular subspace of X.

Proof. (1) ⇒ (2). This is straightforward.

(2) ⇒ (3). Let U be γ-open nbd of $x \in X$. Then $int_{\gamma_X}(cl_{\gamma_X}(U))$ is a γ-regular-open set containing x. By (2), $int_{\gamma_X}(cl_{\gamma_X}(U))$ is a γ-s-regular subspace of X. This proves (3).

(3) ⇒ (4). Let U be γ-open nbd of a point $x \in X$. Then by Lemma 3.5 [10], $scl_{\gamma_X}(U) = int_{\gamma_X}(cl_{\gamma_X}(U))$. This shows that $scl_{\gamma_X}(U)$ is a γ-regular-open set of X containing x. By (3), $scl_{\gamma_X}(U)$ is γ-s-regular subspace of X. This proves (4).

(4) ⇒ (5). Let U be γ-open nbd of x in X. Then by Proposition 3.9 [10], $s_\gamma cl_{\theta_X}(U) = scl_{\gamma_X}(U)$. By (4), $s_\gamma cl_{\theta_X}(U)$ is a γ-s-regular subspace of x. This proves (5).

(5) ⇒ (1). Let $x \in X$ and U a γ-regular-open nbd of x. Then by (5), $s_\gamma cl_{\theta_X}(U) = scl_{\gamma_X}(U) = int_{\gamma_X}(cl_{\gamma_X}(U)) = U$ is γ-s-regular subspace of X. This proves that X is locally γ-s-regular. This completes the proof. □

Theorem 5.3. *A space X is locally γ-s-regular iff for each point $x \in X$, there exists a γ-regular-open set A of X such that $x \in A$ and A is locally γ-s-regular, where γ is regular.*

Proof. Necessity follows from Theorem 5.1.

Sufficiency: Let $x \in X$ and A a γ-regular-open set of X such that $x \in A$ and A is locally γ-s-regular. This gives that there exists a γ-regular-open set U in A such that $x \in U$ and U is γ-s-regular subspace of A. Hence by Lemma 3.5 [10], $U = scl_{\gamma_X}(U) = int_{\gamma_X}(cl_{\gamma_X}(U))$. Thus for each $x \in X$, there exists a γ-regular-open set U in X such that $x \in U$ and U is a γ-s-regular subspace of X. This proves that X is locally γ-s-regular. This completes the proof. □

In [5] a function $f : X \to Y$ is defined to be γ-semi-continuous, if for any γ-open B of Y, $f^{-1}(B)$ is γ-semi-open in X. A function $f : X \to Y$ is defined to be γ-semi-open (respt. γ-semi-closed) [6], if for each γ-open (respt. γ-closed) set U in X, $f(U)$ is γ-semi-open (respt. γ-semi-closed) in Y.

Theorem 5.4. *Let $f : X \to Y$ be γ-semi-continuous, γ-semi-open and γ-semi-closed preserving surjection. If X is γ-s-regular, then Y is γ-s-regular, where γ is an open operation.*

Proof. Let U be a γ-regular-open set in Y and $y \in U$. Let $x \in f^{-1}(y)$. Then $f^{-1}(U)$ is γ-semi-open in X and $x \in f^{-1}(U)$. Since X is γ-s-regular , therefore by Theorem 3.4 [11], there exists a γ-open set V such that $x \in V \subset cl_\gamma(V) \subset f^{-1}(U)$ or $y \in f(V) \subset f(cl_\gamma(V)) \subset f(f^{-1}(U)) \subset U$ or $y \in f(V) \subset f(cl_\gamma(V)) \subset U$, where $f(V)$ is γ-semi-open and $f(cl_\gamma(V))$ is γ-semi-closed [5]. Therefore $cl_\gamma(f(V)) \subset f(cl_\gamma(V))$ and $y \in f(V) \subset cl_\gamma(f(V)) \subset f(cl_\gamma(V)) \subset U$ or $y \in f(V) \subset cl_\gamma(f(V)) \subset U$. This proves that Y is γ-s-regular [11]. Hence the proof. □

Theorem 5.5. *Let $f : X \to Y$ be γ-semi-continuous, γ-semi-open preserving bijection. If X is γ-s-regular, then Y is locally γ-s-regular, where γ is a regular operation.*

Proof. Let U be γ-regular open set in Y such that $y \in U$. Let F be γ-closed set of U such that $y \notin F$. Then there exists a γ-closed set G of Y such that $F = U \cap G$ [2] and $y \notin G$. Then $f^{-1}(G)$ is γ-semi-closed in X and $f^{-1}(y) \notin f^{-1}(G)$. Since X is γ-s-regular, therefore there exists disjoint γ-open sets U_1, U_2 in X such that $f^{-1}(y) \in U_1$, $f^{-1}(G) \subset U_2$ or $y \in f(U_1)$, $G \subset f(U_2)$, where $f(U_1)$ and $f(U_2)$ are γ-semi-open in Y. Clearly $y \in U \cap f(U_1)$ and $F = U \cap G \subseteq U \cap f(U_2)$, where $U \cap f(U_1)$ and $U \cap f(U_2)$ are disjoint γ-semi-open sets in U [5]. This proves that Y is locally γ-s-regular. This completes the proof. □

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ON F-SUPPLEMENTED MODULES

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Abstract. Let R be a ring and M a right R–module. In this paper we prove that if M is weakly F-supplemented, then every factor module and every F-coclosed submodule of M is again weakly F-supplemented. In [5], it is shown that $Rad(M)$ has finite uniform dimension iff M does not contain an infinite direct sum of nonzero small submodules. Here we replace F-small submodules instead of small submodules (which is a weaker condition) and obtain the same result; i.e, we show that if M does not contain an infinite direct sum of F-small submodules, then $Rad(M)$ has finite uniform dimension.

1. Introduction

Throughout this article, R denotes an associative ring with identity, and modules are unitary right R-modules.

We write $N \leq M$ to denote that N is a submodule of the module M while $N \subseteq^{\oplus} M$ means that N is a direct summand of M. A submodule L of M is called *small* in M (denoted by $L \ll M$) if, for every proper submodule K of M, $L + K \neq M$. A module M is called *hollow* if every proper submodule of M is small in M.

We denote the ring of all endomorphisms of M by $End(M)$ and the *Jacobson radical* of M by $Rad(M)$ and the Jacobson radical of the ring R by $J(R)$.

A module M is called *lifting* (or said to *satisfy condition* D_1) if for every submodule N of M, M has a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B \ll B$.

For two submodules N and K of a module M, N is called a *supplement* of K in M if N is minimal with respect to the property $M = K + N$, equivalently $M = K + N$ and $N \cap K \ll N$. Also N is called a *weak supplement* of K in M if, $M = N + K$

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and $N \cap K \ll M$. A module M is called *supplemented* if every submodule of M has a supplement in M. M is called *amply supplemented* if whenever $M = A + B$ for submodules A, B of M, then A has a supplement in M contained in B. Also M is called *weakly supplemented* if any submodule of M has a weak supplement in M.

Let M be a module and $B \leq A \leq M$. If $A/B \ll M/B$, then B is called a *cosmall* submodule of A in M. The submodule A of M is called *coclosed* in M if A has no proper *cosmall* submodule. Also B is called a *coclosure* of A in M if B is a cosmall submodule of A and B is coclosed in M.

Supplemented and lifting modules and some generalizations of these kinds of modules are studied by many authors, see for example [16, 8, 13, 14]. We refer for other basic notions to [6, 17].

2. F-small submodules

The class of small modules and some other classe of modules relative to small modules (for example semiperfect modules, supplemented modules and ...) are studied by many authors. For example, see [10, 4, 11, 12, 1, 9, 15, 7]. In this section we define the F–small class and then in section 3, we investigate the class of F–supplemented modules. Let M be a module and $K \leq M$, then K is called an *F-small* submodule of M, denoted by $K \ll_F M$, if K is a finitely generated small submodule of M. It is clear that any F-small submodule of M is small in M, and in noetherian modules small submodules and F-small submodules coincide.

Lemma 2.1. *Let M be a module and $N \leq M$ such that $N \leq Rad(M)$ and N is finitely generated. Then $N \ll_F M$.*

Proof. It is clear by the proof of [3, Proposition 9.13]. □

It is an immediate conclusion of Lemma 2.1 that the sum of all F-small submodules of the module M is equal to $Rad(M)$.

The proof of following three statements are straightforward and are omitted.

Proposition 2.2. *Let M be a module and A, B submodules of M. If $A \ll_F M$ and $B \ll_F M$, then $A + B \ll_F M$. The converse is true if both A and B are finitely generated. Especially for submodules $A_1, A_2, ..., A_n$ of M, $\bigoplus_{i=1}^{n} A_i \ll_F M$if and only if $A_i \ll_F M$ $(i = 1, 2, ..., n)$.*

Proposition 2.3. *Let M be a module and $K \leq N \leq M$. If $K \ll_F M$ and $N/K \ll_F M/K$, then $N \ll_F M$. Moreover $N \ll_F M$ implies that $N/K \ll_F M/K$.*

Proposition 2.4. *Let M be a module and $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$, sucht that $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \ll_F M_1 \oplus M_2$ iff $K_1 \ll_F M_1$ and $K_2 \ll_F M_2$.*

Example 2.5. Let M denote the $\mathbb{Z}$-module $\bigoplus_{i=1}^{\infty} \mathbb{Z}/4\mathbb{Z}$. Consider the submodule $N = \bigoplus_{i=1}^{\infty} 2\mathbb{Z}/4\mathbb{Z}$ of M. Then N is a small submodule of M but not F-small.

3. F-Supplemented Modules

Let M be a module and N, K submodules of M. Then N is called an *$F-$supplement* of K in M if, $M = N + K$ and $N \cap K \ll_F N$. Similarly N is called a *weak $F-$supplement* of K in M if $M = N + K$ and $N \cap K \ll_F M$. The submodule N of M is called an F-supplement (weak F-supplement, resp.) submodule, if there exists a submodule K of M such that N is an F-supplement (weak F-supplement, resp.) of K in M.

The module M is called *F-supplemented* if every submodule of M has an F-supplement in M. M is called *weakly F-supplemented* if every submodule of M has a weak F-supplement in M and M is called *amply F-supplemented* if whenever $M = A+B$ for submodules A, B of M, then A has an F-supplement in M contained in B.

For two submodules $K \leq N \leq M$ of M, we say that K is an F-*cosmall* submodule of N in M, if $N/K \ll_F M/K$. The submodule N of M is called F-*coclosed* in M if N has no proper F-cosmall submodule, equivalently $N/K \ll_F M/K$ implies $N = K$ for any submodule K of N.

For two submodules N, K of M, we say K is an F-*coclosure* of N in M, if $N/K \ll_F M/K$ (K is an F-cosmall submodule of N in M) and K is F-coclosed in M.

The module M is called *F-lifting* if for any submodule N of M there exists a direct summand A of M such that $A \leq N$ and $N/A \ll_F M/A$. By the definition of F-lifting, we deduce that a module M is F-lifting iff for every submodule N of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll_F M_2$.

In this section we show that in an F-lifting module every F-coclosed submodule is a direct summand. Especially we show that a module M is F-lifting iff M is amply F-supplemented and any F-coclosed submodule of M is a direct summand of M.

Lemma 3.1.

(i) Let M be a module and $A \leq N \leq M$. If N is an F-coclosed submodule of M and $A \ll_F M$, then $A \ll_F N$.

(ii) In any weakly F-supplemented module, every F-coclosed submodule is an F-supplement submodule.

Proof. (i) It is enough to show that $A \ll N$. Suppose that $A + L = N$ for $L \leq N$. Now We prove that $N/L \ll M/L$. Let $N/L + K/L = M/L$ for $L \leq K \leq M$. So $M = N + K = A + L + K$. Since $A \ll M$, we get $M = L + K = K$. Hence $N/L \ll M/L$. Note that $N/L \cong A/A \cap L$ is finitely generated and so $N/L \ll_F M/L$. Therefore $N = L$, i.e. $A \ll N$.

(ii) Suppose that M is a weakly F-supplemented module and $N \leq M$ is an F-coclosed submodule of M. There exists a submodule A of M such that $N + A = M$ and $A \cap N \ll_F M$. By (1), $A \cap N \ll_F N$ and so N is an F-supplement submodule of M. □

Theorem 3.2. *Let M be a module. Then the following are equivalent:*

(i) M is F-lifting;

(ii) Every submodule A of M can be written as $A = N \oplus F$ with $N \subseteq^{\oplus} M$ and $F \ll_F M$;

(iii) M is amply F-supplemented and every F-coclosed submodule of M is a direct summand of M.

Proof. (i) $\Longrightarrow$ (ii) Let $A \leq M$. Since M is F-lifting, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A \cap M_2 \ll_F M_2 \leq M$. By modularity $A = M_1 \oplus A \cap M_2$.

(i) $\Longrightarrow$ (iii) Let $M = X + Y$. By (2) we may assume that $Y \subseteq^{\oplus} M$. Again by (2), $X \cap Y = Y_1 \oplus F$ for $Y_1 \subseteq^{\oplus} M$ and $F \ll_F M$. By Lemma 3.1, $F \ll_F Y$. Clearly $Y_1 \subseteq^{\oplus} Y$ and so write $Y = Y_1 \oplus Y_2$. Let $\pi : Y_1 \oplus Y_2 \longrightarrow Y_2$ denote the projection map. We have $X \cap Y = Y_1 \oplus X \cap Y \cap Y_2$ and also $X \cap Y_2 = X \cap Y \cap Y_2 = \pi(X \cap Y) = \pi(Y_1 + F) = \pi(F)$. Therefore $X \cap Y_2 \ll_F Y_2$ by Proposition 2.3. Finally we obtain $M = X + Y = X + Y_1 + Y_2 = X + Y_2$. So Y_2 is an F-supplement of X contained in Y.

Now suppose that H is an F-coclosed submodule of M. Then $H = A \oplus F$ with $F \ll_F M$. Thus $H/A \ll_F M/A$ and so $H = A$ a direct summand of M.

(iii) $\Longrightarrow$ (i) Let $X \leq M$. Then by (3), X has an F-supplement Y and Y has an F-supplement M_1 such that $M_1 \leq X$ and $M_1 \subseteq^{\oplus} M$. Write $M = M_1 \oplus M_2$. Then $X = M_1 \oplus X \cap M_2$. Furthermore $M = M_1 + Y$ and so $X = M_1 + X \cap Y$. Let $\pi : M_1 \oplus M_2 \longrightarrow M_2$ be the projection map. Then $X \cap M_2 = \pi(X) = \pi(X \cap Y)$. Since $X \cap Y \ll_F M$, $X \cap M_2 \ll_F M$. Therefore M is F-lifting.

$\square$

Example 3.3. Let $M = \mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ as $\mathbb{Z}$-module. Then M is F-lifting and so is F-supplemented by Theorem 3.4. Let $N = p\mathbb{Z}/p^2\mathbb{Z} \oplus p\mathbb{Z}/p^3\mathbb{Z}$. Then N is a submodule of M which is not a direct summand of M, so N is not F-coclosed in M. Now consider the submodule $K = \mathbb{Z}/p^2\mathbb{Z}$ of M. Clearly K is not F-small in M. Also if $K/L \ll_F M/L$ for $L \leq K$, then $K = L$ and so K is an F-coclosed submodule of M.

Let M be a module, then M is called *F-hollow* if every proper submodule of M is F-small in M. By the last Theorem we conclude the following Corollary;

Corollary 3.4. *An indecomposable module M is F-lifting iff it is F-hollow.*

Remark 3.5. Let M be a module and N an F-hollow submodule of M. If N is not F-small in M then N is not small in M and so $N + K = M$ for a proper submodule K of M. Since N is F-hollow, we have $N \cap K \ll_F N$ and so N is an F-supplement submodule of M. Moreover if M is F-lifting, then N is a direct summand of M by Theorem 3.4.

Theorem 3.6. *Let M be a module. Then the following hold*

(i) If M is amply F-supplemented, then for every submodule N of M that is not small in M, there is an F-supplement submodule L of M such that $L \leq N$ and $N/L \ll_F M/L$.

(ii) If A is an F-coclosed submodule of M and $B \leq A$, then A/B is F-coclosed in M/B.

(iii) If L is a supplement submodule of M and $K \leq L$, then K is F-coclosed in L iff K is F-coclosed in M.

Proof. (i) Since N is not small in M we get a proper submodule K of M such that $N + K = M$. Let X be an F-supplement of N contained in K and L be an F-supplement of X contained in N. Since $N \cap X \ll_F X$, we have $(N \cap X)/(L \cap X) \ll_F X/(L \cap X)$. Furthermore $(N \cap X)/(L \cap X) \cong N/L$ and $X/(L \cap X) \cong (X + L)/L = M/L$. So $N/L \ll_F M/L$.

(ii) Let $(A/B)/(C/B) \ll_F (M/B)/(C/B)$ for $B \leq C \leq A$. Then $A/C \ll_F M/C$ and so $A = C$.

(iii) Let L be a supplement submodule of M. If K is F-coclosed in M then obviously K is F-coclosed in L.

For converse assume that K is F-coclosed in L. Let H be a submodule of K such that $K/H \ll_F M/H$. It is clear that L/H is a supplement submodule of M/H. So by Lemma 3.1, $K/H \ll_F L/H$. Hence $K = H$, as K is F-coclosed in L. Therefore K is F-coclosed in M. $\square$

Proposition 3.7. *Every direct summand of an amply F-supplemented module is again amply F-supplemented.*

Proof. Let M be any amply F-supplemented module and $K \subseteq^{\oplus} M$. Write $M = K \oplus K'$. Suppose that $K = C + D$, then $M = C + (D \oplus K')$. So there exists $P \leq C$ such that $M = P + (D \oplus K')$ and $P \cap (D \oplus K') \ll_F P$. Thus $K = K \cap M = P + D$ and $P \cap D = P \cap (D \oplus K') \ll_F P$; i.e., K is amply F-supplemented. $\square$

Suppose that M is a module and $N \leq M$. Then N is said to have *ample F-supplement* in M if N has an F-supplement contained in L, whenever $M = N + L$ for submodule L of M.

Proposition 3.8. *Let M be a module and U_1, U_2 submodules of M such that $M = U_1 + U_2$. If U_1 and U_2 have ample F-supplement in M, then so does $U_1 \cap U_2$.*

Proof. Let $V \leq M$ and $U_1 \cap U_2 + V = M$. Then $U_1 = U_1 \cap U_2 + V \cap U_1$ and $U_2 = U_1 \cap U_2 + V \cap U_2$. So $M = U_1 + V \cap U_2$ and $M = U_2 + V \cap U_1$. Therefore there exist $V_2' \leq V \cap U_2$ and $V_1' \leq V \cap U_1$ such that $U_1 + V_2' = M$ and $U_1 \cap V_2' \ll_F V_2'$, and $U_2 + V_1' = M$ and $U_2 \cap V_1' \ll_F V_1'$. Thus $V_1' + V_2' \leq V$ and $U_1 = U_1 \cap U_2 + V_1'$ and $U_2 = U_1 \cap U_2 + V_2'$. So $(U_1 \cap U_2) + (V_1' + V_2') = M$ and $(U_1 \cap U_2) \cap (V_1' + V_2') = (U_2 \cap V_1') + (U_1 \cap V_2') \ll_F V_1' + V_2'$, that completes the proof. $\square$

Proposition 3.9. *Let M be a module and $U \leq M$. Then the following are equivalent*

(i) There is a decomposition $M = X \oplus X'$ with $X \leq U$ and $X' \cap U \ll_F X'$;
(ii) There is an idempotent $e \in End(M)$ such that $(M)e \leq U$ and $(U)(1-e) \ll_F (M)(1-e)$;
(iii) There is a direct summand X of M such that $X \leq U$ and $U/X \ll_F M/X$;
(iv) U has an F-supplement V in M such that $U \cap V$ is a direct summand of U.

Proof. (i) $\Longrightarrow$ (ii) For $M = X \oplus X'$, there exists an idempotent $e \in End(M)$ such that $(M)e = X$ and $(M)(1-e) = X'$. Since $X \leq U$, we conclude $(U)(1-e) \leq U \cap (M)(1-e) \ll_F (M)(1-e)$.

(ii) $\Longrightarrow$ (iii) Take $X = (M)e$. Then $M = X \oplus (M)(1-e)$ and $U/X \ll_F M/X$.

(iii) $\Longrightarrow$ (i) Write $M = X \oplus X'$. So $U = X \oplus (X' \cap U)$ by modularity. Also we have $X' \cap U \cong U/X \ll_) FM/X \cong X'$. Thus $X' \cap U \ll_F X'$.

(i) $\Longrightarrow$ (iv) By (1), X' is an F-supplement of U in M and also $U = X \oplus (X' \cap U)$.

(iv) $\Longrightarrow$ (i) Let V be an F-supplement of U in M such that $U = X \oplus (V \cap U)$ for some $X \leq U$. Then $M = U + V = X + (V \cap U) + V = X + V$ and $X \cap V = (X \cap U) \cap V = X \cap (U \cap V) = 0$, i.e., X is a direct summand of M. $\square$

Example 3.10. Let M denote the $\mathbb{Z}$-module $\mathbb{Z}/24\mathbb{Z}$ and $U = 4\mathbb{Z}/24\mathbb{Z} \leq M$. Let $X = 8\mathbb{Z}/24\mathbb{Z}$ and $X' = 3\mathbb{Z}/24\mathbb{Z}$ be submodules of M. We have $M = X \oplus X'$. Since $X \leq U$, by Proposition 3.11 (4), there exists an F-supplement V of U in M such that $U \cap V \subseteq^{\oplus} U$. If we get $V = X'$ in this example, then obviously V is an F-supplement of U in M and $U \cap V = 12\mathbb{Z}/24\mathbb{Z}$ is a direct summand of U; as desired.

Proposition 3.11. *Let M be module such that every submodule of M is F-supplemented. Then, M is amply F-supplemented.*

Proof. Suppose that $M = X+Y$ for some submodules X, Y of M. Then there exists a submodule A of X such that $(X \cap Y) + A = X$ and $(X \cap Y) \cap A = Y \cap A \ll_F A$. Thus $M = X+Y = (X \cap Y)+A+Y = A+Y$, and so M is amply F-supplemented. □

Corollary 3.12. *Let R be a ring. Then every R–module is amply F-supplemented if and only if every R–module is F-supplemented.*

Let M be a module. Then M is called $\pi - projective$ if for two submodules X, Y of M, there exists $f \in End(M)$ with $Im(f) \leq X$ and $Im(1-f) \leq Y$.

Proposition 3.13. *Let M be a π–projective module. If M is F-supplemented, then M is amply F-supplemented.*

Proof. Suppose that $M = A + B$ for $A, B \leq M$. Then there is an endomorphism e of M such that $(M)e \leq A$ and $(M)(1-e) \leq B$. Let C be an F-supplement of A in M. Therefore we have $M = (M)e + (M)(1-e) = (M)e + (A+C)(1-e) \leq A + (C)(1-e)$. So $M = A + (C)(1-e)$, where $(C)(1-e) \leq B$. Now we have $A \cap (C)(1-e) = (A \cap C)(1-e) \ll_F (C)(1-e)$; as $A \cap C \ll_F C$. Thus $(C)(1-e)$ is an F-supplement of A in M contained in B; i.e, M is amply F-supplemented. □

Proposition 3.14. *Let M be a weakly F-supplemented module. Then*

(i) Every F-coclosed submodule of M is weakly F-supplemented.

(ii) Every factor module of M is weakly F-supplemented.

Proof. (i) Let K be an F-coclosed submodule of M and $N \leq K$. Since M is weakly F-supplemented, there exists $L \leq M$ such that $M = N + L$ and $N \cap L \ll_F M$. Thus $K = N + (K \cap L)$. Also $N \cap (K \cap L) = N \cap L \ll_F K$ by Lemma 3.1.

(ii) Let N be a submodule of M and $L/N \ll_F M/N$. Since M is weakly F-supplemented, there exists $K \leq M$ such that $M = K + L$ and $K \cap L \ll_F M$. So $M/N = L/N + (K+N)/N$. Let $\pi : M \longrightarrow M/N$ denote the natural epimorphism. Then $L/N \cap (K+N)/N = (N + L \cap K)/N = \pi(L \cap K) \ll_F M/N$ by Proposition 2.3. Therefore M/N is weakly F-supplemented. □

Lemma 3.15. *Let M be a module and $B \leq C \leq M$. Moreover suppose that $M = A + B$. Then $C/B \cong (A \cap C)/(A \cap B)$.*

Proof. Let $B' = A \cap C$, then $B' \cap B = A \cap B$. Now $B'/(B' \cap B) \cong (B + B')/B'$, so that $B + B' = A \cap C + B = C$. Hence $(A \cap C)/(A \cap B) \cong C/B$. □

Lemma 3.16. *Let M be a module such that $M = A + B = (A \cap B) + C$ for submodules A, B, C of M. Then $M = (B \cap C) + A = (A \cap C) + B$.*

Proof. See [5, Lemma 1.2]. □

Lemma 3.17. *Let M be a module such that $M = A+B$, for $A, B \leq M$. If $B \leq C$ and $C/B \ll_F M/B$, then $(A \cap C)/(A \cap B) \ll_F M/(A \cap B)$.*

Proof. By [5, Lemma 1.3], $(A \cap C)/(A \cap B) \ll M/(A \cap B)$ and by Lemma 3.15, $(A \cap B)/(A \cap C)$ is finitely generated. So $(A \cap C)/(A \cap B) \ll_F M/(A \cap B)$. □

Proposition 3.18. *Let M be a module and $B \leq C \leq M$. If C/B is an F-supplement submodule of M/B and B is an F-supplement submodule of M. Then C is an F-supplement submodule of M.*

In particular if M is weakly F-supplemented, then we can replace F-supplement by F-coclosed.

Proof. Let $M/B = C/B + C'/B$ and $C/B \cap C'/B \ll_F C/B$. Also suppose that $M = B + B'$ and $B \cap B' \ll_F B$, for $B \leq C' \leq M$ and $B' \leq M$. Since $B \leq C$ and also $B \leq C'$, we have $M = (C \cap C') + B'$. Also $M = C + C'$. These implies $M = C + (B' \cap C')$ by Lemma 3.16. Therefore it remains we show that $C \cap C' \cap B' \ll_F C$. For this, since $C = C \cap (B + B') = B + (C \cap B')$ and $(C \cap C')/B \ll_F C/B$, we obtain $(C \cap C' \cap B')/(B \cap B') \ll_F C/(B \cap B')$ by Lemma 3.17. Moreover $B \cap B' \ll_F C$. Now by Proposition 2.2, $C \cap C' \cap B' \ll_F C$. The last statement follows immediately from Lemma 3.1. □

Proposition 3.19. *Homomorphic images of amply F-supplemented modules are amply F-supplemented.*

Proof. Suppose that M is an amply F-supplemented module and $f : M \longrightarrow N$ is an epimorphism where N is an arbitrary module. Assume $N = N_1 + N_2$ for two submodules N_1, N_2 of N. Then $M = f^{-1}(N) = f^{-1}(N_1) + f^{-1}(N_2)$. So there exists $X \leq f^{-1}(N_2) \leq M$ such that $M = f^{-1}(N_1) + X$ and $X \cap f^{-1}(N_1) \ll_F X$. Thus $N = N_1 + f(X)$ and $N_1 \cap f(X) = f(f^{-1}(N_1) \cap X) \ll_F f(X)$ and also $f(X) \leq N_2$. Therefore N is amply F-supplemented. □

Let M be a module. Then M is said to have *finite uniform (Goldie) dimension* if, M does not contain an infinite set of independent submodules.

If $Sup\{k \in \mathbb{N} | M \text{ contains } k \text{ independent submodules}\} = n$, then M is said to have uniform dimension n and denoted by $u.dim(M) = n$. In this case M contains n independent uniform submodules $N_1, N_2, ..., N_n$ with $\bigoplus_{i=1}^n N_i \trianglelefteq_e M$. So there exists an essential monomorphism from a direct sum of n uniform modules to M. If $M = 0$ then we denote $u.dim(M) = 0$, else $u.dim(M) \geq 1$.

It is clear that if M has finite uniform dimension and $M = \bigoplus_{i=1}^n N_i$, then $u.dim(M) = \sum_{i=1}^n u.dim(N_i)$.

Suppose that N is a submodule of M and M has finite uniform dimension, then $u.dim(N) \leq u.dim(M)$.

Proposition 3.20. *Let M be a module. Then the following statements are equivalent:*

(i) $Rad(M)$ has finite uniform dimension;
(ii) Every F-small submodule of M has finite uniform dimension and there exists a positive integer n such that $u.dim(N) \leq n$ for every F-small submodule N of M;
(iii) M deos not contain an infinite direct sum of nonzero F-small Submodules.

Proof. $1 \Longrightarrow 2$: This is clear.

$2 \Longrightarrow 3$: Suppose that $N_1 \oplus N_2 \oplus ...$ is an infinite direct sum of non-zero F-small submodules of M. Then $N_1 \oplus N_2 \oplus ... \oplus N_{n+1}$ is F-small in M, and also

$u.dim(N_1 \oplus N_2 \oplus ... \oplus N_{n+1}) \geq n+1$, that is a contradiction by hypothesis. So M does not contain an infinite direct sum of non-zero F-small submodules.

$3 \Longrightarrow 1$: Let $K_1 \oplus K_2 \oplus ...$ be an infinite direct sum of non-zero submodules of $Rad(M)$ and $0 \neq x_i \in K_i$ for each $i \geq 1$. Then $x_i R \ll_F M$ by Lemma 2.1 and so $x_1 R \oplus x_2 R \oplus ...$ is an infinite direct sum of non-zero F-small submodules of M, that is a contradiction. So $Rad(M)$ has finite uniform dimension. □

Example 3.21. Suppose that M is a module with $Rad(M) = M$. Then by Proposition 3.20, M has finite uniform dimension iff M does not contain an infinite direct sum of non-zero F-small submodules. Abelean groups as $\mathbb{Z}$-modules have no maximal submodule, so such modules have finite uniform dimension iff they do not contain an infinite direct sum of non-zero F-small submodules. Especially $\mathbb{Q}_{\mathbb{Z}}$ has finite uniform dimension.

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THE FIXED POINT PROPERTY OF THE PRODUCTS AND HYPERSPACES OF ARBOROIDS

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Abstract. This paper is motivated by the Cauty's result [1] which states that if X is a dendroid, then 2^X and $C(X)$ have the fixed point property. The main purpose of this paper is to study the fixed point property of the hyperspaces of arboroids. It is proved, using the inverse systems method and Cauty's result, that if X is an arboroid and $f : 2^X \to 2^X$ is a mapping, then f has the fixed point property. Similar theorem it is proved for $f : C(X) \to C(X)$.

1. Introduction

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space X is denoted by $\mathrm{w}(X)$. The cardinality of a set A is denoted by $\mathrm{card}(A)$. We shall use the notion of inverse system as in [5, pp. 135-142]. An inverse system is denoted by $\mathbf{X} = \{X_a, p_{ab}, A\}$.

Let A be a partially ordered directed set. We say that a subset $A_1 \subset A$ *majorates* [3, p. 9] another subset $A_2 \subset A$ if for each element $a_2 \in A_2$ there exists an element $a_1 \in A_1$ such that $a_1 \geq a_2$. A subset which majorates A is called *cofinal* in A. A subset of A is said to be a *chain* if every two elements of it are comparable. The symbol $\sup B$, where $B \subset A$, denotes the lower upper bound of B (if such an element exists in A). Let $\tau \geq \aleph_0$ be a cardinal number. A subset B of A is said to be τ-*closed* in A if for each chain $C \subset B$, with $\mathrm{card}(B) \leq \tau$, we have $\sup C \in B$, whenever the element $\sup C$ exists in A. Finally, a directed set A is said to be τ-*complete* if for each chain C of A of elements of A with $\mathrm{card}(C) \leq \tau$, there exists an element $\sup C$ in A.

Suppose that we have two inverse systems $\mathbf{X} = \{X_a, p_{ab}, A\}$ and $\mathbf{Y} = \{Y_b, q_{bc}, B\}$. A *morphism of the system* X *into the system* $\mathbf{Y}$ [3, p. 15] is a family $\{\varphi, \{f_b : b \in B\}\}$ consisting of a nondecreasing function $\varphi : B \to A$ such that $\varphi(B)$ is cofinal in A, and of maps $f_b : X_{\varphi(b)} \to Y_b$ defined for all $b \in B$ such that the following

$$\begin{array}{ccc} X_{\varphi(b)} & \overset{p_{\varphi(b)\varphi(c)}}{\longleftarrow} & X_{\varphi(c)} \\ \downarrow f_b & & \downarrow f_c \\ Y_b & \overset{q_{bc}}{\longleftarrow} & Y_c \end{array}$$

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diagram commutes. Any morphism $\{\varphi, \{f_b : b \in B\}\} : \mathbf{X} \to \mathbf{Y}$ induces a map, called the *limit map of the morphism*

$$\lim\{\varphi, \{f_b : b \in B\}\} : \lim \mathbf{X} \to \lim \mathbf{Y}$$

In the present paper we deal with the inverse systems defined on the same indexing set A. In this case, the map $\varphi : A \to A$ is taken to be the identity and we use the following notation $\{f_a : X_a \to Y_a; a \in A\} : \mathbf{X} \to \mathbf{Y}$.

We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *factorizing* [3, p. 17] if for each real-valued mapping $f : \lim \mathbf{X} \to \mathbb{R}$ there exist an $a \in A$ and a mapping $f_a : X_a \to \mathbb{R}$ such that $f = f_a p_a$.

An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be *σ-directed* if for each sequence $a_1, a_2, ..., a_k, ...$ of the members of A there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

Lemma 1.1. [3, Corollary 1.3.2, p. 18]. *If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a σ-directed inverse system of compact spaces with surjective bonding mappings, then it is factorizing.*

An inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is said to be *τ-continuous* [3, p. 19] if for each chain B in A with $\mathrm{card}(B) < \tau$ and $\sup B = b$, the diagonal product $\Delta\{p_{ab} : a \in B\}$ maps the space X_b homeomorphically into the space $\lim\{X_a, p_{ab}, B\}$.

An inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is said to be a *τ-system* [3, p. 19] if:

a) $w(X_a) \leq \tau$ for every $a \in A$,

b) The system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is τ-continuous,

c) The indexing set A is τ-complete.

A *σ-system* is τ-system, where $\tau = \aleph_0$. The following theorem is called the *Spectral Theorem* [3, p. 19].

Theorem 1.2. [3, Theorem 1.3.4, p. 19]. *If a τ-system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ with surjective limit projections is factorizing, then each map of its limit space into the limit space of another τ-system $\mathbf{Y} = \{Y_a,\ q_{ab},\ A\}$ is induced by a morphism of cofinal and τ-closed subsystems. If two factorizing τ-systems with surjective limit projections and the same indexing set have homeomorphic limit spaces, then they contain isomorphic cofinal and τ-closed subsystems.*

Let us remark that the requirement of surjectivity of the limit projections of systems in Theorem 1.2 is essential [3, p. 21].

A *fixed point of a function* $f : X \to X$ is a point $p \in X$ such that $f(p) = p$. A space X is said to have the *fixed point property* provided that every mapping $f : X \to X$ has a fixed point.

In the sequel we shall use the following result.

Theorem 1.3. *Let $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ be a σ-system of compact spaces with limit X and onto projections $p_a : X \to X_a$. Let $\{f_a : X_a \to X_a\} : \mathbf{X} \to \mathbf{X}$ be a morphism. Then the induced mapping $f = \lim\{f_a\} : X \to X$ has a fixed point if and only if each mapping $f_a : X_a \to X_a$, $a \in A$, has a fixed point.*

As an immediate consequence of this theorem and the Spectral theorem 1.2 we have the following result.

Theorem 1.4. *Let a non-metric continuum X be the inverse limit of an inverse σ-system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ such that each X_a has the fixed point property and each bonding mapping p_{ab} is onto. Then X has the fixed point property.*

The following result enables to prove Theorem 1.6 as the application of Theorem 1.4.

Theorem 1.5. [7, Theorem 1.6, p. 402]. *If X is the Cartesian product $X = \prod\{X_s : s \in S\}$, where* $\mathrm{card}(S) > \aleph_0$ *and each X_s is compact, then there exists a σ-directed inverse system $\mathbf{X} = \{Y_a, P_{ab}, A\}$ of the countable products $Y_a = \prod\{X_\mu : \mu \in a\}$,* $\mathrm{card}(a) = \aleph_0$, *such that X is homeomorphic to* $\lim \mathbf{X}$.

Theorem 1.6. *Let S be an infinite set and $Q = \Pi\{X_s : s \in S\}$ Cartesian product of compact spaces. If each product $X_{s_1} \times X_{s_2} \times ... \times X_{s_n}$ of finitely many spaces X_s has the fixed point property, then Q has the fixed point property.*

Proof. We shall consider the following cases.

Case 1 $\mathrm{card}(S) = \aleph_0$. We may assume that $S = \mathbb{N}$. The proof is a straightforward modification of the proof of [10, Corollary 3.5.3, pp. 106-107]. Let $f : Q \to Q$ be continuous. For every $n \in \mathbb{N}$ define

$$K_n = \{x \in Q : (x_1, ..., x_n) = (f(x)_1, ..., f(x)_n)\}.$$

It is clear that for every n the set K_n is closed in Q and that $K_{n+1} \subset K_n$. For every $n \in \mathbb{N}$, let o_n be a given point of X_n and $p_n : Q \to X_1 \times ... \times X_n$ be the projection. Define continuous function $f_n : X_1 \times ... \times X_n \to X_1 \times ... \times X_n$ by

$$f_n(x_1, ..., x_n) = (p_n f)(x_1, ..., x_n, o_{n+1}, o_{n+2}, ...).$$

By assumption of Theorem f_n has the fixed point property, say $(x_1, ..., x_n)$. It follows that

$$(x_1, ..., x_n, o_{n+1}, o_{n+2}, ...) \in K_n.$$

We conclude that $\{K_n : n \in \mathbb{N}\}$ is a decreasing collection of nonempty closed subsets of Q. By compactness of Q we have that

$$K = \cap\{K_n : n \in \mathbb{N}\}$$

is nonempty. It is clear that every point in K is a fixed point of f.

Case 2 $\mathrm{card}(A) \geq \aleph_1$. By Theorem 1.5 there exists a σ-directed inverse system $\mathbf{X} = \{Y_a, P_{ab}, A\}$ of the countable products $Y_a = \prod\{X_\mu : \mu \in a\}$, $\mathrm{card}(a) = \aleph_0$, such that Q is homeomorphic to $\lim \mathbf{X}$. By Case 1 each Y_a has the fixed point property. Finally, by Theorem 1.4 we infer that Q has the fixed point property. □

A space X is said to be *rim-metrizable* if it has a basis $\mathcal{B}$ such that $\mathrm{Bd}(U)$ is metrizable for each $U \in \mathcal{B}$.

Theorem 1.7. [8, Theorem 9, p. 205]. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces and surjective bonding mappings p_{ab}. Then:*

- **1):** *There exists an inverse system $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ of compact spaces such that m_{ab} are monotone surjections and* $\lim \mathbf{X}$ *is homeomorphic to* $\lim M(\mathbf{X})$,
- **2):** *If $\mathbf{X}$ is σ-directed, then $M(\mathbf{X})$ is σ-directed,*
- **3):** *If $\mathbf{X}$ is σ-complete, then $M(\mathbf{X})$ is σ-complete,*
- **4):** *If every X_a is a metric space and* $\lim \mathbf{X}$ *is locally connected (a rim-metrizable continuum), then every M_a is metrizable.*

REMARK. Let us observe that the projections $m_a : \lim M(\mathbf{X}) \to M_a, a \in A$, are monotone. In the case of the locally connected spaces or the rim-metrizable continua, we have the following result.

Theorem 1.8. [8, Theorem 10, p. 207]. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a* σ*-system of compact spaces and surjective bonding mappings* p_{ab}. *If* $\lim \mathbf{X}$ *is a locally connected space (a rim-metrizable continuum), then there exists an* $a \in A$ *such that the projection* p_b *is monotone, for every* $b \geq a$.

For a compact space X we denote by 2^X the hyperspace of all nonempty closed subsets of X equipped with the Vietoris topology. $C(X)$ and $X(n)$, where n is a positive integer, stand for the sets of all connected members of 2^X and of all nonempty subsets consisting of at most n points, respectively, both considered as subspaces of 2^X.

For a mapping $f : X \to Y$ define $2^f : 2^X \to 2^Y$ by $2^f(F) = f(F)$ for $F \in 2^X$. It is known that 2^f is continuous, $2^f(C(X)) \subset C(Y)$ and $2^f(X(n)) \subset Y(n)$. The restriction $2^f|C(X)$ is denoted by $C(f)$.

An element $\{x_a\}$ of the Cartesian product $\prod\{X_a : a \in A\}$ is called a *thread* of $\mathbf{X}$ if $p_{ab}(x_b) = x_a$ for any a, $b \in A$ satisfying $a \leq b$. The subspace of $\prod\{X_a : a \in A\}$ consisting of all threads of $\mathbf{X}$ is called the limit of the inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ and is denoted by $\lim \mathbf{X}$ or by $\lim\{X_a, p_{ab}, A\}$ [5, p. 135].

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces with the natural projections $p_a : \lim \mathbf{X} \to X_a$, for $a \in A$. Then $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$, $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ and $\mathbf{X}(n) = \{X_a(n), 2^{p_{ab}}|\, X_b(n), A\}$ form inverse systems. For each $F \in 2^{\lim \mathbf{X}}$, i.e., for each closed $F \subseteq \lim \mathbf{X}$ the set $p_a(F) \subseteq X_a$ is closed and compact. Thus, we have a mapping $2^{p_a} : 2^{\lim \mathbf{X}} \to 2^{X_a}$ induced by p_a for each $a \in A$. Define a mapping $M : 2^{\lim \mathbf{X}} \to \lim 2^{\mathbf{X}}$ by $M(F) = \{p_a(F) : a \in A\}$. Since $\{p_a(F) : a \in A\}$ is a thread of the system $2^{\mathbf{X}}$, the mapping M is continuous and one-to-one. It is also onto since for each thread $\{F_a : a \in A\}$ of the system $2^{\mathbf{X}}$ the set $F' = \bigcap\{p_a^{-1}(F_a) : a \in A\}$ is non-empty and $p_a(F') = F_a$. Thus, M is a homeomorphism. If $P_a : \lim 2^{\mathbf{X}} \to 2^{X_a}, a \in A$, are the projections, then $P_a M = 2^{p_a}$. Identifying F with $M(F)$ we have $P_a = 2^{p_a}$.

Lemma 1.9. . *Let* $X = \lim \mathbf{X}$. *Then* $2^X = \lim 2^{\mathbf{X}}$, $C(X) = \lim C(\mathbf{X})$ *and* $X(n) = \lim \mathbf{X}(n)$.

2. The arboroids as the inverse limit space of dendroids

A continuum X with precisely two non-separating points is called a *generalized arc.*

A *simple* n*-od* is the union of n generalized arcs $A_1O, A_2O, ..., A_\alpha O$, each two of which have only the point O in common. The point O is called the *vertex* or the *top* of the n-od.

By a *branch point* of a compact space X we mean a point p of X which is the vertex of a simple triod lying in X. A point $x \in X$ is said to be *end point* of X if for each neighborhood U of x there exists a neighborhood V of x such that $V \subset U$ and $\text{card}(Bd(V)) = 1$.

Let S be the set of all end points and of all branch points of a continuum X. An arc pq in X is called a *free arc* in X if $pq \cap S = \{p, q\}$.

A continuum is a *graph* if it is the union of a finite number of metric free arcs. A *tree* is an acyclic graph.

A continuum X is *tree-like* (*arc-like*) if for each open cover $\mathcal{U}$ of X, there is a tree (arc) $X_{\mathcal{U}}$ and a $\mathcal{U}$-mapping $f_{\mathcal{U}} : X \to X_{\mathcal{U}}$ (the inverse image of each point is contained in a member of $\mathcal{U}$).

Every tree-like continuum is hereditarily unicoherent.

A non-metric hereditarily unicoherent continuum which is arcwise connected by generalized arcs is said to be an *arboroid*. A metrizable hereditarily unicoherent continuum which is arcwise connected is said to be a *dendroid*. Every arboroid is tree-like [4, Corollary, p. 20]. If X is an arboroid and $x, y \in X$, then there exists a unique arc $[x, y]$ in X with endpoints x and y. If $[x, y]$ is an arc, then $[x, y] \setminus \{x, y\}$ is denoted by (x, y).

A point t of an arboroid X is said to be a *ramification point* of X if t is the only common point of some three arcs such that it is the only common point of any two, and an end point of each of them.

A point e of an arboroid X is said to be *end point* of X if there exists no arc $[a, b]$ in X such that $x \in [a, b] \setminus \{a, b\}$.

Let Y^X be the set of all mappings of X to Y. If Y is a metric space with a metric d, then on the set Y^X one can define a metric $\widehat{d}$ by letting

$$\widehat{d}(f, g) = \sup_{x \in X} d\left(f(x), g(x)\right).$$

Proposition 1. *Let X be any tree-like continuum, let P be a polyhedron with a given metric d, $r > 0$ a real number and $f : X \to P$ a mapping. Then there exist a tree Q, a mapping $g : X \to Q$ and a mapping $p : Q \to P$ such that $g(X) = Q$ and $\widehat{d}(f, pg) \leq r$.*

Proof. Let K be a triangulation of P of mesh not greater than $r/2$. Let a_i be the vertices of K, and let St a_i be the open star of K around the vertex a_i. Hence, $\{\text{St } a_i\}$ is an open covering for P, and so is $\mathcal{U} = \{f^{-1}(\text{St } a_i)\}$ for X. There exist a tree Q and a mapping $g : X \to Q$ such that g is an $\mathcal{U}$-mapping and $g(X) = Q$. There exists a triangulation L of Q with vertices b_j such that the cover $\mathcal{V} = \{g^{-1}(\text{St } b_j)\}$ refines the cover $\mathcal{U}$. Let x be a point of X and let s be a simplex of Q with vertices $b_{j_1}, ..., b_{j_k}$ containing $g(x)$. This means that $\left\{g^{-1}(\text{St } b_{j_1}), ..., g^{-1}(\text{St } b_{j_k})\right\}$ is a collection of some $g^{-1}(\text{St } b_j)$ containing x. It follows that $g^{-1}(\text{St } b_{j_1}) \cap ... \cap g^{-1}(\text{St } b_{j_k}) \neq \emptyset$. We infer that St $b_{j_1} \cap ... \cap$ St $b_{j_k} \neq \emptyset$. Let $p : Q \to P$ be a simplicial mapping sending each vertex b_j of Q into a vertex a_i having the property that $g^{-1}(\text{St } b_i) \subset f^{-1}(\text{St } a_i)$. It remains to prove that $d(f, pg) \leq r$. Now, for each $g^{-1}(\text{St } b_{i_j})$ we have some $f^{-1}(\text{St } a_{i_j})$ with $g^{-1}(\text{St } b_{i_j}) \subset f^{-1}(\text{St } a_{i_j})$. From $g^{-1}(\text{St } b_{j_1}) \cap ... \cap g^{-1}(\text{St } b_{j_k}) \neq \emptyset$ it follows that $f^{-1}(\text{St } b_{j_1}) \cap ... \cap f^{-1}(\text{St } b_{j_k}) \neq \emptyset$, i.e., that there exists a simplex σ of K with vertices $b_{j_1}, ..., b_{j_k}$ such that $f(x) \in \text{St } \sigma$. Clearly, $pg(x) \in \text{St } \sigma$. Finally, $\widehat{d}(f, pg) \leq r$. □

Proposition 2. *If $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is an inverse system of tree-like continua and if p_{ab} are onto mappings, then the limit $X = \lim \mathbf{X}$ is a tree-like continuum.*

Proof. Let $\mathcal{U} = \{U_1, ..., U_n\}$ be an open covering of X. There exist an $a \in A$ and an open covering $\mathcal{U}_a = \{U_{1a}, ..., U_{ka}\}$ such that $\left\{p_a^{-1}(U_{1a}), ..., p_a^{-1}(U_{ka})\right\}$ refines the covering $\mathcal{U}$. There exist a tree T_a and a $\mathcal{U}_a$-mapping $f_{\mathcal{U}_a} : X_a \to T_a$ since X_a is tree-like. It is clear that $f_{\mathcal{U}_a} p_a : X \to T_a$ is a $\mathcal{U}$-mapping. Hence, X is tree-like. □

Proposition 3. *If X is a tree-like continuum, Q a tree and $f : X \to Q$ is a mapping, then $f(X)$ also is a tree.*

Proof. This follows from the fact that a subcontinuum of a tree is a tree. □

The following result is an expanding theorem of tree-like continua into inverse σ-systems of metric tree-like continua.

Theorem 2.1. *If X is a tree-like non-metric continuum, then there exists a σ-system $\mathbf{X}_\sigma = \{X_\Delta,\, P_{\Delta\Gamma},\, A_\sigma\}$ of metric tree-like continua X_Δ and onto mappings $P_{\Delta\Gamma}$ such that X is homeomorphic to* $\lim \mathbf{X}_\sigma$.

Proof. Let us observe that Propositions 1-3 are the conditions (A)-(C) in [9, p. 220]. Then from Mardešić's General Expansion Theorem [9, Theorem 2] it follows that there exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric tree-like continua X_a and onto bonding mappings p_{ab} such that X is homeomorphic to $\lim \mathbf{X}$. It remains to prove that there exists such σ-system. The proof is broken into several steps.

Step 1. For each subset Δ_0 of $(A, \leq)$ we define sets Δ_n, $n = 0, 1, ...$, by the inductive rule $\Delta_{n+1} = \Delta_n \bigcup \{m(x, y) : x,\, y \in \Delta_n\}$, where $m(x, y)$ is a member of A such that $x, y \leq m(x, y)$. Let $\Delta = \bigcup\{\Delta_n : n \in \mathbb{N}\}$. It is clear that $\mathrm{card}(\Delta) = \mathrm{card}(\Delta_0)$. Moreover, Δ is directed by $\leq$. For each directed set $(A, \leq)$ we define

$$A_\sigma = \{\Delta : \emptyset \neq \Delta \subset A,\, \mathrm{card}(\Delta) \leq \aleph_0 \quad \text{and } \Delta \text{ is directed by } \leq\}.$$

Step 2. *If A is a directed set, then A_σ is σ-directed and σ-complete.* Let $\{\Delta^1, \Delta^2, ..., \Delta^n, ...\}$ be a countable subset of A_σ. Then $\Delta_0 = \cup\{\Delta^1, \Delta^2, ..., \Delta^n, ...\}$ is a countable subset of A_σ. Define sets Δ_n, $n = 0, 1, ...$, by the inductive rule $\Delta_{n+1} = \Delta_n \bigcup \{m(x, y) : x,\, y \in \Delta_n\}$, where $m(x,\, y)$ is a member of A such that $x, y \leq m(x, y)$. Let $\Delta = \bigcup\{\Delta_n\colon\, n \in \mathbb{N}\}$. It is clear that $\mathrm{card}(\Delta) = \mathrm{card}(\Delta_0)$. This means that Δ is countable. Moreover $\Delta \supseteq \Delta^i$, $i \in \mathbb{N}$. Hence A_σ is σ-directed. Let us prove that A_σ is σ-complete. Let $\Delta^1 \subset \Delta^2 \subset ... \subset \Delta^n \subset ...$ be a countable chain in A_σ. Then $\Delta = \cup\{\Delta^i : i \in \mathbb{N}\}$ is countable and directed subset of A, i.e., $\Delta \in A_\sigma$. It is clear that $\Delta \supseteq \Delta^i$, $i \in \mathbb{N}$. Moreover, for each $\Gamma \in A_\sigma$ with property $\Gamma \supseteq \Delta^i$, $i \in \mathbb{N}$, we have $\Gamma \supseteq \Delta$. Hence $\Delta = \sup\{\Delta^i : i \in \mathbb{N}\}$. This means that A_σ is σ-complete.

Step 3. If $\Delta \in A_\sigma$, let $\mathbf{X}^\Delta = \{X_b,\, p_{bb'},\, \Delta\}$ and $X_\Delta = \lim \mathbf{X}^\Delta$. If $\Delta, \Gamma \in A_\sigma$ and $\Delta \subseteq \Gamma$, let $P_{\Delta\Gamma}\colon\; X_\Gamma \to X_\Delta$ denote the map induced by the projections $p^\Gamma_\delta : X_\Gamma \to X_\delta$, $\delta \in \Delta$, of the inverse system $\mathbf{X}^\Gamma$.

Step 4. *If $\mathbf{X} = \{X_a,\, p_{ab},\, A\}$ is an inverse system, then $\mathbf{X}_\sigma = \{X_\Delta,\, P_{\Delta\Gamma},\, A_\sigma\}$ is a σ-directed and σ-complete inverse system such that $\lim \mathbf{X}$ and $\lim \mathbf{X}_\sigma$ are homeomorphic.* Each thread $x = (x_a : a \in A)$ induces the thread $(x_a : a \in \Delta)$ for each $\Delta \in A_\sigma$, i.e., the point $x_\Delta \in X_\Delta$. This means that we have a mapping $H : \lim \mathbf{X} \to \lim \mathbf{X}_\sigma$ such that $H(x) = (x_\Delta : \Delta \in A_\sigma)$. It is obvious that H is continuous and 1-1. The mapping H is onto since the collections of the threads $\{x_\Delta : \Delta \in A_\sigma\}$ induces the thread in $\mathbf{X}$. We infer that H is a homeomorphism since $\lim \mathbf{X}$ is compact.

Step 5. *Every X_Δ is a metric tree-like continuum.* Apply Proposition 2.

Step 6. *Every projection $P_\Delta : \lim \mathbf{X}_\sigma \to X_\Delta$ is onto.* This follows from the assumption that the bonding mappings p_{ab} are surjective.

Finally, $\mathbf{X}_\sigma = \{X_\Delta,\, P_{\Delta\Gamma},\, A_\sigma\}$ is a desired σ-system. $\square$

Now we shall prove an expanding theorem of arboroids into inverse σ-systems of dendroids.

Theorem 2.2. *If X is an arboroid, then there exists a σ-system $\mathbf{X} = \{X_a,\, p_{ab},\, A\}$ of dendroids X_a and onto mappings p_{ab} such that X is homeomorphic to* $\lim \mathbf{X}$.

Proof. Firstly we recall that each arboroid is tree-like [4, Corollary, p. 20]. Then from Theorem 2.1 it follows that there exists an inverse σ-system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ of metric tree-like continua X_a and onto bonding mappings p_{ab} such that X is homeomorphic to $\lim \mathbf{X}$. It remains to prove that there every X_a is a dendroid. By the fact that X_a is tree-like it follows that X_a is unicoherent. Moreover, it is metric. It remains to prove that X_a is arcwise connected. Let a, b be a pair of points of X_a. There exists a pair x, y of points of X such that $a = P_a(x)$ and $b = P_a(y)$. There exist a unique arc xy in X with end points x and y since X is arcwise connected. Now, $P_a(xy)$ is arcwise connected [13]. This means that there is an arc ab with end points a and b. Thus, X_a is a dendroid. □

A *non-metric or generalized dendrite* is a locally connected arboroid. From Theorem 2.2 we obtain the following result.

Theorem 2.3. *If X is a generalized dendrite, then there exists a σ-system $\mathbf{X}_\sigma = \{X_\Delta,\ P_{\Delta\Gamma},\ A_\sigma\}$ of metric dendrites X_Δ and onto monotone projections P_Δ such that X is homeomorphic to* $\lim \mathbf{X}_\sigma$.

Proof. From Theorem 2.2 we have a σ-system $\mathbf{X}_\sigma = \{X_\Delta,\ P_{\Delta\Gamma},\ A_\sigma\}$ of metric dendroids X_Δ and onto mappings $P_{\Delta\Gamma}$ such that X is homeomorphic to $\lim \mathbf{X}_\sigma$. It suffices to prove that every X_Δ is locally connected. This follows from [14, Lemma 1.5, p. 70]. Moreover, by Theorem 1.8 it follows that there exists an $a \in A$ such that the projection P_b is monotone, for every $b \geq a$. □

By similar method of proof we have the following result.

Theorem 2.4. *If X is a rim-metrizable arboroid, then there exists a σ-system $\mathbf{X}_\sigma = \{X_\Delta,\ P_{\Delta\Gamma},\ A_\sigma\}$ of dendroids X_Δ and onto monotone projections P_Δ such that X is homeomorphic to* $\lim \mathbf{X}_\sigma$.

A λ-*dendroid* is an hereditarily decomposable, hereditarily unicoherent continuum. A λ-dendroid is tree-like [4, Corollary, p. 20].

Theorem 2.5. *If X is a non-metric rim-metrizable λ-dendroid, then there exists a σ-system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ of metric λ-dendroids X_a and onto mappings p_{ab} such that X is homeomorphic to* $\lim \mathbf{X}$.

Proof. From Theorem 2.1 it follows that there exists a σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric tree-like continua X_a and onto mappings p_{ab} such that X is homeomorphic to $\lim \mathbf{X}$ since λ-dendroid X is tree-like.We infer that each X_a is unicoherent. It remains to prove that every X_a is hereditarily decomposable. By Theorem 1.8 there exists an $a \in A$ such that the projection p_b is monotone, for every $b \geq a$. Using Theorem 14 of [2, p. 217] (see also [12, p. 297]) we conclude that every X_a is λ-dendroid. □

3. The fixed point property of the hyperspaces of arboroids

Now we shall investigate the fixed point property of the hyperspaces of arboroids. Let us recall the following known results.

Theorem 3.1. [1, Theorem 1, p. 1]. *For every dendroid X, every tree T_0 contained in X and every $\varepsilon > 0$, there exists a tree T contained in X and containing T_0 and an ε-retraction of X onto T.*

Theorem 3.2. [1, Corollaire 1, p. 1]. *If X is a dendroid, then 2^X and $C(X)$ have the fixed point property.*

By this theorem and Theorems 1.4 and 1.9 we shall prove the following result.

Theorem 3.3. *If X is an arboroid, then 2^X has the fixed point property.*

Proof. By Theorem 2.2 there exists a σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of dendroids such that all the bonding mappings p_{ab} are surjective and the limit $\lim \mathbf{X}$ is homeomorphic to X. Now we have the inverse system $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$ whose limit is 2^X (Lemma 1.9). It is clear that the mappings $2^{p_{ab}}$ are onto if the bonding mappings p_{ab} are onto. Now we can apply Theorem 1.4 since, by Theorem 3.2, every 2^{X_a} has the fixed point property. Hence, 2^X has the fixed point property. □

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-system. If we consider the inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$, then $C(p_{ab})$ are not always the surjections. This is the case only if p_{ab} are weakly confluent mappings [11, Theorem (0.49.1), p. 24]. This means that we can apply Theorem 1.4 on the system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ only if p_{ab} are weakly confluent mappings. Let us recall that a mapping $f : X \to Y$ is *weakly confluent* provided that for each subcontinuum K of Y there exists a component A of $f^{-1}(K)$ such that $f(A) = K$ [11, (0.45.4), p. 22]. Each monotone mapping is weakly confluent. It follows that expanding Theorem 2.2 is not enough for proving the fixed point property of $C(X)$ when X is an arboroid. For this reason we shall consider the fixed point property for 2^X and $C(X)$ if X is a rim-metrizable arboroid.

Theorem 3.4. *If X is a rim-metrizable arboroid, then $C(X)$ has the fixed point property.*

Proof. By Theorem 2.4 there exists a σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of dendroids such that all the bonding mappings p_{ab} are monotone surjections and the limit $\lim \mathbf{X}$ is homeomorphic to X. It is clear that the mappings $C(p_a)$ are onto if the bonding mappings p_a are monotone. Now we can apply Theorem 1.4 since, by Theorem 3.2, every $C(X_a)$ has the fixed point property. Hence, $C(X)$ has the fixed point property. □

Similarly, by Theorem 2.3, on can prove the following result.

Theorem 3.5. *If X is a generalized dendrite, then 2^X and $C(X)$ have the fixed point property.*

Let Y be a topological space. The *cone* $\mathrm{Cone}(Y)$ over Y is the quotient space obtained from $Y \times [0,1]$ by shrinking $Y \times \{1\}$ to a point. This point is called the *vertex* of $\mathrm{Cone}(Y)$. The subset $Y \times \{1\}$ of $\mathrm{Cone}(Y)$ is called the *base* of $\mathrm{Cone}(Y)$.

The following result generalize Theorem 22.15 of [6, p. 195].

Theorem 3.6. *Let $X = \mathrm{Cone}(Y)$, where Y is an arboroid. Then 2^X has the fixed point property. Moreover, if Y is rim-metrizable, then $C(X)$ has the fixed point property.*

The *suspension* $\Sigma(Y)$ *over a topological space* Y is the quotient space obtained from $Y \times [-1,1]$ by shrinking $Y \times \{-1\}$ and $Y \times \{1\}$ to different points point.

By the similar method of proof one can get the following result which generalize Theorem 22.16 of [6, p. 196].

Theorem 3.7. *Let $X = \Sigma(Y)$, where Y is an arboroid. Then 2^X has the fixed point property. Moreover, if Y is rim-metrizable, then $C(X)$ has the fixed point property.*

4. Fixed point property for a product of arboroids

In the sequel we shall use the following result.

Proposition 4. [5, Exercise 2.5.D.(b), p. 143]. *Let $\mathbf{S}(s) = \{X(s)_\alpha, p(s)_{ab}, A\}$ be an inverse system for every $s \in S$. Then*

$$\Pi\{\mathbf{S}(s) : s \in S\} = \{\Pi\{X(s)_a : s \in S\}, \Pi\{p(s)_{ab} : s \in S\}, A\}$$

is an inverse system and $\lim(\Pi\{\mathbf{S}(s) : s \in S\})$ *is homeomorphic to* $\Pi\{\lim \mathbf{S}(s) : s \in S\}$.

In this Section we shall generalize the following result in two directions.

Theorem 4.1. [1, Corollaire 2, p. 1]. *Each product of dendroids has the fixed point property.*

Theorem 4.2. *Let $X = \prod\{X(s) : s \in S\}$ be a product of arboroids such that $\mathrm{w}(X(s)) = \tau$ for every $s \in S$ and for cardinal number τ. Then X has the fixed point property.*

Proof. If for every $s \in S$ we have an arboroid $X(s)$, then, for every $s \in S$, there exists a σ-directed inverse system $\mathbf{X}(s) = \{X_a(s), p_{ab}(s), A(s)\}$ such that $X(s)$ is homeomorphic to $\lim \mathbf{X}(s)$ and every $X_a(s)$ is a dendroid (Theorem 2.2). If $\mathrm{w}(X(s_1)) = \mathrm{w}(X(s_2)), s_1, s_2 \in S$, then $A(s_1) = A(s_2)$ and we may suppose that $A(s) = A$ for every $s \in S$. By Theorem 4 the family $\Pi\{\mathbf{X}(s) : s \in S\} = \{\Pi\{X_a(s) : s \in S\}, \Pi\{ p_{ab}(s) : s \in S\}, A\}$ is an inverse system and $\lim(\Pi\{\mathbf{X}(s) : s \in S\})$ is homeomorphic to $\Pi\{\lim \mathbf{X}(s) : s \in S\}$. From Theorem 4.1 it follows that each $\Pi\{X_a(s) : s \in S\}$ has the fixed point property. Finally, from Theorem 1.4 it follows that $\Pi\{X(s) : s \in S\}$ has the fixed point property. □

For $\mathrm{card}(A) = 1$ we have the following result.

Corollary 4.3. *Every arboroid has the fixed point property.*

QUESTION. Is it true that the assumption "*of the same weight*" in Theorem 4.2 can be omitted?

We close this section with the result which generalize Theorem 4.1 in the another direction.

Theorem 4.4. *Let X be an arboroid and let $\{D_m : m \in M\}$ be a family of dendroids. The $X \times \Pi\{D_m : m \in M\}$ has the fixed point property.*

Proof. By Theorem 2.2 there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that X is homeomorphic to $\lim \mathbf{X}$ and every X_a is a dendroid. From Theorem 4 it follows that $X \times \Pi\{D_m : m \in M\}$ is homeomorphic to the limit of the system

$$\mathbf{X} \times \Pi\{D_m : m \in M\} = \{X_a \times \Pi\{D_m : m \in M\}, p_{ab} \times identity, A\}.$$

Each $X_a \times \Pi\{D_m : m \in M\}$ has the fixed point property (Theorem 4.1). Finally, by Theorem 1.4, $X \times \Pi\{D_m : m \in M\}$ has the fixed point property □

5. The fixed point property of the hyperspaces of the products of arboroids

In this section we shall use the following result from [6, Exercise 22.20, p. 197].

Proposition 5. *Let $X = \Pi\{X_i : i \leq n \leq \infty\}$ be a product of metric continua. Assume that for each i and each $\varepsilon > 0$ there is a continuous function $f_{i,\varepsilon} : X_i \to f_{i,\varepsilon}(X_i) \subset X_i$, where $f_{i,\varepsilon}(X_i)$ is locally connected and $f_{i,\varepsilon}$ is within ε of the identity map on X. Then 2^X and $C(X)$ have the fixed point property.*

We shall prove that Proposition 5 is true for every product of metric continua.

Theorem 5.1. *Let $X = \Pi\{X_a : a \in A\}$ be a product of metric continua. Assume that for each a and each $\varepsilon > 0$ there is a continuous function $f_{a,\varepsilon} : X_a \to f_{a,\varepsilon}(X_a) \subset X_a$, where $f_{a,\varepsilon}(X_a)$ is locally connected and $f_{a,\varepsilon}$ is within ε of the identity map on X. Then 2^X and $C(X)$ have the fixed point property.*

Proof. By Theorem 1.5 if $X = \prod\{X_a : a \in A\}$, where $\mathrm{card}(A) \ > \aleph_0$ and each X_a is compact, then there exists a σ-directed inverse system $\mathbf{X} = \{Y_a, P_{ab}, A\}$ of the countable products $Y_a = \prod\{X_\mu : \mu \in a\}$, $\mathrm{card}(a) = \aleph_0$, such that X is homeomorphic to $\lim \mathbf{X}$. Moreover, $P_{ab} : Y_b \to Y_a$ is a projection. This means that if $X_a, a \in A$, are continua, then $P_{ab}, a \leq b$, are monotone. We infer that the systems $2^{\mathbf{X}} \ = \{2^{X_a}, 2^{p_{ab}}, A\}$ and $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ have the the surjective bonding mappings. This means that one can apply Theorem 1.4 since each 2^{X_a} and $C(X_a)$ has the fixed point property (Theorem 5) and the projections $P_a : \lim \mathbf{X} \to Y_a$ are surjections. □

Applying Theorems 3.1 and 5.1 one can get the following result which generalize Theorems 3.2 and 3.3.

Theorem 5.2. *Let $X = \Pi\{X_a : a \in A\}$ be a product of dendroids. Then 2^X and $C(X)$ have the fixed point property.*

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ON SUPERELLIPTIC CURVES OF LEVEL n AND THEIR QUOTIENTS, I.

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Abstract. We study families of superelliptic curves with fixed automorphism groups. Such families are parametrized with invariants expressed in terms of the coefficients of the curves. Algebraic relations among such invariants determine the lattice of inclusions among the loci of superelliptic curves and their field of moduli. We give a Maple package of how to compute the normal form of an superelliptic curve and its invariants. A complete list of all superelliptic curves of genus $g \leq 10$ defined over any field of characteristic $\neq 2$ is given in a subsequent paper [3].

1. Introduction

Let $\mathcal{X}_g$ be an algebraic curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p \neq 2$. What is the group of automorphisms of $\mathcal{X}_g$ over k? Given the group of automorphisms G of a genus g curve, can we determine the equation of the curve? These two questions have been studied for a long time and a complete answer is not known for either one. There are some families of

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curves where we can answer completely the above questions, such as the hyperelliptic curves. The Klein's curve was the first celebrated example of a non-hyperelliptic curve where the the automorphisms of the curve and its equation are shown explicitly. The main purpose of this paper is to show that we can do this for a larger family of curves.

In characteristic zero, the first question is answered by work of Magaard, Shaska, Shpectorov, Völklein, et al. Based on previous work of Breuer and using computer algebra systems as GAP, they show how one can compute the list of full automorphism groups for any fixed genus $g \geq 2$. It is still an unsettled question the case of positive characteristic, where many tedious cases of wild ramifications need to be considered. The second question is unsettled even in characteristic zero. It is much harder to determine a parametric equation for the curve, given its group of automorphisms G.

However, if we go through the lists of groups G which occur as automorphism groups of genus g curves we notice, as to be expected, that the majority of them have the following property; there is a central element $\tau \in G$ such that the quotient space $\mathcal{X}_g / \langle \tau \rangle$ has genus zero. Such curves in the literature are called *superelliptic curves* or *cyclic curves*. For the purposes of this paper we will use the term **superelliptic curves of level** n.

Hence, for a fixed genus g, certain families of curves have equation $y^n = f(x)$, for some $n \in \mathbb{Z}$ and a generic polynomial $f(x) \in k[x]$. The values of n depend solely on the genus g and the field k. Such cases we call them *root cases* or *fundamental cases*. For a given n let $\mathcal{H}_n$ denote a connected component of the space of genus g curves with equation as above. Isomorphism classes of curves in $\mathcal{H}_n$ are determined by the invariants of degree n binary forms. Such invariants were the main focus of classical invariant theory in the 19-th century and they are only known for $n \leq 8$. Even for $n \leq 8$ the expressions of such invariants in terms of the coefficients of $f(x)$ are quite long and not so convenient for computations.

If the curve has an additional automorphism then this automorphism has to permute the roots of $f(x)$. In this case, additional invariants can be defined in terms of the coefficients of $f(x)$. These invariants were first discovered by Shaska for genus two curves in [29] and then generalized by Shaska/Gutierrez for all hyperelliptic curves in [13], where they were called **dihedral invariants**. Moreover, in [13] was determined a relation among such invariants, for any genus g, in the case of hyperelliptic curves with an extra involution. In [12] algebraic relations among such invariants were computed for the case of genus three hyperelliptic curves and a method was described how to compute such relations in general. Extending work done by Gutierrez/Shaska in [13], Antoniadis and Kontogeorgis defined these invariants [1] for cyclic covers of $\mathbb{P}^1(k)$ for positive characteristic. In these paper we will call them **$\mathfrak{s}$-invariants** and will describe how to compute them for any genus g superelliptic curve of level n.

Superelliptic curves are quite important in many applications. They are the only curves where we fully understand the automorphism groups for every characteristic and can associate an equation of the curve in each case of the group. The full groups of automorphisms of superelliptic curves defined over a field of characteristic zero followed from previous work of Magaard et al, [14]. However, for the first time a complete list of full automorphism groups of superelliptic curves for odd characteristic was determined by Sanjeewa in [16]. The equations for each family, when the full automorphism group was fixed, were determined by Sanjeewa and Shaska in [15]. Such curves were further studied in [2, 4], where singular subloci of $\mathcal{M}_2$ were studied, in applications in coding in [9]. The $\mathfrak{s}$-invariants which we study in section 4 were discovered in [29] and used by several authors in many applications since them. For further applications of such invariants one can check [3, 6, 7, 17–19, 19–27, 29, 30].

In this paper we give a list of automorphism groups of superelliptic curves of genus g and the corresponding equation for each group. We define invariants for such curves and give algorithms how to compute such invariants and how to determine algebraic relations among them. Such computations are completely done in the case of genus 3, in order to provide some general idea of the genus $g > 3$ case.

Notation: Throughout this paper by g we denote an integer ≥ 2 and k denotes an algebraically closed field of characteristic $\neq 2$. Unless otherwise noted, by a "curve" we always mean the isomorphism class of an algebraic curve defined over k. The automorphism group of a curve always means the full automorphism group of the curve.

2. Preliminaries on automorphisms of the projective line.

In this section we set the notation and describe briefly some general facts. Fix an integer $g \geq 2$. Let $\mathcal{X}_g$ denote a genus g generic curve defined over an algebraically closed field k of characteristic $p \geq 0$. We denote by G the full automorphism group of $\mathcal{X}_g$. Hence, G is a finite group. Denote by K the function field of $\mathcal{X}_g$ and assume that the affine equation of $\mathcal{X}_g$ is given some polynomial in terms of x and y.

Let $H = \langle \tau \rangle$ be a cyclic subgroup of G such that $|H| = n$ and $H \lhd G$, where $n \geq 2$. Moreover, we assume that the quotient curve $\mathcal{X}_g/H$ has genus zero. The **reduced automorphism group of $\mathcal{X}_g$ with respect to H** is called the group $\bar{G} := G/H$, see [10, 16].

Assume $k(x)$ is the genus zero subfield of K fixed by H. Hence, $[K : k(x)] = n$. Then, the group $\bar{G}$ is a subgroup of the group of automorphisms of a genus zero field. Hence, $\bar{G} < PGL_2(k)$ and $\bar{G}$ is finite. It is a classical result that every finite subgroup of $PGL_2(k)$ is isomorphic to one of the following: C_m, D_m, A_4, S_4, A_5, *semidirect product of elementary Abelian group with cyclic group*, $PSL(2, q)$ and $PGL(2, q)$, see [31].

The group $\bar{G}$ acts on $k(x)$ via the natural way. The fixed field of this action is a genus 0 field, say $k(z)$. Thus, z is a degree $|\bar{G}| := m$ rational function in x, say $z = \phi(x)$. We illustrate with the following diagram:

$$
\begin{tikzcd}
K = k(x,y) \arrow[d, dash, "n"', "H"] & \mathcal{X}_g \arrow[d, "\phi_0"', "H"] \\
k(x) = k(x, y^n) \arrow[d, dash, "m"', "\bar{G}"] & \mathbb{P}^1(k) \arrow[d, "\phi"', "\bar{G}"] \\
E = k(z) & \mathbb{P}^1(k)
\end{tikzcd}
$$

Figure 1. The automorphism groups and the corresponding covers

It obvious that G is a degree n extension of $\bar{G}$ and $\bar{G}$ is a finite subgroup of $PGL_2(k)$. Hence, if we know all the possible groups that occur as $\bar{G}$ then hopefully we can figure out G and the equation for K.

To do this we have to recall some classical result on finite subgroups of the projective linear group $PGK_2(k)$ and their fixed fields. First we define a semidirect product of an elementary Abelian group with a cyclic group as follows, see [31] for details.

Let char $k = p$ and $k = \mathbb{F}_q$ for $q = p^r$. For each $m \,|\, (p^t - 1)$, $t = 1, \dots, r$, we define $\mathcal{U}_m$ as follows

$$\mathcal{U}_m := \{a \in k \,|\, (a \prod_{j=0}^{\frac{p^t-1}{m}-1} (a^m - b_j)) = 0,\, b_j \in k^*\}.$$

Obviously $\mathcal{U}_m$ is a subgroup of the additive group of k. Let

$$K_m := \langle \{\sigma_a(x) = x + a,\, \tau(x) = \xi^2 x \,|\, \forall a \in \mathcal{U}_m\} \rangle,$$

where ξ is a primitive $2m$-th root of unity. Now we are ready to state the following classical result.

Theorem 1. *i) Let k be an algebraically closed field of characteristic $p \neq 2$ of size q when k is finite and G be a finite subgroup of $PGL_2(k)$. Then, G is isomorphic to one of the following groups*

$$C_m, D_m, A_4, S_4, A_5, U = C_p^t, K_m, PSL_2(q), \text{ and } PGL_2(q),$$

where $(m,p) = 1$ and C_m (resp. D_m) denotes the cyclic (resp. dihedral) group of size m.

ii) Let G act on $k(x)$ in the natural way. The fixed field of G is a genus zero subfield $k(z)$, where z is given as in Table 1, with $\alpha = \frac{q(q-1)}{2}$, $\beta = \frac{q+1}{2}$ and H_t is a subgroup of the additive group of k with order $|H_t| = p^t$ and $b_j \in k^*$.

Proof of the first part can be found in [31] and verifying the second part is an easy computational exercise. Next, we continue with our tasks of determining G and an equation for K.

Let $\phi_0 : \mathcal{X}_g \to \mathbb{P}^1(k)$ be the cover which corresponds to the degree n extension $K/k(x)$. Then $\Phi := \phi \circ \phi_0$ has monodromy group $G :=$ Aut $(\mathcal{X}_g)$. From the basic covering theory, the group G is embedded in the group S_l where $l = \deg \Phi = nm$. There is an r-tuple $\overline{\sigma} := (\sigma_1, \dots, \sigma_r)$, where $\sigma_i \in S_l$ such that $\sigma_1, ..., \sigma_r$ generate G and $\sigma_1, \dots, \sigma_r = 1$. The signature of Φ is an r-tuple of conjugacy classes

$$\sigma := (C_1, \dots, C_r)$$

in S_l such that C_i is the conjugacy class of σ_i. We use the notation i to denote the conjugacy class of permutations which is cycle of length i. Using the signature of $\phi : \mathbb{P}^1(k) \to \mathbb{P}^1(k)$ one finds out the signature of $\Phi : \mathcal{X}_g \to \mathbb{P}^1(k)$ for any given g and G.

For the extension K/E, from the Hurwitz genus formula we have that

$$2(g_K - 1) = 2(g_E - 1)|G| + deg(\mathfrak{D}_{K/E}) \tag{1}$$

with g_K and g_E the genera of K and E respectively and $\mathfrak{D}_{K/E}$ the different of K/E. Let $\overline{P}_1, \overline{P}_2, ..., \overline{P}_r$ be ramified primes of E. If we set $d_i = deg(\overline{P}_i)$ and let

e_i be the ramification index of the $\overline{P}_i$ and let β_i be the exponent of $\overline{P}_i$ in $\mathfrak{D}_{K/E}$. Hence, (1) may be written as

$$2(g_K - 1) = 2(g_E - 1)|G| + |G| \sum_{i=1}^{r} \frac{\beta_i}{e_i} d_i \tag{2}$$

If $\overline{P}_i$ is tamely ramified then $\beta_i = e_i - 1$ or if $\overline{P}_i$ is wildly ramified then $\beta_i = e_i^* q_i + q_i - 2$ with $e_i = e_i^* q_i$, e_i^* relatively prime to p, q_i a power of p and $e_i^* | q_i - 1$.

For a fixed G and σ the family of covers $\Phi : \mathcal{X}_g \to \mathbb{P}^1(k)$ is a Hurwitz space $\mathcal{H}(G, \sigma)$. $\mathcal{H}(G, \sigma)$ is an irreducible algebraic variety of dimension $\delta(G, \sigma)$. Using Eq. (2) and signature σ one can find out the dimension for each G.

Case	$\bar{G}$	$z = \phi(x)$	*Ramification*
1	C_m, $(m,p) = 1$	x^m	(m, m)
2	D_{2m}, $(m,p) = 1$	$x^m + \frac{1}{x^m}$	$(2, 2, m)$
3	A_4, $p \neq 2, 3$	$\frac{x^{12} - 33x^8 - 33x^4 + 1}{x^2(x^4-1)^2}$	$(2, 3, 3)$
4	S_4, $p \neq 2, 3$	$\frac{(x^8+14x^4+1)^3}{108(x(x^4-1))^4}$	$(2, 3, 4)$
5	A_5, $p \neq 2, 3, 5$	$\frac{(-x^{20}+228x^{15}-494x^{10}-228x^5-1)^3}{(x(x^{10}+11x^5-1))^5}$	$(2, 3, 5)$
	A_5, $p = 3$	$\frac{(x^{10}-1)^6}{(x(x^{10}+2ix^5+1))^5}$	$(6, 5)$
6	U	$\prod_{a \in H_t} (x + a)$	(p^t)
7	K_m	$(x \prod_{j=0}^{\frac{p^t-1}{m}-1} (x^m - b_j))^m$	(mp^t, m)
8	$PSL(2,q)$, $p \neq 2$	$\frac{((x^q-x)^{q-1}+1)^{\frac{q+1}{2}}}{(x^q-x)^{\frac{q(q-1)}{2}}}$	(α, β)
9	$PGL(2,q)$	$\frac{((x^q-x)^{q-1}+1)^{q+1}}{(x^q-x)^{q(q-1)}}$	$(2\alpha, 2\beta)$

TABLE 1. Rational functions for each finite $G < PGL_2(k)$

3. Superelliptic curves

The superelliptic curves by definition have the group H as a subgroup of their automorphism group. However, the curve might have more automorphisms. Determining the full automorphism group is equivalent to determine degree n extensions of $\bar{G}$, where G is as above.

The following theorems give us all possible automorphism groups of genus $g \geq 2$ superelliptic curves defined over any k such that *char*$k \neq 2$, see [10, 15, 16] for details.

Theorem 2 (Sanjeewa, 2010). *Let $\mathcal{X}_g$ be a genus $g \geq 2$ irreducible superelliptic curve defined over an algebraically closed field k, char $(k) = p \neq 2$. Let $G = Aut(\mathcal{X}_g)$, $\bar{G}$ its reduced automorphism group with respect to H, where $|H| = n$. Then, G is isomorphic to one of the following:*

(1) *If $\bar{G} \cong C_m$ then $G \cong C_{mn}$ or G is isomorphic to*
$$\langle \gamma, \sigma | \, \gamma^n = 1, \sigma^m = 1, \sigma\gamma\sigma^{-1} = \gamma^l \rangle,$$
where $(l, n) = 1$ and $l^m \equiv 1 \mod n$.

(2) *If $\bar{G} \cong D_{2m}$ for some $m \in \mathbb{Z}$, then $G \cong D_{2m} \times C_n$, $G \cong D_{2mn}$, or G is isomorphic to*

i) $\langle \gamma, \sigma, \tau | \, \gamma^n = 1, \sigma^2 = \gamma, \tau^2 = 1, (\sigma\tau)^m = 1, \sigma\gamma\sigma^{-1} = \gamma, \tau\gamma\tau^{-1} = \gamma^{n-1} \rangle$

ii) $\langle \gamma, \sigma, \tau | \, \gamma^n = 1, \sigma^2 = \gamma, \tau^2 = \gamma^{n-1}, (\sigma\tau)^m = 1, \sigma\gamma\sigma^{-1} = \gamma, \tau\gamma\tau^{-1} = \gamma \rangle$

iii) $\langle \gamma, \sigma, \tau | \, \gamma^n = 1, \sigma^2 = \gamma, \tau^2 = 1, (\sigma\tau)^m = \gamma^{\frac{n}{2}}, \sigma\gamma\sigma^{-1} = \gamma, \tau\gamma\tau^{-1} = \gamma^{n-1} \rangle$

iv) $\langle \gamma, \sigma, \tau | \, \gamma^n = 1, \sigma^2 = \gamma, \tau^2 = \gamma^{n-1}, (\sigma\tau)^m = \gamma^{\frac{n}{2}}, \sigma\gamma\sigma^{-1} = \gamma, \tau\gamma\tau^{-1} = \gamma \rangle$

(3) *If $\bar{G} \cong A_4$ and $p \neq 3$ then $G \cong A_4 \times C_n$ or G is isomorphic to*

i) $\langle \gamma, \sigma, \tau | \, \gamma^n = 1, \sigma^2 = 1, \tau^3 = 1, (\sigma\tau)^3 = 1, \sigma\gamma\sigma^{-1} = \gamma, \tau\gamma\tau^{-1} = \gamma^l \rangle$

ii) $\langle \gamma, \sigma, \tau | \, \gamma^n = 1, \sigma^2 = 1, \tau^3 = \gamma^{\frac{n}{3}}, (\sigma\tau)^3 = \gamma^{\frac{n}{3}}, \sigma\gamma\sigma^{-1} = \gamma, \tau\gamma\tau^{-1} = \gamma^l \rangle$

where $(l, n) = 1$ and $l^3 \equiv 1 \mod n$ or
$$\langle \gamma, \sigma, \tau | \, \gamma^n = 1, \sigma^2 = \gamma^{\frac{n}{2}}, \tau^3 = \gamma^{\frac{n}{2}}, (\sigma\tau)^5 = \gamma^{\frac{n}{2}}, \sigma\gamma\sigma^{-1} = \gamma, \tau\gamma\tau^{-1} = \gamma \rangle$$
or

iii) $\langle \gamma, \sigma, \tau | \, \gamma^n = 1, \sigma^2 = 1, \tau^3 = 1, (\sigma\tau)^3 = 1, \sigma\gamma\sigma^{-1} = \gamma, \tau\gamma\tau^{-1} = \gamma^k \rangle$

iv) $\langle \gamma, \sigma, \tau | \, \gamma^n = 1, \sigma^2 = \gamma^{\frac{n}{2}}, \tau^3 = 1, (\sigma\tau)^3 = 1, \sigma\gamma\sigma^{-1} = \gamma, \tau\gamma\tau^{-1} = \gamma^k \rangle$

where $(k, n) = 1$ and $k^3 \equiv 1 \mod n$.

(4) *If $\bar{G} \cong S_4$ and $p \neq 3$ then $G \cong S_4 \times C_n$ or G is isomorphic to*

i) $\langle \gamma, \sigma, \tau | \, \gamma^n = 1, \sigma^2 = 1, \tau^3 = 1, (\sigma\tau)^4 = 1, \sigma\gamma\sigma^{-1} = \gamma^l, \tau\gamma\tau^{-1} = \gamma \rangle$

ii) $\langle \gamma, \sigma, \tau | \, \gamma^n = 1, \sigma^2 = 1, \tau^3 = 1, (\sigma\tau)^4 = \gamma^{\frac{n}{2}}, \sigma\gamma\sigma^{-1} = \gamma^l, \tau\gamma\tau^{-1} = \gamma \rangle$

iii) $\langle \gamma, \sigma, \tau | \, \gamma^n = 1, \sigma^2 = \gamma^{\frac{n}{2}}, \tau^3 = 1, (\sigma\tau)^4 = 1, \sigma\gamma\sigma^{-1} = \gamma^l, \tau\gamma\tau^{-1} = \gamma \rangle$

iv) $\langle \gamma, \sigma, \tau | \, \gamma^n = 1, \sigma^2 = \gamma^{\frac{n}{2}}, \tau^3 = 1, (\sigma\tau)^4 = \gamma^{\frac{n}{2}}, \sigma\gamma\sigma^{-1} = \gamma^l, \tau\gamma\tau^{-1} = \gamma \rangle$

where $(l,n)=1$ *and* $l^2\equiv 1 \mod n$.

(5) *If* $\bar{G}\cong A_5$ *and* $p\neq 5$ *then* $G\cong A_5\times C_n$ *or* G *is isomorphic to*

$$\langle \gamma,\sigma,\tau \,|\, \gamma^n=1, \sigma^2=\gamma^{\frac{n}{2}}, \tau^3=\gamma^{\frac{n}{2}}, (\sigma\tau)^5=\gamma^{\frac{n}{2}}, \sigma\gamma\sigma^{-1}=\gamma, \tau\gamma\tau^{-1}=\gamma\rangle$$

(6) *If* $\bar{G}\cong U$ *then* $G\cong U\times C_n$ *or* G *is isomorphic to*

$$\langle \gamma,\sigma_1,\dots,\sigma_t|\gamma^n=\sigma_1^p=\cdots=\sigma_t^p=1, \sigma_i\sigma_j=\sigma_j\sigma_i, \sigma_i\gamma\sigma_i^{-1}=\gamma^l, 1\leq i,j\leq t\rangle$$

where $(l,n)=1$ *and* $l^p\equiv 1 \mod n$.

(7) *If* $\bar{G}\cong K_m$ *then* G *is isomorphic to one of the following*

i) $\langle \gamma,\sigma_1,...,\sigma_t,v|\gamma^n=\sigma_1^p=...=\sigma_t^p=v^m=1, \sigma_i\sigma_j=\sigma_j\sigma_i, v\gamma v^{-1}=\gamma, \sigma_i\gamma\sigma_i^{-1}=\gamma^l, \sigma_i v\sigma_i^{-1}=v^k, 1\leq i,j\leq t\rangle$ *where* $(l,n)=1$ *and* $l^p\equiv 1 \mod n, (k,m)=1$ *and* $k^p\equiv 1 \mod m$.

ii) $\langle \gamma,\sigma_1,\dots,\sigma_t \,|\, \gamma^{nm}=\sigma_1^p=\cdots=\sigma_t^p=1, \sigma_i\sigma_j=\sigma_j\sigma_i, \sigma_i\gamma\sigma_i^{-1}=\gamma^l, i\geq 1, j\leq t\rangle$, *where* $(l,nm)=1$ *and* $l^p\equiv 1 \mod nm$.

(8) *If* $\bar{G}\cong PSL_2(q)$ *then* $G\cong PSL_2(q)\times C_n$ *or* $SL_2(3)$.

(9) *If* $\bar{G}\cong PGL_2(q)$ *then* $G\cong PGL_2(q)\times C_n$.

Proof. See [16] for all the details.

For sake of completennes and also because of the fact that the signatures of Φ were crucial in determining all cases of the theorem above, we display all these signatures. The proof can be found in [16].

Lemma 1. *The signature of cover* $\Phi:\mathcal{X}\to\mathcal{X}^G$ *and dimension* δ *is given in Table 3, where* $m=|PSL_2(q)|$ *for cases 38-41 and* $m=|PGL_2(q)|$ *for cases 42-45.*

Case	$\bar{G}$	$\delta(G,C)$	$\mathbf{C}=(C_1,...,C_r)$
a		$\frac{g+n-1}{30(n-1)}-1$	$(6,5,n,...,n)$
b		$\frac{g+5n-5}{30(n-1)}-1$	$(6,5n,n,...,n)$
c	A_5	$\frac{g+6n-6}{30(n-1)}-1$	$(6n,5,n,...,n)$
d		$\frac{g}{30(n-1)}-1$	$(6n,5n,n,...,n)$

TABLE 2. δ for $\bar{G}\cong A_5$, $p=3$

Remark 1. *The above Lemma gives signatures and dimensions for* $p>5$. *Since* $\bar{G}\cong C_m, D_m, A_4, S_4, U, K_m, PSL(2,q), PGL(2,q)$ *when* $p=5$ *and* $\bar{G}\cong C_m, D_m, A_5$, U, K_m, $PSL(2,q)$, $PGL(2,q)$ *when* $p=3$, *then all cases except* $\bar{G}\cong A_5$ *have ramification as* $p>5$. *However,* $\bar{G}\cong A_5$ *has different ramification. Hence, that case has signatures and dimensions as in Table 2.*

#	$\bar{G}$	$\delta(G, C)$	δ, n, g	$\mathbf{C} = (C_1, \dots, C_r)$
1	$(p, m) = 1$	$\frac{2(g+n-1)}{m(n-1)} - 1$	$n < g + 1$	$(m, m, n, \dots, n)$
2	C_m	$\frac{2g+n-1}{m(n-1)} - 1$		$(m, mn, n, \dots, n)$
3		$\frac{2g}{m(n-1)} - 1$	$n < g$	$(mn, mn, n, \dots, n)$
4	$(p, m) = 1$	$\frac{g+n-1}{m(n-1)}$		$(2, 2, m, n, \dots, n)$
5		$\frac{2g+m+2n-nm-2}{2m(n-1)}$		$(2n, 2, m, n, \dots, n)$
6	D_{2m}	$\frac{g}{m(n-1)}$		$(2, 2, mn, n, \dots, n)$
7		$\frac{g+m+n-mn-1}{m(n-1)}$	$n < g + 1$	$(2n, 2n, m, n, \dots, n)$
8		$\frac{2g+m-mn}{2m(n-1)}$	$g \neq 2$	$(2n, 2, mn, n, \dots, n)$
9		$\frac{g+m-mn}{m(n-1)}$	$n < g$	$(2n, 2n, mn, n, \dots, n)$
10		$\frac{n+g-1}{6(n-1)}$		$(2, 3, 3, n, \dots, n)$
11	A_4	$\frac{g-n+1}{6(n-1)}$		$(2, 3n, 3, n, \dots, n)$
12		$\frac{g-3n+3}{6(n-1)}$		$(2, 3n, 3n, n, \dots, n)$
13		$\frac{g-2n+2}{6(n-1)}$	$\delta \neq 0$	$(2n, 3, 3, n, \dots, n)$
14		$\frac{g-4n+4}{6(n-1)}$		$(2n, 3n, 3, n, \dots, n)$
15		$\frac{g-6n+6}{6(n-1)}$	$\delta \neq 0$	$(2n, 3n, 3n, n, \dots, n)$
16		$\frac{g+n-1}{12(n-1)}$		$(2, 3, 4, n, \dots, n)$
17		$\frac{g-3n+3}{12(n-1)}$		$(2, 3n, 4, n, \dots, n)$
18		$\frac{g-2n+2}{12(n-1)}$		$(2, 3, 4n, n, \dots, n)$
19		$\frac{g-6n+6}{12(n-1)}$		$(2, 3n, 4n, n, \dots, n)$
20	S_4	$\frac{g-5n+5}{12(n-1)}$		$(2n, 3, 4, n, \dots, n)$
21		$\frac{g-9n+9}{12(n-1)}$		$(2n, 3n, 4, n, \dots, n)$
22		$\frac{g-8n+8}{12(n-1)}$		$(2n, 3, 4n, n, \dots, n)$
23		$\frac{g-12n+12}{12(n-1)}$		$(2n, 3n, 4n, n, \dots, n)$
24		$\frac{g+n-1}{30(n-1)}$		$(2, 3, 5, n, \dots, n)$
25		$\frac{g-5n+5}{30(n-1)}$		$(2, 3, 5n, n, \dots, n)$
26		$\frac{g-15n+15}{30(n-1)}$		$(2, 3n, 5n, n, \dots, n)$
27		$\frac{g-9n+9}{30(n-1)}$		$(2, 3n, 5, n, \dots, n)$
28	A_5	$\frac{g-14n+14}{30(n-1)}$		$(2n, 3, 5, n, \dots, n)$
29		$\frac{g-20n+20}{30(n-1)}$		$(2n, 3, 5n, n, \dots, n)$
30		$\frac{g-24n+24}{30(n-1)}$		$(2n, 3n, 5, n, \dots, n)$
31		$\frac{g-30n+30}{30(n-1)}$		$(2n, 3n, 5n, n, \dots, n)$
32		$\frac{2g+2n-2}{p^t(n-1)} - 2$		$(p^t, n, \dots, n)$
33	U	$\frac{2g+np^t-p^t}{p^t(n-1)} - 2$	$(n, p) = 1, n \mid p^t - 1$	$(np^t, n, \dots, n)$
34		$\frac{2(g+n-1)}{mp^t(n-1)} - 1$	$(m, p) = 1, m \mid p^t - 1$	$(mp^t, m, n, \dots, n)$
35		$\frac{2g+2n+p^t-np^t-2}{mp^t(n-1)} - 1$	$(m, p) = 1, m \mid p^t - 1$	$(mp^t, nm, n, \dots, n)$
36	K_m	$\frac{2g+np^t-p^t}{mp^t(n-1)} - 1$	$(nm, p) = 1, nm \mid p^t - 1$	$(nmp^t, m, n, \dots, n)$
37		$\frac{2g}{mp^t(n-1)} - 1$	$(nm, p) = 1, nm \mid p^t - 1$	$(nmp^t, nm, n, \dots, n)$
38		$\frac{2(g+n-1)}{m(n-1)} - 1$	$\left(\frac{q-1}{2}, p\right) = 1$	$(\alpha, \beta, n, \dots, n)$
39	$PSL_2(q)$	$\frac{2g+q(q-1)-n(q+1)(q-2)-2}{m(n-1)} - 1$	$\left(\frac{q-1}{2}, p\right) = 1$	$(\alpha, n\beta, n, \dots, n)$
40		$\frac{2g+nq(q-1)+q-q^2}{m(n-1)} - 1$	$\left(\frac{n(q-1)}{2}, p\right) = 1$	$(n\alpha, \beta, n, \dots, n)$
41		$\frac{2g}{m(n-1)} - 1$	$\left(\frac{n(q-1)}{2}, p\right) = 1$	$(n\alpha, n\beta, n, \dots, n)$
42		$\frac{2(g+n-1)}{m(n-1)} - 1$	$(q - 1, p) = 1$	$(2\alpha, 2\beta, n, \dots, n)$
43	$PGL_2(q)$	$\frac{2g+q(q-1)-n(q+1)(q-2)-2}{m(n-1)} - 1$	$(q - 1, p) = 1$	$(2\alpha, 2n\beta, n, \dots, n)$
44		$\frac{2g+nq(q-1)+q-q^2}{m(n-1)} - 1$	$(n(p - 1), p) = 1$	$(2n\alpha, 2\beta, n, \dots, n)$
45		$\frac{2g}{m(n-1)} - 1$	$(n(q - 1), p) = 1$	$(2n\alpha, 2n\beta, n, \dots, n)$

TABLE 3. The signature of curves and dimensions δ for $char > 5$

3.1. **Equations of superelliptic curves.** Next we give the parametric equations of superelliptic curves based on their group of automorphisms. Such equations for the first time were computed in [15]. It is exactly the fact that their equations are easily determined that makes superelliptic curves quite attractive in applications. Let δ be given as in Table 3 and $M, \Lambda, Q, B, \Delta, \Theta$ and Ω are as follows:

$$\begin{aligned} M =& \prod_{i=1}^{\delta} \big(x^{24} + \lambda_i x^{20} + (759 - 4\lambda_i)x^{16} + 2(3\lambda_i + 1228)x^{12} \\ & + (759 - 4\lambda_i)x^8 + \lambda_i x^4 + 1\big) \end{aligned}$$

$$\begin{aligned} \Lambda =& \prod_{i=1}^{\delta} \big(-x^{60} + (684 - \lambda_i)x^{55} - (55\lambda_i + 157434)x^{50} - (1205\lambda_i - 12527460)x^{45} \\ & - (13090\lambda_i + 77460495)x^{40} + (130689144 - 69585\lambda_i)x^{35} \\ & + (33211924 - 134761\lambda_i)x^{30} + (69585\lambda_i - 130689144)x^{25} \\ & - (13090\lambda_i + 77460495)x^{20} - (12527460 - 1205\lambda_i)x^{15} \\ & -(157434 + 55\lambda_i)x^{10} + (\lambda_i - 684)x^5 - 1\big) \end{aligned}$$

$$Q = x^{30} + 522x^{25} - 10005x^{20} - 10005x^{10} - 522x^5 + 1$$

$$B = \prod_{i=1}^{\delta} \prod_{a \in H_t} ((x + a) - \lambda_i)$$

$$\Theta = \prod_{i=1}^{\delta} G_{\lambda_i}(x), \ \ \textit{where } G_{\lambda_i} = \left(x \cdot \prod_{j=1}^{\frac{p^t-1}{m}} (x^m - b_j) \right)^m - \lambda_i$$

$$\Delta = \prod_{i=1}^{\delta} (((x^q - x)^{q-1} + 1)^{\frac{q+1}{2}} - \lambda_i (x^q - x)^{\frac{q(q-1)}{2}})$$

$$\Omega = \prod_{i=1}^{\delta} (((x^q - x)^{q-1} + 1)^{q+1} - \lambda_i (x^q - x)^{q(q-1)})$$

Then we have the following result.

Theorem 3. *Let $\mathcal{X}_g$ be e genus $g \geq 2$ algebraic curve defined over an algebraically closed field k, G its automorphism group over k, and C_n a cyclic normal subgroup of G such that $g(X_g^{C_n}) = 0$. Then, the equation of $\mathcal{X}_g$ can be written as in one of the following cases as in Table 4.*

#	$\bar{G}$	$y^n = f(x)$
1		$x^{m\delta} + a_1 x^{m(\delta-1)} + ... + a_\delta x^m + 1$
2	C_m	$x^{m\delta} + a_1 x^{m(\delta-1)} + ... + a_\delta x^m + 1$
3		$x(x^{m\delta} + a_1 x^{m(\delta-1)} + ... + a_\delta x^m + 1)$
4		$F(x) := \prod_{i=1}^{\delta}(x^{2m} + \lambda_i x^m + 1)$
5		$(x^m - 1) \cdot F(x)$
6		$x \cdot F(x)$
7	D_{2m}	$(x^{2m} - 1) \cdot F(x)$
8		$x(x^m - 1) \cdot F(x)$
9		$x(x^{2m} - 1) \cdot F(x)$
10		$G(x) := \prod_{i=1}^{\delta}(x^{12} - \lambda_i x^{10} - 33x^8 + 2\lambda_i x^6 - 33x^4 - \lambda_i x^2 + 1)$
11		$(x^4 + 2i\sqrt{3}x^2 + 1) \cdot G(x)$
12	A_4	$(x^8 + 14x^4 + 1) \cdot G(x)$
13		$x(x^4 - 1) \cdot G(x)$
14		$x(x^4 - 1)(x^4 + 2i\sqrt{3}x^2 + 1) \cdot G(x)$
15		$x(x^4 - 1)(x^8 + 14x^4 + 1) \cdot G(x)$
16		$M(x)$
17		$\big(x^8 + 14x^4 + 1\big) \cdot M(x)$
18		$x(x^4 - 1) \cdot M(x)$
19		$\big(x^8 + 14x^4 + 1\big) \cdot x(x^4 - 1) \cdot M(x)$
20	S_4	$\big(x^{12} - 33x^8 - 33x^4 + 1\big) \cdot M(x)$
21		$\big(x^{12} - 33x^8 - 33x^4 + 1\big) \cdot \big(x^8 + 14x^4 + 1\big) \cdot M(x)$
22		$\big(x^{12} - 33x^8 - 33x^4 + 1\big) \cdot x(x^4 - 1) \cdot M(x)$
23		$\big(x^{12} - 33x^8 - 33x^4 + 1\big) \cdot \big(x^8 + 14x^4 + 1\big) \cdot x(x^4 - 1)M(x)$
24		$\Lambda(x)$
25		$x(x^{10} + 11x^5 - 1) \cdot \Lambda(x)$
26		$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1)(x(x^{10} + 11x^5 - 1)) \cdot \Lambda(x)$
27		$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1) \cdot \Lambda(x)$
28	A_5	$Q(x) \cdot \Lambda(x)$
29		$x(x^{10} + 11x^5 - 1).\psi(x) \cdot \Lambda(x)$
30		$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1) \cdot \psi(x) \cdot \Lambda(x)$
31		$(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1)(x(x^{10} + 11x^5 - 1)) \cdot \psi(x) \cdot \Lambda(x)$
32	U	$B(x)$
33		$B(x)$
34		$\Theta(x)$
35	K_m	$x \prod_{j=1}^{\frac{p^t-1}{m}} (x^m - b_j) \cdot \Theta(x)$
36		$\Theta(x)$
37		$x \prod_{j=1}^{\frac{p^t-1}{m}} (x^m - b_j) \cdot \Theta(x)$
38		$\Delta(x)$
39	$PSL_2(q)$	$((x^q - x)^{q-1} + 1) \cdot \Delta(x)$
40		$(x^q - x) \cdot \Delta(x)$
41		$(x^q - x)((x^q - x)^{q-1} + 1) \cdot \Delta(x)$
42		$\Omega(x)$
43	$PGL_2(q)$	$((x^q - x)^{q-1} + 1) \cdot \Omega(x)$
44		$(x^q - x) \cdot \Omega(x)$
45		$(x^q - x)((x^q - x)^{q-1} + 1) \cdot \Omega(x)$

TABLE 4. The equations of the curves related to the cases in Table 3

Each case in the above table correspond to a δ-dimensional family, where δ can be found in [15]. Moreover, our parametrizations are exact in the sense that the number of parameters in each case is equal to the dimension. We would like to find invariants to classify isomorphism classes of these curves.

4. Isomorphism classes of superelliptic curves

A superelliptic curve $\mathcal{X}_g$ is given by an equation of the form $y^n = f(x)$ for some degree d polynomial $f(x)$. Let us assume that

$$y^n = f(x) = \prod_{i=1}^{s} (x-\alpha_i)^{d_i}, \quad 0 < d_i < d.$$

We have that $\sum_{i=1}^{s} d_i = d$. We call this the **standard form** of the curve. The only places of $F_0 = k(x)$ that ramify are the places which correspond to the points $x = \alpha_i$. We denote such places by $Q_1, \dots, Q_s$ and by $\mathcal{B} := \{Q_1, \dots, Q_s\}$ the set of these places. The ramification indexes are $e(Q_i) = \frac{n}{(n,d_i)}$.

Hence, every set $\mathcal{B}$ determines a genus g superelliptic curve $\mathcal{X}_g$. However, the correspondence between the sets $\mathcal{B}$ and the isomorphism classes of $\mathcal{X}_g$ is not a one to one correspondence. Obviously the set of roots of $f(x)$ does not determine uniquely the isomorphism class of $\mathcal{X}_g$ since every coordinate change in x would change the set of these roots. Such isomorphism classes are classified by the invariants of binary forms. Invariants of binary forms of of degree up to eight are known by classical work of many invariant theorists and some more recent work, see [7, 11, 28].

4.1. Invariants of binary forms.

In this section we define the action of $GL_2(k)$ on binary forms and discuss the basic notions of their invariants. Let $k[X, Z]$ be the polynomial ring in two variables and let V_d denote the $(d+1)$-dimensional subspace of $k[X, Z]$ consisting of homogeneous polynomials.

$$f(X, Z) = a_0 X^d + a_1 X^{d-1} Z + \dots + a_d Z^d \tag{3}$$

of degree d. Elements in V_d are called *binary forms* of degree d. We let $GL_2(k)$ act as a group of automorphisms on $k[X, Z]$ as follows:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(k), \ \textit{then} \quad M \begin{pmatrix} X \\ Z \end{pmatrix} = \begin{pmatrix} aX + bZ \\ cX + dZ \end{pmatrix} \tag{4}$$

This action of $GL_2(k)$ leaves V_d invariant and acts irreducibly on V_d.

Remark 2. *It is well known that $SL_2(k)$ leaves a bilinear form (unique up to scalar multiples) on V_d invariant. This form is symmetric if d is even and skew symmetric if d is odd.*

Let $A_0, A_1, \dots, A_d$ be coordinate functions on V_d. Then the coordinate ring of V_d can be identified with $k[A_0, ..., A_d]$. For $I \in k[A_0, ..., A_d]$ and $M \in GL_2(k)$, define $I^M \in k[A_0, ..., A_d]$ as follows

$$I^M(f) := I(M(f)) \tag{5}$$

for all $f \in V_d$. Then $I^{MN} = (I^M)^N$ and Eq. (5) defines an action of $GL_2(k)$ on $k[A_0, ..., A_d]$. A homogeneous polynomial $I \in k[A_0, \dots, A_d, X, Z]$ is called a *covariant* of index s if

$$I^M(f) = \delta^s I(f),$$

where $\delta = \det(M)$. The homogeneous degree in $a_1, \dots, a_n$ is called the *degree* of I, and the homogeneous degree in X, Z is called the *order* of I. A covariant of order zero is called *invariant*. An invariant is a $SL_2(k)$-invariant on V_d.

We will use the symbolic method of classical theory to construct covariants of binary forms. Let

$$f(X,Z) := \sum_{i=0}^{n} \binom{n}{i} a_i X^{n-i} Z^i, \quad \textit{and} \quad g(X,Z) := \sum_{i=0}^{m} \binom{m}{i} b_i X^{n-i} Z^i$$

be binary forms of degree n and m respectively with coefficients in k. We define the **r-transvection**

$$(f,g)^r := \frac{(m-r)!\,(n-r)!}{n!\,m!} \sum_{k=0}^{r} (-1)^k \binom{r}{k} \cdot \frac{\partial^r f}{\partial X^{r-k}\,\partial Z^k} \cdot \frac{\partial^r g}{\partial X^k\,\partial Z^{r-k}}$$

It is a homogeneous polynomial in $k[X,Z]$ and therefore a covariant of order $m + n - 2r$ and degree 2. In general, the r-transvection of two covariants of order m, n (resp., degree p, q) is a covariant of order $m + n - 2r$ (resp., degree $p + q$).

For the rest of this paper $F(X,Z)$ denotes a binary form of order $d := 2g + 2$ as below

$$F(X,Z) = \sum_{i=0}^{d} a_i X^i Z^{d-i} = \sum_{i=0}^{d} \binom{n}{i} b_i X^i Z^{n-i} \tag{6}$$

where $b_i = \frac{(n-i)!\ i!}{n!} \cdot a_i$, for $i = 0, \dots, d$. We denote invariants (resp., covariants) of binary forms by I_s (resp., J_s) where the subscript s denotes the degree (resp., the order). We define the following covariants and invariants:

$$\begin{aligned}
&I_2 := (F,F)^d, && J_{4j} := (F,F)^{d-2j},\ j = 1, \dots, g,\\
&I_4 := (J_4, J_4)^4, && I_4' := (J_8, J_8)^8,\\
&I_6 := ((F,J_4)^4, (F,J_4)^4)^{d-4}, && I_6' := ((F,J_8)^8, (F,J_8)^8)^{d-8},\\
&I_6^* := ((F,J_{12})^{12}, (F,J_{12})^{12})^{d-12}, && I_3 := (F, J_d)^d,\\
&M := ((F,J_4)^4, (F,J_8)^8)^{d-10}, && I_{12} := (M,M)^8
\end{aligned} \tag{7}$$

Absolute invariants are called $GL_2(k)$-invariants. We define the following absolute invariants:

$$i_1 := \frac{I_4'}{I_2^2},\ i_2 := \frac{I_3^2}{I_2^3},\ i_3 := \frac{I_6^*}{I_2^3},\ j_1 := \frac{I_6'}{I_3^2},\ j_2 := \frac{I_6}{I_3^2},\ u_1 := \frac{I_6^2}{I_{12}},\ u_2 := \frac{(I_6')^2}{I_{12}}$$

$$\mathfrak{v}_1 := \frac{I_6}{I_6^*},\ \mathfrak{v}_2 := \frac{(I_4')^3}{I_3^4},\ \mathfrak{v}_3 := \frac{I_6}{I_6'},\ \mathfrak{v}_4 := \frac{(I_6^*)^2}{I_4^3}.$$

In the case $g = 10$ and $I_{12} = 0$ we define

$$\begin{aligned}
I_6^\star &:= ((F,J_{16})^{16}, (F,J_{16})^{16})^{d-16}),\\
S &:= (J_{12}, J_{16})^{12},\\
I_{12}^* &:= (\,(J_{16}, S)^4,\ (J_{16}, S)^4\,)^{12}
\end{aligned} \tag{8}$$

and $\mathfrak{v}_5 := \frac{I_6^\star}{I_{12}^*}$.

For a given curve $\mathcal{X}_g$ we denote by $I(\mathcal{X}_g)$ or $i(\mathcal{X}_g)$ the corresponding invariants. When the above invariants are a good set of invariants to study the small genus curves, they are not a set of complete invariants for curves of arbitrary genus.

Example 1. *Let C be a genus 4 curve with equation*

$$y^3 = x^6 + a_5 x^5 + \cdots + a_1 x + a_0,$$

defined over $\mathbb{C}$. This curve has automorphism group C_3. The family V of such curves is a 3-dimensional variety. The isomorphism classes of curves in this variety are determined by Igusa invariants J_2, J_4, J_6, J_{10}, see [11, 21] *for their definitions. Two curves C and C' in V are isomorphic if and only if*

$$(J_2(C), J_4(C), J_6(C), J_{10}(C)) = \lambda \cdot (J_2(C'), J_4(C'), J_6(C'), J_{10}(C'))$$

for some $\lambda \neq 0$.

Lemma 2. *Let $\mathcal{X}_g$ be a superelliptic curves of genus $g \geq 2$. The following statements are true.*

i) If $\bar{G} \equiv A_4$ then $I_4(\mathcal{X}_g) = 0$.

ii) If $\bar{G} \equiv A_5$ then $(J_i, J_i)^i = 0$ for $i = 4, 8, 16, 28$.

Proof. See [7] for the proof of these and other properties of superelliptic curves in terms of invariants of binary forms.

□

5. $\mathfrak{s}$-INVARIANTS OF SUPERELLIPTIC CURVES

In this section we will introduce $\mathfrak{s}$-invariants of superelliptic curves. These invariants were introduced in [13] for hyperelliptic curves and generalized in [1] for superelliptic curves. Here we simply follow the approach from [1].

Let k be an algebraic closed field of characteristic $p \geq 0$. Let $F_0 = k(x)$ be the function field of the projective line $\mathbb{P}^1(k)$. We consider a cyclic extension of F_0 of degree n of the form $F := k(x, y)$ where

$$y^n = \prod_{i=1}^{s} (x - \rho_i)_i^d =: f(x), \quad o < d_i < n. \tag{9}$$

If $d := \sum_{i=1}^{s} d_i \equiv 0 \mod n$ then the place at infinity does not ramify at the above extension. The only places at F_0 that are ramified are the places P_i that correspond to the points $x = \rho_i$ and the corresponding ramification indices are given by

$$e_i = \frac{n}{(n, d_i)}.$$

Moreover if $(n, d_i) = 1$ then the places P_i are ramified completely and the Riemann-Hurwitz formula implies that the function field F has genus

$$g = \frac{(n-1)(s-2)}{2}.$$

Notice that the condition $g \geq 2$ is equivalent to $s \geq 2\frac{n+1}{n-1}$. In particular, $s > 2$.

For the proof of the following Lemmas se [1].

Lemma 3. *Let $G = Aut(F)$. Suppose that a cyclic extension F/F_0 of the rational function field F_0 is ramified completely at s places and $n := |Gal(F/F_0)|$. If $2n < s$ then $Gal(F/F_0) \triangleleft G$.*

Lemma 4. *Suppose that τ is an extra automorphism of F, and let s be the number of ramified places at the extension F/F_0 and let d be the degree of the defining polynomial. Then $\delta|s, \delta|d$ and the defining equation of F can be written as*

$$y^n = \sum_{i-0}^{d/\delta} a_i x^{\delta\cdot i},$$

where $a_0 = 1$.

We will say that the superelliptic curve is in **normal form** if and only if it is given by an equation:

$$y^n = x^s + \sum_{i-1}^{\frac{d}{\delta}} a_i x^{\delta\cdot i} + 1.$$

Parametrizing superelliptic curves that admit an extra automorphism of order δ, is the set of coefficients $\{a_{s/\delta-1}, \cdots, a_1\}$ of a normal form up to a change of coordinate in x. The condition $\tau(x) = \zeta x$, implies that $\bar{\tau}$ fixes the places $0, \infty$. Moreover we can change the defining equation by a morphism $\gamma \in PGL(2,k)$ of the form $\gamma : x \to mx$ or $\gamma : x \to \frac{m}{x}$ so that the new equation is again in normal form. Substituting $a_0 = (-1)^{d/s} \prod_{i=1}^{d/s} \beta_i^s$ we have

$$(-1)^{s/\delta} \prod_{i=1}^{s/\delta} \gamma(\beta_i)^\delta = 1$$

and this gives $m^s = (-1)^{s/\delta}$. Then, x is determined up to a coordinate change by the subgroup $D_{s/\delta}$ generated by

$$\tau_1 : x \to \epsilon x, \; \tau_2 : x \to \frac{1}{x}$$

where ϵ is a primitive s/δ-root of one, see [13] for details.

The action of $D_{s/\delta}$ on the parameter space $k(a_1, \ldots, a_{s/\delta})$ is given by

$$\begin{aligned} \tau_1 &: a_i \to \epsilon^{\delta i} a_i, \; \textit{for } i = 1, \ldots s/\delta \\ \tau_2 &: a_i \to a_{d/\delta - i}, \; \textit{for } i = 1, \ldots [s/\delta] \end{aligned}$$

Notice that if $s/\delta = 1$ then the above actions are trivial, therefore the normal form determines the equivalence class. If $s/\delta = 2$ then

$$\tau_1(a_1) = -a_1, \tau_1(a_2) = a_2, \tau_2 = 1$$

and the action is not dihedral but cyclic on the first vector.

Lemma 5. *Assume that $s/\delta > 2$. The fixed field $k(a_1, a_2, \cdots a_{s/\delta})^{D_{s/\delta}}$ is the same as the function field of the variety $\mathcal{L}_{n,s,\delta}$.*

Proof. See [1] for the proof.

Lemma 6. *Let $r := s/\delta > 2$ The elements*

$$\mathfrak{s}_i := a_1^{r-i} a_1 + a_{r-1}^{r-i} a_{r-i}, \; \textit{for } i = 1, \ldots, r$$

are invariants under the action of the group $D_{s/\delta}$ defined as above.

Proof. See [1] for the proof.

The elements $\mathfrak{s}_i$ are called the **dihedral invariants** or **$\mathfrak{s}$-invariants** of $D_{s/\delta}$.

Theorem 4. *Let $\mathfrak{s} = (\mathfrak{s}_1, \dots, \mathfrak{s}_r)$ be the $r-$ tuple of $\mathfrak{s}$-invariants. Then $k(\mathcal{L}_{s,n,\delta}) = k(\mathfrak{s}_1, \dots, \mathfrak{s}_r)$.*

Example 2. *For genus $g = 2$ all curves are hyperelliptic and therefore superelliptic. The generic curve of genus 2 is given by $y^2 = f(x)$, where $\deg(f) = 5$ or 6. The space is determined by the invariants of binary sextics. When the curves have extra automorphisms then we have two main cases.*

i) The first case is when there is an automorphism of degree 5. Then the full automorphism group G is isomorphic to $\mathbb{Z}_{10}$ and the corresponding space has dimension zero. There is only one curve in this case (up to isomorphism), which is given by $y^2 = x^5 - 1$.

ii) In the second case, the extra automorphism is an involution. Then, G is isomorphic to the Klein four-group V_4 and the curve has equation

$$y^2 = x^6 + a_1 x^4 + a_2 x^2 + 1.$$

The $\mathfrak{s}$-invariants are

$$\mathfrak{s}_1 = a_1^3 + a_2^3, \qquad \mathfrak{s}_2 = 2a_1 a_2, \tag{10}$$

see [29] *for a detailed study of this case. In* [29] *was the first time that such invariants were defined and later generalized in* [13].

6. Computational aspects of invariants of superelliptic curves

In this part we give a quick introduction to some computational aspects of $\mathfrak{s}$-invariants. A more detailed study of superelliptic curves and their computational invariants will appear in [8].

Problem Given a genus $g \geq 3$.

1) Find the lattice of inclusions of all the cases based on the automorphism groups.

2) Compute relations among $\mathfrak{s}$-invariants for every group of the table.

In other words, we would like to characterize for every group G the locus of the curves in each case in the Table 2, in terms of invariants of these curves and determine the inclusions among such loci. While such lattice can be computed using only group theory methods, from the computational viewpoint this is really not very useful. Instead such lattice and such loci need to be computed in terms of coefficients of the curves, or more precisely invariants of the curves. A step further would be to characterize the Jacobians of curves in these loci. This can be done through the theory of theta functions as in [5].

6.1. A Maple package for computing with superelliptic curves.

Computing the $\mathfrak{s}$-invariants we first need the equation of the curve in the normal form

$$y^n = f(x).$$

Once the normal form is determined then it is rather straight forward to compute the $\mathfrak{s}$-invariants. We have implemented some of these tasks in Maple and display the codes below.

```
normalpol:=proc(f,x)      # Computes the normal form of a polynomial.

    local a,n,f1;
    n:=degree(f,x); f1:=f/coeff(f,x,0);  a:=coeff(f1,x,n)^(1/n);

RETURN(subs(x=x/a,f1));
end:

s_inv := proc(f, x)            # Computing the s-invariants.

       local i, a, g, s;

     g:=(degree(f, x) - 2)/2;

     for i to g do
        a[i]:=coeff(f, x, 2*i)
     od:

     for i to g do
         s[i]:=factor(a[1]^(g-i+1)*a[i]+a[g]^(g-i+1)*a[g-i+1])
     od;

RETURN([seq(s[i],i=1..g)]);
end;

fg_s:=proc (f,g,x,y,s)       # Computing the s-transvection of
    local n,m,fg,k;              # binary forms f and g.

  n:=degree(f,{x,y});
  m:=degree(g,{x,y});

  fg:=(n-s)!*(m-s)!/(n!*m!)*add((-1)^k*( s!/(k!*(s-k)!))
       *diff(f,x$(s-k),y$k) *diff(g,x$k,y$(s-k)),k=0..s );

  RETURN(expand(fg));
end:

fg_s2:=proc (f,g,x,y)    # fg_s2(f,g,x,y) := fg_s(f,g,x,y,deg(f))
   local n,f2,g2,k;         # deg(g)=deg(f)

  n:=degree(f,{x,y});    # returns an invariant (which means order=0)

  f2:=collect(f,[x,y]);
  g2:=collect(g,[x,y]);

  1/(n!*n!)*add((-1)^k*( n!/(k!*(n-k)!))*(n-k)!*k!
   *coeff(coeff(f2,x,n-k),y,k)
```

```
   * (n-k)!*k!*coeff(coeff(g2,x,k),y,n-k) ,k=0..n );

  RETURN(expand(%));
end:

homogpol:=proc(f,x,y)    # Converts a polynomial to a homogenous one.
  RETURN(expand(subs(x=x/y,f*y^degree(f,x))));
end:

J_i:=proc(F,x,y,i)
  fg_s(F,F,x,y,degree(F,{x,y})-i/2);

  RETURN(%);
end:

I4prime:=proc(F,x,y)
  J_i(F,x,y,8);
  fg_s2(%,%,x,y);

  RETURN(%);
end:

I2:=proc(F,x,y)
  fg_s2(F,F,x,y);

  RETURN(%);
end:

I3:=proc(F,x,y)
  J_i(F,x,y,degree(F,{x,y}));
  fg_s2(F,%,x,y);

  RETURN(%);
end:
```

7. Genus 3

In this section we will determine all the superelliptic curves of genus 3. Completing the case in positive characteristic is a natural extension of the methods used here.

7.1. **Automorphism groups of genus 3 superelliptic curves.** Applying Thm. 2 we obtain the automorphism groups of a genus 3 superelliptic curves defined over algebraically closed field of characteristic $p \neq 2$. Below we list GAP group ID's of those groups.

Lemma 7. *Let $\mathcal{X}_g$ be a genus 3 superelliptic curve defined over a field of characteristic $p \neq 2$. Then the automorphism groups of $\mathcal{X}_g$ are as follows.*

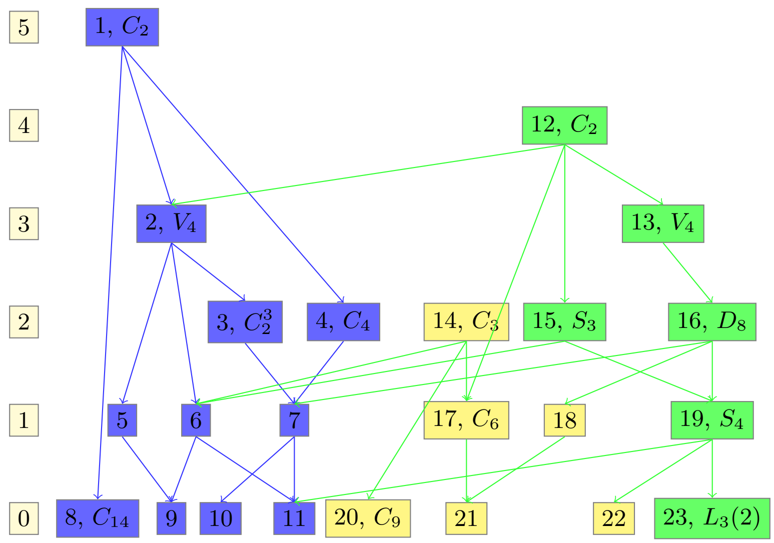

FIGURE 2. The lattice of genus 3 case. The blue items correspond to hyperelliptic curves, the yellow ones to the other superelliptic cases.

- **i):** $p = 3$*:* $(2,1)$, $(4,2)$, $(3,1)$, $(4,1)$, $(8,2)$, $(8,3)$, $(7,1)$, $(14,2)$, $(6,2)$, $(8,1)$, $(8,5)$, $(16,11)$, $(16,10)$, $(32,9)$, $(30,2)$, $(16,7)$, $(16,8)$, $(6,2)$.
- **ii):** $p = 5$*:* $(2,1)$, $(4,2)$, $(3,1)$, $(4,1)$, $(8,2)$, $(8,3)$, $(7,1)$, $(21,1)$, $(14,2)$, $(6,2)$, $(12,2)$, $(9,1)$, $(8,1)$, $(8,5)$, $(16,11)$, $(16,10)$, $(32,9)$, $(42,3)$, $(12,4)$, $(16,7)$, $(24,5)$, $(18,3)$, $(16,8)$, $(48,33)$, $(48,48)$.
- **iii):** $p = 7$*:* $(2,1)$, $(4,2)$, $(3,1)$, $(4,1)$, $(8,2)$, $(8,3)$, $(7,1)$, $(21,1)$, $(6,2)$, $(12,2)$, $(9,1)$, $(8,1)$, $(8,5)$, $(16,11)$, $(16,10)$, $(32,9)$, $(30,2)$, $(42,3)$, $(12,4)$, $(16,7)$, $(24,5)$, $(18,3)$, $(16,8)$, $(48,33)$, $(48,48)$.
- **iv):** $p = 0$ *or* $p > 7$*:* $(2,1)$, $(4,2)$, $(3,1)$, $(4,1)$, $(8,2)$, $(14,2)$, $(6,2)$, $(9,1)$, $(8,5)$, $(16,11)$, $(32,9)$, $(12,4)$, $(16,13)$, $(24,5)$, $(48,33)$, $(48,48)$, *(96, 64)*.

Recall that the list for $p = 0$ is the same as for $p > 7$. In the diagram below we display the inclusion among the loci in the case of genus 3. We will briefly discuss the superelliptic curves and display their equations.

7.2. **Equations of hyperelliptic curves of genus three.** Let $\mathcal{X}_3$ be a hyperelliptic curve of genus 3. In Tab. 5 we list the automorphism groups of genus 3 hyperelliptic curves. The first column is the case number, in the second column the groups which occur as full automorphism groups are given, and the third column indicates the reduced automorphism group for each case. The dimension δ of the locus and the equation of the curve are also are given in the next two columns. The last column is the GAP identity of each group in the library of small groups in GAP.

Case 1: $\mathbb{Z}_2$-hyperelliptic: Then, the equation of $\mathcal{X}_3$ is given by

$$y^2 = f(x)$$

	Aut $(\mathcal{X}_g)$	$\overline{\text{Aut}}(\mathcal{X}_g)$	δ	equation $y^2 = f(x)$	Id.
1	$\mathbb{Z}_2$	$\{1\}$	5	$x(x-1)(x^5+ax^4+bx^3+cx^2+dx+e)$	$(2,1)$
2	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2$	3	$x^8+a_3x^6+a_2x^4+a_1x^2+1$	$(4,2)$
3	$\mathbb{Z}_2^3$	D_4	2	$(x^4+ax^2+1)(x^4+bx^2+1)$	$(8,5)$
4	$\mathbb{Z}_4$	$\mathbb{Z}_2$	2	$x(x^2-1)(x^4+ax^2+b)$	$(4,1)$
5	$\mathbb{Z}_2\times\mathbb{Z}_4$	D_4	1	$(x^4-1)(x^4+ax^2+1)$	$(8,2)$
6	D_{12}	D_6	1	$x(x^6+ax^3+1)$	$(12,4)$
7	$\mathbb{Z}_2\times D_8$	D_8	1	x^8+ax^4+1	$(16,11)$
8	$\mathbb{Z}_{14}$	$\mathbb{Z}_7$	0	x^7-1	$(14,2)$
9	U_6	D_{12}	0	$x(x^6-1)$	$(24,5)$
10	V_8	D_{16}	0	x^8-1	$(32,9)$
11	$\mathbb{Z}_2\times S_4$	S_4	0	x^8+14x^2+1	$(48,48)$

TABLE 5. Aut $(\mathcal{X}_3)$ for hyperelliptic $\mathcal{X}_3$

where $\deg f = 7$ or 8. To have an explicit way of describing a point in the moduli space of hyperelliptic curves of genus 3 we need the generators of the field of invariants of binary octavics. These invariants are described in terms of covariants of binary octavics. Such covariants were first constructed by van Gall who showed that the graded ring of covariants is generated by 70 covariants and explicitly constructed them, see [28].

Let $f(X,Y)$ be the binary octavic

$$f(X,Y)=\sum_{i=0}^{8} a_i X^i Y^{8-i}.$$

We define the following covariants:

$$\begin{aligned} &g=(f,f)^4, \quad k=(f,f)^6, \quad h=(k,k)^2, \quad m=(f,k)^4,\\ &n=(f,h)^4, \quad p=(g,k)^4, \quad q=(g,h)^4. \end{aligned}$$

Then the following

$$\begin{aligned} &J_2=(f,f)^8, \quad J_3=(f,g)^8, \quad J_4=(k,k)^4, \quad J_5=(m,k)^4,\\ &J_6=(k,h)^4, \quad J_7=(m,h)^4 \quad J_8=(p,h)^4, \quad J_9=(n,h)^4, \quad J_{10}=(q,h)^4 \end{aligned} \tag{11}$$

are $SL_2(k)$-invariants. Shioda has shown that the ring of invariants is a finitely generated module of $k[J_2,\dots,J_7]$, see [28] for more details.

Case 2: V_4-hyperelliptic: Then, $\mathcal{X}_3$ has normal equation

$$Y^2=X^8+a_3\,X^6+a_2\,X^4+a_1\,X^2+1,$$

see [13]. The $\mathfrak{s}$-invariants of $\mathcal{X}_3$ are

$$\mathfrak{s}_1=a_1^4+a_3^4, \quad \mathfrak{s}_2=(a_1^2+a_3^2)\,a_2, \quad \mathfrak{s}_3=\,2\,a_1\,a_3.$$

If $a_1 = a_3 = 0$, then $\mathfrak{s}_1 = \mathfrak{s}_2 = \mathfrak{s}_3 = 0$. In this case

$$w := a_2^2$$

is invariant. Thus, we define

$$\mathfrak{s}(\mathcal{X}_3) = \begin{cases} w & \text{if} \quad a_1 = a_3 = 0, \\ (\mathfrak{s}_1, w, \mathfrak{s}_3) & \text{if} \quad a_1^2 + a_3^2 = 0 \text{ and } a_2 \neq 0, \\ (\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3) & \text{otherwise.} \end{cases} \tag{12}$$

The expressions of these covariants are very large in terms of the coefficients of the curve and difficult to compute. However, in terms of the $\mathfrak{s}$-invariants $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3$ these expressions are smaller. Analogously, J_{14} is the discriminant of the octavic. All these invariants are determined explicitly in terms of the $\mathfrak{s}$-invariants in [12].

We denote by $\mathcal{L}_3$ the sublocus of $\mathcal{M}_3$ of hyperelliptic curves with automorphism group V_4. This is a closed subvariety of $\mathcal{M}_3$ determined as below as shown in [12]. The following are true, see [12] for their proofs.

Remark 3. *i)* $k(\mathcal{L}_3) = k(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3)$.

ii) The relations among the $\mathfrak{s}$-invariants for other hyperelliptic curves of genus 3 are given in the Fig. 3.

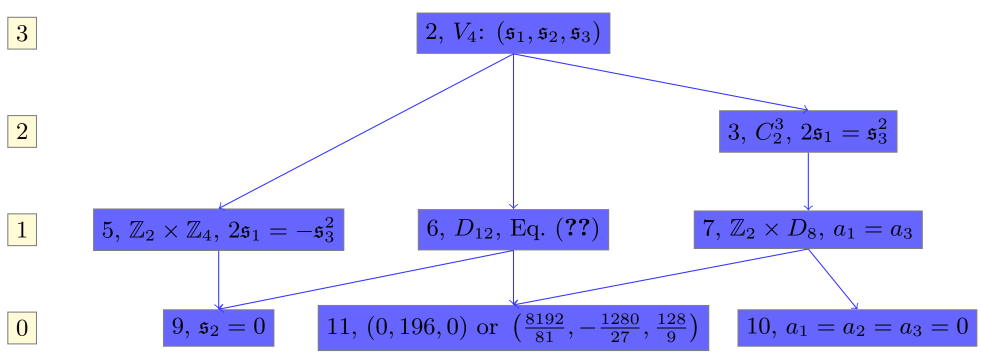

FIGURE 3. Relations among $\mathfrak{s}$-invariants for hyperelliptic curves of genus 3 with extra involutions.

7.3. **Equations of other superelliptic curves of genus 3.** In this section we take a quick glance of all superelliptic curves of genus 3 in all positive characteristics $\neq 2$. Similar tables are computed for all genus $g \leq 10$ in all characteristics and include for each curve the normal equation of the curve, the automorphism group, the invariants of the corresponding binary forms, $\mathfrak{s}$-invariants, the dimension of the corresponding moduli space. In a current project the half-integer theta characteristics will be computed in each case and the equation of the corresponding curve in terms of these characteristics.

In the Table below we present these curves for $p = 7$ for the cases 1-32 of the Table 2 so we can give an idea how this tables will look on the website with all the

TABLE 6. Superelliptic curves of genus three

#	Case	$\bar{G}$	n	Equation
				Genus 3, $p = 7$
1	1		n=2	$x^7 + a_1x^6 + a_2x^5 + a_3x^4 + a_4x^3 + a_5x^2 + a_6x + 1$
2	4			$(x^2 + a_1x + 1)(x^2 + a_2x + 1)(x^2 + a_3x + 1)(x^2 + a_4x + 1)$
3	4			$(x^4 + a_1x^2 + 1)(x^4 + a_2x^2 + 1)$
4	4			$x^8 + a_1x^4 + 1$
5	2, 7			$x^8 - 1$
6	6			$x(x^2 + a_1x + 1)(x^2 + a_2x + 1)(x^2 + a_3x + 1)$
7	6			$x(x^6 + a_1x^3 + 1)$
8	7			$(x^2 - 1)(x^2 + a_1x + 1)(x^2 + a_2x + 1)(x^2 + a_3x + 1)$
9	7			$(x^4 - 1)(x^4 + a_1x^2 + 1)$
10	8,9			$x(x^6 - 1)$
11	8			$x(x^2 - 1)(x^4 + a_1x^2 + 1)$
12	9			$x(x^2 - 1)(x^2 + a_1x + 1)(x^2 + a_2x + 1)$
13	12,17			$x^8 + 14x^4 + 1$
14	1		n=3	$x^4 + a_1x^3 + a_2x^2 + a_3x + 1$
15	8			$x(x - 1)(x^2 + a_1x + 1)$
16	8			$x(x^3 - 1)$
17	4		n=4	$(x^2 + a_1x + 1)(x^2 + a_2x + 1)$
18	4			$x^4 + a_1x^2 + 1$
19	2,7			$x^4 - 1$
20	6			$x(x^2 + a_1x + 1)$
21	7			$(x^2 - 1)(x^2 + a_1x + 1)$
22	8,9			$x(x^2 - 1)$
23	11			$x^4 + 21\sqrt{3}x^2 + 1$
24	8		n=7	$x(x - 1)$

data. Clicking on each curve will display all the information about the curve such as the automorphism group, invariants of the ninary form, $\mathfrak{s}$-invariants, an equation of the curve in terms of the theta-nulls, etc.

8. Concluding remarks

Finally, we are able to compute for a given genus $g \geq 2$ all full automorphism groups, equations, of genus g superelliptic curves defined over any algebraically closed field of characteristic different from two. We organize them according to their level n.

These tables are computed for all genus $g \leq 10$ in all characteristics and include for each curve the normal equation of the curve, the automorphism group, the invariants of the corresponding binary forms, $\mathfrak{s}$-invariants, the dimension of the corresponding moduli space. Such results will be presented in a continuation of this paper, [3] where some of the algorithms will be described in more detail.

In a current project we study superelliptic curves defined over $\mathbb{C}$. The half-integer theta characteristics will be computed in each case and the equation of the corresponding curve in terms of these characteristics, see [5].

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ON THE IMPLEMENTATION OF PUBLIC KEY ALGORITHM BASED ON GRAPHS AND THEIR SYMMETRIES

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Abstract. The paper devoted to implementation of the public key algorithm based on directed algebraic graphs over finite commutative ring K and their symmetries. First we expand the key space K^n of graph based encryption algorithm in such way that arbitrary chosen plaintext can be converted to arbitrary chosen ciphertext. Second, we conjugate chosen encryption map, which is a composition of several “elementary” cubical polynomial automorphisms of a free module K^n with special invertible affine transformation of K^n . Finally we compute symbolically corresponding cubic public map g of K^n onto K^n. We evaluate time for the generation of g, time of execution of public map, number of monomial expression in the list of corresponding public rules.

1. Introduction

Cloud computing systems open a new perspective in various aspects of computing.

Some security issues raised by cloud computing are motivated by virtualization. Dynamic scalability or “elasticity” will help generalize high-performance computing and very large data sets in applications. But the real gains in performance depend heavily on the predictability of physical and virtualized resources. It means that the balancing of performance against security and the adaptation of HPC or VLDB techniques to cloud computing are important issues and will have long-lasting scientific content.

The direction of Key Dependent Message (KDM) secure encryption in Cryptography can bring an appropriate security tools for Cloud Computing.

In publications [4] were proposed classes of stream ciphers and public key algorithms based on explicit construction of families of algebraic graphs of large girth.

It was shown that for each finite commutative ring K we can create a cubical polynomial map f of K^n onto K^n depending on string of regular elements (non zero divisors $(\alpha_1\alpha_2,\ldots,\alpha_t)$ password). If $t \leq (n+5)/2$ then different strings produce different ciphertext. One can use such a map as a stream cipher. It is possible to combine f with two invertible sparse affine transformations τ_1 and τ_2 and use the composition $g = \tau_1 f \tau_2$ as a public rule. Public user is not able to decrypt without knowledge of τ_1, τ_2 and string $(\alpha_1\alpha_2,\ldots,\alpha_t)$.

One can set τ_2 as the inverse of τ_1 and use the "symbolic" generator g and related cyclic groupfor the Diffie -Hellman key exchange protocol. We can prove that the order of g is growing with the grows of parameter n

This publication is devoted to the implementation of generalisation of the above algorithm. We consider linear transformations T_{a} depending on the string $\mathrm{a} = (\beta_1,\beta_2,\ldots,\beta_d)$, where$d = [n/4]$ and use fT_{a} instead of f.

The constraction of transformation f use graphs $D(n,K)$ (graphs of large girth for $K = F_q$, which was very useful for creation of good LDPS codes in Coding Theory. The transformation T_{a} is a special automorphism of graph $D(n,K)$.

In fact the key space of all passwords $g = fT_{\mathrm{a}}$ has the following property in case of char k-for each pair plaintext p - ciphertext c there is a transform g sending p to c. So we hope that usage of families of large girth and their automprphism may lead to good public keys.

Classical problems on Turan type problems on studies of the maximal size of simple graphs without prohibited cycles are attractive for mathematicians because they are beautiful and difficult (see [2], [9]). The concept of a family of simple graphs of large girth appears as an important tool to study such problems. Later the applications of these problems in Networking [1], Coding Theory and Cryptography were found (see [11] and further references).

Section 2 is devoted to the concept of the girth indicator and the family of large girth for digraphs.

In Section 3 we consider the definition of a family of affine algebraic digraphs of large girth over commutative rings. Explicit constructions of such families of graphs can be used for the development of public keys and a key exchange protocol. We discuss the connection of these algorithms with the group theoretical discrete logarithm problem.

The known examples of families of simple algebraic graphs were constructed just in the case of finite fields (see [5]). In section 4 we consider an explicit construction of a family of affine algebraic digraphs of large girth over each finite commutative ring containing at least 3 regular elements. Different properties of this family are investigated in [12], [11], [13], [14] , [8], [7].

Section 5 is devoted to the latest implementation of the public key algorithm based on one of the family described in section 4.

2. On the families of directed graphs of large girth

The missing theoretical definitions on directed graphs the reader can find in [6]. Let Φ be an irreflexive binary relation over the set V, i.e., $\Phi \in V \times V$ and for each v the pair (v,v) is not the element of Φ.

We say that u is the neighbour of v and write $v \to u$ if $(v,u) \in \Phi$. We use the term *balanced binary relation graph* for the graph Γ of irreflexive binary relation ϕ over a finite set V such that for each $v \in V$ the sets $\{x|(x,v) \in \phi\}$ and $\{x|(v,x) \in \phi\}$

have the same cardinality. It is a directed graph without loops and multiple edges. We say that a balanced graph Γ is k-regular if for each vertex $v \in \Gamma$ the cardinality of $\{x|(v,x) \in \phi\}$ is k.

Let Γ be the graph of binary relation. The *path* between vertices a and b is the sequence $a = x_0 \to x_1 \to \dots x_s = b$ of length s, where x_i, $i = 0, 1, \dots s$ are distinct vertices.

We say that the pair of paths $a = x_0 \to x_1 \to \cdots \to x_s = b$, $s \geq 1$ and $a = y_0 \to y_1 \to \cdots \to y_t = b$, $t \geq 1$ form an (s,t)- commutative diagram $O_{s,t}$ if $x_i \neq y_j$ for $0 < i < s$, $0 < j < t$. Without loss of generality we assume that $s \geq t$.

We refer to the number $\max(s,t)$ as the rank of $O_{s,t}$. It is ≥ 2, because the graph does not contain multiple edges.

Notice that the graph of antireflexive binary relation may have a directed cycle $O_s = O_{s,0}$: $v_0 \to v_1 \to \dots v_{s-1} \to v_0$, where v_i, $i = 0, 1, \dots, s-1$, $s \geq 2$ are distinct vertices.

We will count directed cycles as commutative diagrams.

For the investigation of commutative diagrams we introduce *girth indicator* gi, which is the minimal value for $\max(s,t)$ for parameters s,t of a ommutative diagram $O_{s,t}$, $s+t \geq 3$. The minimum is taken over all pairs of vertices (a,b) in the digraph. Notice that two vertices v and u at distance $<$ gi are connected by the unique path from u to v of length $<$ gi.

We assume that the *girth* $g(\Gamma)$ of a directed graph Γ with the girth indicator $d+1$ is $2d+1$ if it contains a commutative diagram $O_{d+1,d}$. If there are no such diagrams we assume that $g(\Gamma)$ is $2d+2$.

In case of a symmetric binary relation gi $= d$ implies that the girth of the graph is $2d$ or $2d-1$. It does not contain an even cycle $2d-2$. In general case gi $= d$ implies that $g \geq d+1$. So in the case of the family of graphs with unbounded girth indicator, the girth is also unbounded. We also have gi $\geq g/2$.

In the case of symmetric irreflexive relations the above mentioned general definition of the girth agrees with the standard definition of the girth of simple graph, i.e., the length of its minimal cycle.

We will use the term *the family of graphs of large girth* for the family of balanced directed regular graphs Γ_i of degree k_i and order v_i such that $\mathrm{gi}(\Gamma_i)$ is $\geq c\log_{k_i} v_i$, where c' is a constant independent of i.

As it follows from the definition $g(\Gamma_i) \geq c'\log_{k_i}(v_i)$ for an appropriate constant c'. So, it agrees with the well known definition for the case of simple graphs.

The diameter of the strongly connected digraph [6] is the minimal length d of the shortest directed path $a = x_0 \to x_1 \to x_2 \cdots \to x_d$ between two vertices a and b. Recall that a graph is k-regular, if each vertex of G has exactly k outputs. Let F be the infinite family of k_i regular graphs G_i of order v_i and diameter d_i. We say, that F is a family of small world graphs if $d_i \leq C\log_{k_i}(v_i)$, $i = 1, \dots$ for some constant C independent on i. The definition of small world simple graphs and related explicit constructions the reader can find in [3]. For the studies of small world simple graphs without small cycles see [9], [12].

3. On the K-theory of affine graphs of high girth and its cryptographical motivations

Let K be a commutative ring. A *directed algebraic graph* ϕ over K consists of two things, such as the *vertex set* Q being a quasiprojective variety over K of nonzero dimension and the *edge set* being a quasiprojective variety ϕ in $Q \times Q$. We assume that ($x\phi y$ means $(x, y) \in \phi$).

The graph ϕ is *balanced* if for each vertex $v \in Q$ the sets $\mathrm{Im}(v) = \{x \mid v\phi x\}$ and $\mathrm{Out}(v) = \{x \mid x\phi v\}$ are quasiprojective varieties over K of the same dimension.

The graph ϕ is *homogeneous* (or (r, s)-homogeneous) if for each vertex $v \in Q$ the sets $\mathrm{Im}(v) = \{x|v\phi x\}$ and $\mathrm{Out}(v) = \{x|x\phi v\}$ are quasiprojective varieties over F of fixed nonzero dimensions r and s, respectively.

In the case of *balanced homogeneous algebraic graphs* for which $r = s$ we will use the term r-homogeneous graph. Finally, *regular algebraic graph* is a balanced homogeneous algebraic graph over the ring K if each pair of vertices v_1 and v_2 is a pair of isomorphic algebraic varieties.

Let $\mathrm{Reg}(K)$ be the totality of regular elements (or nonzero divisors) of K, i.e., nonzero elements $x \in K$ such that for each nonzero $y \in K$ the product xy is different from 0. We assume that the $\mathrm{Reg}(K)$ contains at least 3 elements. We assume here that K is finite, thus the vertex set and the edge set are finite and we get a usual finite directed graph.

We apply the term *affine graph* for the regular algebraic graph such that its vertex set is an affine variety in Zarisski topology.

Let G be r-regular affine graph with the vertex $V(G)$, such that Out v, $v \in V(G)$ is isomorphic to the variety $R(K)$. Let the variety $E(G)$ be its arrow set (a binary relation in $V(G) \times V(G)$). We use the standard term *perfect algebraic colouring of edges* for the polynomial map ρ from $E(G)$ onto the set $R(K)$ (the set of colours) if for each vertex v different output arrows $e_1 \in \mathrm{Out}(v)$ and $e_2 \in \mathrm{Out}(v)$ have distinct colours $\rho(e_1)$ and $\rho(e_2)$ and the operator $N_\alpha(v)$ of taking the neighbour u of vertex v ($v \to u$) is a polynomial map of the variety $V(G)$ into itself.

We will use the term *rainbow-like colouring* in the case when the perfect algebraic colouring is a bijection. Let $\mathrm{dirg}(G)$ be a directed girth of the graph G, i.e., the minimal length of a directed cycle in the graph. Obviously $\mathrm{gi}(G) \leq \mathrm{dirg}(G)$.

Studies of infinite families of directed affine algebraic digraphs over commutative rings K of large girth with the rainbow-like colouring is a nice and a difficult mathematical problem. Good news is that such families do exist. In the next section we consider the example of such a family for each commutative ring with more than 2 regular elements.

Here, at the end of section, we consider cryptographical motivations for studies of such families.

1) Let G be a finite group and $g \in G$. The discrete logarithm problem for group G is about finding a solution for the equation $g^x = b$ where x is unknown positive number. If the order $|g| = n$ is known we can replace G on a cyclic group C_n. So we may assume that the order of g is sufficiently large to make unfeasible the computation of n. For many finite groups the discrete logarithm problem is NP complete.

Let K be a finite commutative ring and M be an affine variety over K. Then the Cremona group $C(M)$ of all polynomial automorphism of the variety M can be large. For example, if K is a finite prime field F_p and $M = {F_p}^n$ then $C(M)$ is a symmetric group S_{p^n}.

Let us consider the family of affine graphs $G_i(K)$, $i = 1, 2, \ldots$ with the rainbow-like algebraic colouring of edges such that $V(G_i(K)) = V_i(K)$, where K is a commutative ring, and the colour sets are algebraic varieties $R_i(K)$. Let us choose a constant k. The operator $N_\alpha(v)$ of taking the neighbour of a vertex v corresponding to the output arrow of colour α are elements of $C_i = C(V_i(K))$. We can chose a relatively small number k to generate $h = h_i = N_{\alpha_1} N_{\alpha_2} \dots N_{\alpha_k}$ in each group C_i, $i = 1, 2, \dots$

Let us assume that the family of graphs $G_i(K)$ is the family of graphs of large girth. It means that the girth indicator $\mathrm{gi}_i = \mathrm{gi}(G_i(K))$ and the parameter $\mathrm{dirg}_i = \mathrm{dirg}(G_i(K))$ are growing with the growth of i. Notice that $|h_i|$ is bounded below by dirg_i/k. So there is j such that for $i \geq j$ the computation of $|h_i|$ is impossible. Finally we can take the base $g = u^{-1} h_j u$ where u is a chosen element of C_j to hide the graph up to conjugation. We may use some package of symbolic computations to express the polynomial map g via the list of polynomials in many unknowns. For example, if $V_j(K)$ is a free module K^n then we can write g in a public mode fashion

$x_1 \to g_1(x_1, x_2, \dots, x_n)$, $x_2 \to g_2(x_1, x_2, \dots, x_n)$, $\dots$, $x_n \to g_n(x_1, x_2, \dots, x_n)$.

The symbolic map g can be used for Diffie - Hellman *key exchange protocol* (see [3] for the details). Let Alice and Bob be correspondents. Alice computes the symbolic map g and send it to Bob via open channel. So the variety and the map are known for the adversary (Cezar).

Let Alice and Bob choose natural numbers n_A and n_B, respectively.

Bob computes g^{n_B} and sends it to Alice, who computes $(g^{n_B})^{n_A}$, while Alice computes g^{n_A} and sends it to Bob, who is getting $(g^{n_A})^{n_B}$. The common information is $g^{n_A n_B}$ given in "public mode fashion".

Bob can be just a public user (no information on the way in which the map g were cooked) , so he and Cezar are making computations much slower than Alice who has the decomposition $g = u^{-1} N_{\alpha_1} N_{\alpha_2} \dots N_{\alpha_k} u$.

We may modify slightly the Diffie - Hellman protocol using the action of the group on the variety. Alice chooses a rather short password $\alpha_1, \alpha_2, \dots, \alpha_k$, computes the public rules for the encryption map g and sends them to Bob via an open channel together with some vertex $v \in V_j(K)$.

Then Alice and Bob choose natural numbers n_A and n_B, respectively.

Bob computes $v_B = g^{n_B}(v)$ and sends it openly to Alice, who computes $(g^{n_A})(v_B)$, while Alice computes $v_A = g^{n_A}(v)$ and sends it to Bob, who is getting $(g^{n_B})(v_A)$.

The common information is the vertex $g^{n_A \times n_B}(v)$.

In both cases Cezar has to solve one of the equations $E^{n_B}(u_A) = z$ or $E^{n_A}(u_B) = w$ for unknowns n_B or n_A, where z and w are known points of the variety.

2) We can construct the *public key* map in the following manner:

The key holder (Alice) chooses the variety $V_j(K)$ and the sequence $\alpha_1, \alpha_2, \dots, \alpha_t$ of length $t = t(j)$ to determine the encryption map g as above. Let $\dim(V_j(K) = n = n(j)$ and each element of the variety be determined by independent parameters $x_1, x_2, \dots, x_n$. Alice presents the map in the form of public rules, such as

$x_1 \to f_1(x_1, x_2, \dots, x_n)$, $x_2 \to f_2(x_1, x_2, \dots, x_n)$, $\dots$, $x_n \to f_n(x_1, x_2, \dots, x_n)$.

We can assume (at least theoretically) that the public rule depending on parameter j is applicable to encryption of potentially infinite text (parameter t is a linear function on j now).

For the computation she may use the Gröbner base technique or alternative methods, special packages for the symbolic computation (popular "Mathematica" or "Maple", package "Galois" for "Java" as well special fast symbolic software). So Alice can use the decomposition of the encryption map into u^{-1}, maps of kind N_α

and u to encrypt fast. For the decryption she can use the inverse graph $G_j(K)^{-1}$ for which $VG_j(K)^{-1} = VG_j(K)$ and vertices w_1 and w_2 are connected by an arrow if and only if w_2 and w_1 are connected by an arrow in $G_j(K)$. Let us assume that colours of $w_1 \to w_2$ in $G_j(K)^{-1}$ and $w_2 \to w_1$ in $G_j(K)$ are of the same colour. Let $N'_\alpha(x)$ be the operator of taking the neighbour of vertex x in $G_j(K)^{-1}$ of colour α. Then Alice can decrypt applying consequently $u^{-1}, N'_{\alpha_t}, N'_{\alpha_{t-1}}, \ldots, N_{\alpha_1}$ and u to the ciphertext. So the decryption and the encryption for Alice take the same time. She can use a numerical program to implement her symmetric algorithm.

Bob can encrypt with the public rule but for a decryption he needs to invert the map. Let us consider the case $t_j = kl$, where k is a small number and the sequence $\alpha_1, \alpha_2, \ldots, \alpha_{t_j}$ has the period k and the transformation $h = u^{-1}N_{\alpha_1}N_{\alpha_2}\ldots N_{\alpha_k}u$ is known for Bob in the form of public key mode. In such a case a problem to find the inverse for g is equivalent to a discrete logarithm problem with the base h in related Cremona group of all polynomial bijective transformations.

Of course for further cryptoanalysis we need to study the information on possible divisors of order of the base of related discrete logarithm problem, alternative methods to break the encryption. In the next section the family of digraphs $RE_n(K)$ will be described.

3) We may study security of the private key algorithm used by Alice in the algorithm of the previous paragraph but with a parameter t bounded by the girth indicator of graph $G_j(K)$. In that case different keys produce distinct ciphertexts from the chosen plaintext. In that case we prove that if the adversary has no access to plaintexts then he can break the encryption via the brut-force search via all keys from the key space. The encryption map has no fixed points.

4. On the family of affine digraph of large girth over commutative rings

E. Moore used term *tactical configuration* of order (s, t) for biregular bipartite simple graphs with bidegrees $s + 1$ and $r + 1$. It corresponds to the incidence structure with the point set P, the line set L and the symmetric incidence relation I. Its size can be computed as $|P|(s + 1)$ or $|L|(t + 1)$.

Let $F = \{(p, l)|p \in P, l \in L, pIl\}$ be the totality of flags for the tactical configuration with partition sets P (point set) and L (line set) and an incidence relation I. We define the following irreflexive binary relation ϕ on the set F:

Let (P, L, I) be the incidence structure corresponding to regular tactical configuration of order t.

Let $F_1 = \{(l, p)|l \in L, p \in P, lIp\}$ and $F_2 = \{[l, p]|l \in L, p \in P, lIp\}$ be two copies of the totality of flags for (P, L, I). Brackets and parenthesis allow us to distinguish elements from F_1 and F_2. Let $DF(I)$ be the directed graph (double directed flag graph) on the disjoint union of F_1 with F_2 defined by the following rules

$(l_1, p_1) \to [l_2, p_2]$ if and only if $p_1 = p_2$ and $l_1 \neq l_2$,

$[l_2, p_2] \to (l_1, p_1)$ if and only if $l_1 = l_2$ and $p_1 \neq p_2$.

Below we consider the family of graphs $D(k, K)$, where $k > 5$ is a positive integer and K is a commutative ring. Such graphs are disconnected and their connected components were investigated in [13] (for the case when K is a finite field F_q see [5]).

Let P and L be two copies of Cartesian power K^N, where K is the commutative ring and N is the set of positive integer numbers. Elements of P will be called *points* and those of L *lines*.

To distinguish points from lines we use parentheses and brackets. If $x \in V$, then $(x) \in P$ and $[x] \in L$. It will also be advantageous to adopt the notation for co-ordinates of points and lines introduced in [15] for the case of general commutative ring K:

$$\begin{aligned}(p) &= (p_{0,1}, p_{1,1}, p_{1,2}, p_{2,1}, p_{2,2}, p'_{2,2}, p_{2,3}, \ldots, p_{i,i}, p'_{i,i}, p_{i,i+1}, p_{i+1,i}, \ldots),\\ [l] &= [l_{1,0}, l_{1,1}, l_{1,2}, l_{2,1}, l_{2,2}, l'_{2,2}, l_{2,3}, \ldots, l_{i,i}, l'_{i,i}, l_{i,i+1}, l_{i+1,i}, \ldots].\end{aligned}$$

The elements of P and L can be thought as infinite ordered tuples of elements from K, such that only a finite number of components are different from zero.

We now define an incidence structure (P, L, I) as follows. We say that the point (p) is incident with the line $[l]$, and we write $(p)I[l]$, if the following relations between their co-ordinates hold:

$$\begin{aligned}l_{i,i} - p_{i,i} &= l_{1,0}p_{i-1,i}\\ l'_{i,i} - p'_{i,i} &= l_{i,i-1}p_{0,1}\\ l_{i,i+1} - p_{i,i+1} &= l_{i,i}p_{0,1}\\ l_{i+1,i} - p_{i+1,i} &= l_{1,0}p'_{i,i}\end{aligned}$$

(These four relations are defined for $i \geq 1$, $p'_{1,1} = p_{1,1}$, $l'_{1,1} = l_{1,1}$). This incidence structure (P, L, I) we denote as $D(K)$. We identify it with the bipartite *incidence graph* of (P, L, I), which has the vertex set $P \cup L$ and the edge set consisting of all pairs $\{(p), [l]\}$ for which $(p)I[l]$.

For each positive integer $k \geq 2$ we obtain an incidence structure (P_k, L_k, I_k) as follows. First, P_k and L_k are obtained from P and L, respectively, by simply projecting each vector onto its k initial coordinates with respect to the above order. The incidence I_k is then defined by imposing the first $k-1$ incidence equations and ignoring all others. The incidence graph corresponding to the structure (P_k, L_k, I_k) is denoted by $D(k, K)$.

For each positive integer $k \geq 2$ we consider the *standard* graph homomorphism ϕ_k of (P_k, L_k, I_k) onto $(P_{k-1}, L_{k-1}, I_{k-1})$ defined L_k by simply projection of each vector from P_k and L_k onto its $k-1$ initial coordinates with respect to the above order. The transformation $t'_{m,m}(x)$ acts on vertices of $D(K)$ by the following rules.

(a) $l'_{m,m} \to l'_{m,m} + x$, $p'_{m,m} \to p'_{m,m} + x$.

(b) $l_{m+1,m} \to l_{m+1,m} + l_{1,0}x$.

(c) $l_{m+1,m+1} \to l_{m+1,m+1} + l_{1,1}x$, $p_{m+1,m+1} \to l_{m+1,m+1} + p_{1,1}x$

(d) $l_{m+r,m+r} \to l_{m+r,m+r} + l'_{r,r}x$, $p_{m+r,m+r} \to p_{m+r,m+r} + p'_{r,r}x$, $r \geq 2$.

(e) $l_{m+r+1,m+r} \to l_{m+r+1,m+r} + l_{r+1,r}x$, $p_{m+r+1,m+r} \to p_{m+r+1,m+r} + p_{r+1,r}x$, $r \geq 2$.

(f) All other components are unchanged.

We define the transformation T_{a}, where $\mathrm{a} = (\beta_{22}, \beta_{33}, \ldots)$ as a product of all transformations $t'_{i,i}(\beta_{ii})$

Let $DE_n(K)$ $(DE(K))$ be the double directed graph of the bipartite graph $D(n, K)$ $(D(K)$, respectively). Remember, that we have the arc e of kind $(l^1, p^1) \to [l^2, p^2]$ if and only if $p^1 = p^2$ and $l^1 \neq l^2$. Let us assume that the colour $\rho(e)$ of the arc e is $l^1_{1,0} - l^2_{1,0}$.

Recall, that we have the arc e' of kind $[l^2, p^2] \to (l^1, p^1)$ if and only if $l^1 = l^2$ and $p^1 \neq p^2$. Let us assume that the colour $\rho(e')$ of arc e' is $p^1_{1,0} - p^2_{1,0}$. It is easy to see that ρ is a perfect algebraic colouring.

If K is finite, then the cardinality of the colour set is $(|K| - 1)$. Let Reg$K$ be the totality of regular elements, i.e., not zero divisors. Let us delete all arrows with colour, which is a zero divisor. We will show that a new graph $RE_n(K)$ ($RE(K)$) with the induced colouring into colours from the alphabet $\mathrm{Reg}(K)$ is vertex transitive. Really, according to [9] graph $D(n, K)$ is an edge transitive. This fact had been established via the description of regular on the edge set subgroup $U(n, K)$ of the automorphisms group $\mathrm{Aut}(G)$. The vertex set for the graph $DE_n(K)$ consists of two copies F_1 and F_2 of the edge set for $D(n, K)$. It means that Group $U(n, K)$ acts regularly on each set F_i, $i = 1, 2$. An explicit description of generators for $U(n, K)$ implicates that this group is a colour preserving group for the graph $DE_n(K)$ with the above colouring.

If K is finite, then the cardinality of the colour set is $(|K| - 1)$. Let Reg$K$ be the totality of regular elements, i.e., non-zero divisors. Let us delete all arrows with colour, which is a zero divisor. We can show that a new affine graph $RE_n(K)$ ($RE(K)$) with the induced colouring into colours from the alphabet $\mathrm{Reg}(K)$ is vertex transitive (see [14]).

Notice, that each T_{a} acts naturally on the flags, it is an automorphism of $RE_n(K)$.

5. On the implementation of the public key algorithm based on $RE(t, K)$

The graphs $CRE_n(K)$ have the best known speed of growth of the girth indicator evaluated in the previous section. It turns out that for the computer implementation of the public key algorithm described in the section 4 the family $RE_n(K)$ of "enveloping" for $CRE_n(K)$ graphs were chosen first. It is also a family of digraphs of large girth but the speed of the growth of girth indicator for the family is less of those for $RE_n(K)$. Graphs $RE_n(K)$ were defined via the family of graphs $D(n, K)$ in the way described in the previous section. So, in some publications the description of the algorithm was done in terms of $D(n, K)$. We introduced here a speed evaluation of the algorithm for its latest implementation.

The set of vertices of the graph $RE_n(K)$ is a union of two copies free module K^{n+1}. So the Cremona group of the variety is the direct product of $C(K^{n+1})$ with itself, expanded by polarity π. In the simplest case of a finite field F_p, where p is a prime number $C(F_p)$ is a symmetric group $S_{p^{n+1}}$. The Cremona group $C(K^{n+1})$ contains the group of all affine invertible transformations, i.e., transformation of kind $\mathrm{x} \to \mathrm{x}A + \mathrm{b}$, where $\mathrm{x} = (x_1, x_2, \ldots, x_{n+1}) \in C(K^{n+1})$, $\mathrm{b} = (b_1, b_2, \ldots, b_{n+1})$ is a chosen vector from $C(K^{n+1})$ and A is a matrix of a linear invertible transformation of K^{n+1}.

Graph $RE_n(K)$ is a bipartite directed graph. We assume that the plaintext K^{n+1} is a point $(p_1, p_2, \ldots, p_{n+1})$. We choose two affine transformations T_1 and T_2 and a linear transformation u will be of kind $p_1 \to p_1 + a_1p_2 + a_3p_3 + \cdots + a_{n+1}$. We slightly modify a general scheme, so Alice computes symbolically of chosen T_1 and T_2, chooses a string $(\beta_1, \beta_2, \ldots, \beta_l)$ of colours for $RE_n(K)$, such that $\beta_i \neq -\beta_{i+1}$ for $i = 1, 2, \ldots, l-1$. She computes $N_l = N_{\beta_1} \times N_{\beta_2} \cdots \times N_{\beta_l}$. Recall that N_α,

$\alpha \in \mathrm{Reg}(K)$ is the operator of taking the neighbour of the vertex v alongside the arrow with the colour α in the graph $RE_n(K)$. Alice chooses additionally string a.

Alice keeps chosen parameters secret and computes the public rule g as the symbolic composition of T_1, N, T_{a} and T_2.

In case $K = F_q$, $q = 2^n$ this public key rule has a certain similarity to the Imai-Matsomoto public rule, which is computed as a composition $T_1 E T_2$ of two linear transformations T_1 and T_2 of the vector space $F_{2^n F_{2^s}}$, where F_{2^s} is a special subfield, and E is a special Frobenius automorphism of F_{2^n}. The public rule corresponding to $T_1 E T_2$ is a quadratic polynomial map (see [3] for the detailed description of the algorithm, its cryptoanalisis and generalisations by J. Patarin)

In the case of $RE_n(K)$ the degree of transformation N_l is 3, independently on the choice of length l [16]. So the public rule is a cubical polynomial map of the free module K^{n+1} onto itself. In case of a finite field the algorithm is equivalent to the public rule considered in [10].

5.1. **On the time evaluation for the public rule.** Recall, that we combine a graph transformation N_l with two affine transformation T_1 and T_2 and transformation T_a. Alice can use $T_1 N_l T_a T_2$ for the construction of the following public map of

$$y = (F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n))$$

$F_i(x_1, \ldots, x_n)$ are polynomials of n variables written as the sums of monomials of kind $x_{i+1} \ldots x_{i_3}$, where $i_1, i_2, i_3 \in 1, 2, \ldots, n_1$ with the coefficients from $K = F_q$. As we mention before the polynomial equations $y_i = F_i(x_1, x_2, \ldots, x_n)$, which are made public, have the degree 3. Hence the process of an encryption and a decryption can be done in polynomial time $O(n^4)$ (in one $y_i,\ i = 1, 2 \ldots, n$ there are $2(n^3 - 1)$ additions and multiplications). But the cryptoanalyst Cezar, having only a formula for y, has a very hard task to solve the system of n equations of n variables of degree 3. It is solvable in exponential time $O(3^{n^4})$ by the general algorithm based on Gröbner basis method. Anyway studies of specific features of our polynomials could lead to effective cryptoanalysis. This is an open problem for specialists.

We have written a program for generating a public key and for encrypting text using the generated public key. The program is written in C++ and compiled with the gcc compiler (version 4.4.1).

We have implemented three cases:

- T_1 and T_2 are identities,
- T_1 and T_2 are of kind $x_1 \to x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_{n+1} x_{n+1}$ (linear time of computing T_1 and T_2),
- $T_1 = A_1 x + b_1$, $T_2 = A_2 x + b_2$; matrices A_1, A_2 and vectors b_1, b_2 has mostly nonzero elements.

The table 1 applies to the second case. It presents the time (in milliseconds) of the generation of the public key depending on the number of variables (n) and the password length (p). It also presents the time of computing the transformation T_a.

The time of encryption process depends linearly on the number of monomials (the number of nonzero coefficients) in cubic polynomials F_1, $F_2 \ldots F_n$ in the public map $y = (F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n))$. For $n = 120$ and $p = 60$ this number is about 8500 in the first case, about 780000 in the second case and about 2600000 in the third case.

TABLE 1. Time of public key generation ($K = F_{2^{32}}$)

	$p = 20$		$p = 40$		$p = 60$	
		T_a		T_a		T_a
$n = 10$	16	0	16	0	31	0
$n = 20$	141	16	280	0	437	0
$n = 30$	562	0	1217	0	1888	16
$n = 40$	1513	16	3464	16	5678	16
$n = 50$	3261	63	8346	62	13089	62
$n = 60$	6271	125	16239	140	26426	156
$n = 70$	11139	328	29032	328	47440	312
$n = 80$	17301	531	47315	546	79279	561
$n = 90$	26582	1435	72415	1560	122866	1513
$n = 100$	38173	2418	104053	2418	180790	2418
$n = 110$	53149	3557	144987	3634	251380	3572
$n = 120$	70169	4867	189479	3151	338258	3308

Applying the transformation T_a has the biggest impact on the time of encryption in the first case — about 16% for $n = 120$ and $p = 60$. In the second case it is about 3.5% and in the third case it has no impact at all. The tables 2 and 3 apply to the second case. They present the number of monomials in a public map depending on n and p. The table 2 shows the number of monomials in a public map without transformation T_a and the table 3 — the number of monomials in a public map with T_a.

TABLE 2. Number (percentage) of nonzero coefficients ($T_1 N T_2$)

	$p = 20$	$p = 40$	$p = 60$
$n = 10$	435 (15.21%)	435 (15.21%)	435 (15.21%)
$n = 20$	3327 (9.39%)	3327 (9.39%)	3327 (9.39%)
$n = 30$	11795 (7.21%)	11795 (7.21%)	11795 (7.21%)
$n = 40$	27426 (5.56%)	27427 (5.56%)	27427 (5.56%)
$n = 50$	49995 (4.27%)	54855 (4.68%)	54855 (4.68%)
$n = 60$	77245 (3.24%)	93552 (3.93%)	93552 (3.93%)
$n = 70$	110395 (2.54%)	150865 (3.47%)	150865 (3.47%)
$n = 80$	149445 (2.03%)	222951 (3.03%)	222952 (3.03%)
$n = 90$	194395 (1.66%)	307015 (2.63%)	321075 (2.75%)
$n = 100$	245245 (1.39%)	398140 (2.25%)	436877 (2.47%)
$n = 110$	301995 (1.17%)	501165 (1.95%)	586735 (2.28%)
$n = 120$	364645 (1.00%)	616090 (1.70%)	756576 (2.08%)

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Table 3. Number (percentage) of nonzero coefficients ($T_1NT_aT_2$)

	$p = 20$	$p = 40$	$p = 60$
$n = 10$	435 (15.21%)	435 (15.21%)	435 (15.21%)
$n = 20$	3367 (9.51%)	3367 (9.51%)	3367 (9.51%)
$n = 30$	12070 (7.37%)	12070 (7.37%)	12070 (7.37%)
$n = 40$	28126 (5.70%)	28127 (5.70%)	28127 (5.70%)
$n = 50$	52009 (4.44%)	56505 (4.82%)	56505 (4.82%)
$n = 60$	81653 (3.43%)	96412 (4.05%)	96412 (4.05%)
$n = 70$	118894 (2.73%)	155865 (3.58%)	155865 (3.58%)
$n = 80$	158948 (2.16%)	230346 (3.13%)	230347 (3.13%)
$n = 90$	221795 (1.90%)	318860 (2.73%)	332275 (2.85%)
$n = 100$	287881 (1.63%)	416661 (2.36%)	452057 (2.56%)
$n = 110$	361270 (1.40%)	529460 (2.06%)	607860 (2.36%)
$n = 120$	448691 (1.24%)	646858 (1.78%)	783666 (2.16%)

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GENERALIZED HYERS-ULAM STABILITY OF DERIVATIONS IN PROPER LIE CQ^*-ALGEBRAS

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Abstract. In this paper, we obtain the general solution and the generalized Hyers-Ulam stability for the following functional equation

$$f\Big(\frac{\sum_{i=1}^{m} x_i}{m}\Big) + \sum_{\substack{i=1\\ i\neq j}}^{m} f\Big(\frac{x_j - x_i}{m}\Big) = f(x_j).$$

This is applied to investigate derivations and their stability in proper Lie CQ^*-algebras.

1. Introduction and preliminaries

Ulam [42] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these problems was the following question concerning the stability of homomorphisms.

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Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies the inequality

$$d(f(x * y), f(x) \diamond f(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $T : G_1 \to G_2$ with

$$d(f(x), T(x)) < \epsilon$$

for all $x \in G_1$?

If the answer is affirmative, we say that the equation of homomorphism $T(xy) = T(x)T(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

Hyers [18] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

Hyers' theorem was generalized by Aoki [3] for additive mappings and independently by Th.M. Rassias [36] for linear mappings by considering an *unbounded Cauchy difference*. In 1994, a generalization of Th.M. Rassias' theorem was obtained by Găvruta [15]. J.M. Rassias [31]-[34] generalized Hyers result. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [11]-[13], [20], [24]-[28],[30], [37]-[39]). We also refer the readers to the books [1], [10], [19], [21] and [37].

We recall some basic facts concerning quasi $*$-algebras.

Definition 1.1. Let A be a linear space and A_0 be a $*$-algebra contained in A as a subspace. We say that A is a *quasi $*$-algebra* over A_0 if

- (i) the right and left multiplications of an element of A and an element of A_0 are always defined and linear;
- (ii) $x_1(x_2a) = (x_1x_2)a$, $(ax_1)x_2 = a(x_1x_2)$ and $x_1(ax_2) = (x_1a)x_2$ for all $x_1, x_2 \in A_0$ and all $a \in A$;

(iii) an involution $*$, which extends the involution of A_0, is defined in A with the property $(ab)^* = b^*a^*$, whenever the multiplication is defined.

Quasi $*$-algebras [22, 23] arise in natural way as completions of locally convex $*$-algebras whose multiplication is not jointly continuous; in this case one has to deal with topological quasi $*$-algebras.

A quasi $*$-algebra (A, A_0) is called *topological* if a locally convex topology τ on A is given such that:

(i) the involution $a \mapsto a^*$ is continuous for each $a \in A$,

(ii) the mappings $a \mapsto ab$ and $a \mapsto ba$ are continuous for each $a \in A$ and $b \in A_0$,

(iii) A_0 is dense in $A[\tau]$.

Throughout this paper, we suppose that a locally convex quasi $*$-algebra (A, A_0) is complete. For an overview on partial $*$-algebra and related topics we refer to [2].

In a series of papers [4], [5], [6], [7] many authors have considered a special class of quasi $*$-algebras, called proper CQ^*-algebras, which arise as completions of C^*-algebras. They can be introduced in the following way:

Definition 1.2. Let A be a Banach module over the C^*-algebra A_0 with involution $*$ and C^*-norm $\|.\|_0$ such that $A_0 \subset A$. We say that (A, A_0) is a *proper CQ^*-algebra* if

(i) A_0 is dense in A with respect to its norm $\|.\|$;

(ii) $(ab)^* = b^*a^*$ whenever the multiplication is defined;

(iii) $\|y\|_0 = \max\{\sup_{a\in A,\|a\|\le 1} \|ay\|, \sup_{a\in A,\|a\|\le 1} \|ya\|\}$ for all $y \in A_0$.

A proper CQ^*-algebra (A, A_0) is said to have a unit e if there exists an element $e \in A_0$ such that $ae = ea = a$ for all $a \in A$. In this paper we will always assume that the proper CQ^*-algebra under consideration have an identity.

Definition 1.3. A proper CQ^*-algebra (A, A_0), endowed with a bilinear multiplication $[,] : (A \times A_0) \cup (A_0 \times A) \to A$, called the bracket, which satisfies two simple properties:

(i) $[x_1, x_2] = -[x_2, x_1]$ for all $(x_1, x_2) \in (A \times A_0) \cup (A_0 \times A)$;

(ii) $[x_1, [x_2, x_3]] = [[x_1, x_2], x_3] + [x_1, [x_2, x_3]]$ for all $x_1, x_2, x_3 \in A_0$

is called a *proper Lie CQ^*-algebra.*

Definition 1.4. Let (A, A_0) be a proper Lie CQ^*-algebras. A $\mathbb{C}$-linear mapping $\delta : A_0 \to A$ is called a *Lie derivation* if

$$\delta([z, x]) = [\delta(z), x] + [z, \delta(x)]$$

for all $x, z \in A_0$ (see [28]).

Throughout this paper, we assume that m and j are fixed positive integers with $m \geq 2$.

In this paper, we obtain the general solution and the generalized Ulam-Hyers stability for the following functional equation

$$f\Big(\frac{\sum_{i=1}^{m} x_i}{m}\Big) + \sum_{\substack{i=1 \\ i\neq j}}^{m} f\Big(\frac{x_j - x_i}{m}\Big) = f(x_j) \tag{1.1}$$

where m is a fixed positive integer with $m \geq 2$. This is applied to investigate derivations and their stability on proper Lie CQ^*-algebras.

2. Solution of functional equation (1.1)

Throughout this section, let both X and Y be real vector spaces. We here present the general solution of (1.1).

Theorem 2.1. *A mapping $f : X \to Y$ satisfies (1.1) if and only if the mapping $f : X \to Y$ is additive.*

We first assume that the mapping $f : X \to Y$ satisfies (1.1). Setting $x_j = x$ and $x_i = 0$ for all $1 \leq i \leq m$ and $i \neq j$ in (1.1), we get

$$f\left(\frac{x}{m}\right) = \frac{1}{m} f(x) \tag{2.1}$$

for all $x \in X$. Setting $x_j = x$, $x_{j+1} = y$ and $x_i = 0$ for $i \neq j, j+1$ in (1.1) and using (2.1), we get

$$f\left(\frac{x+y}{m}\right) + f\left(\frac{x-y}{m}\right) = \frac{2}{m} f(x) \tag{2.2}$$

for all $x, y \in X$. Replacing x and y by mx and my in (2.2), we get

$$f(x+y) + f(x-y) = 2f(x) \tag{2.3}$$

for all $x, y \in X$. Setting $y = x$ in (2.3), we get

$$f(2x) = 2f(x) \tag{2.4}$$

for all $x \in X$. Replacing x by $\frac{x+y}{2}$ and y by $\frac{x-y}{2}$ in (2.3), and using (2.4) we get

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$. So the mapping $f : X \to Y$ is additive.

Conversely, let the mapping $f : X \to Y$ be additive. By a simple computation, one can show that the mapping f satisfies the functional equation (1.1).

3. Stability of derivation on proper Lie CQ^*-algebras

Throughout this section, assume that (A, A_0) is a proper Lie CQ^*-algebra with C^*-norm $\|.\|_{A_0}$ and norm $\|.\|_A$. For convenience, we use the following abbreviation for a given mapping $f : \underbrace{A_0 \times A_0 \times ... \times A_0}_{m-\text{times}} \to A$

$$D_\mu f(x_1, ..., x_m) := f\left(\frac{\sum_{i=1}^m \mu x_i}{m}\right) + \sum_{\substack{i=1\\ i\neq j}}^m f\left(\frac{\mu x_j - \mu x_i}{m}\right) - \mu f(x_j)$$

for all $x_1, \cdots, x_m \in A_0$, where $\mu \in \mathbb{T}^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$.

We will use the following lemma:

Lemma 3.1. [29] *Let $f : A_0 \to A$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in A_0$ and all $\mu \in \mathbb{T}^1$. Then the mapping f is $\mathbb{C}$-linear.*

Theorem 3.2. *Let $\varphi : \underbrace{A_0 \times A_0 \times ... \times A_0}_{m-times} \to [0, \infty)$ and $\psi : A_0 \times A_0 \to [0, \infty)$ be mappings such that*

$$\lim_{n\to\infty} \frac{1}{m^n}\varphi(m^n x_1, ..., m^n x_m) = 0, \tag{3.1}$$

$$\lim_{n\to\infty} \frac{1}{m^{2n}}\psi(m^n x_1, m^n x_2) = 0, \tag{3.2}$$

$$\widetilde{\varphi_j}(x) := \sum_{i=1}^{\infty} \frac{1}{m^i}\varphi(0, ..., \underbrace{m^i x}_{j\,th}, ...0) < \infty \tag{3.3}$$

for all $x, x_1, \cdots, x_m \in A_0$. Suppose that $f : A_0 \to A$ is a mapping such that

$$\|D_\mu f(x_1, ..., x_m)\|_A \leq \varphi(x_1, ..., x_m), \tag{3.4}$$

$$\big\|f([x_1, x_2]) - [f(x_1), x_2] - [x_1, f(x_2)]\big\|_A \leq \psi(x_1, x_2) \tag{3.5}$$

for all $x_1, \cdots, x_m \in A_0$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique Lie derivation $\delta : A_0 \to A$ such that

$$\|f(x) - \delta(x)\|_A \leq \widetilde{\varphi_j}(x) \tag{3.6}$$

for all $x \in A_0$.

Letting $\mu = 1$, $x_j = mx$ and $x_i = 0$ for all $1 \leq i \leq m$ with $i \neq j$ in (3.4) , we get

$$\|f(mx) - mf(x)\|_A \leq \varphi(0, ..., \underbrace{mx}_{j\,th}, ...0) \tag{3.7}$$

for all $x \in A_0$. Replacing x by $m^n x$ in (3.7) and dividing both sides of (3.7) by m^{n+1}, we get

$$\Big\|\frac{1}{m^{n+1}}f(m^{n+1}x) - \frac{1}{m^n}f(m^n x)\Big\|_A \leq \frac{1}{m^{n+1}}\varphi(0, ..., \underbrace{m^{n+1}x}_{j\,th}, ..., 0) \tag{3.8}$$

for all $x \in A_0$ and all non-negative integers n. Hence

$$\begin{aligned}\left\|\frac{1}{m^{n+1}}f(m^{n+1}x)-\frac{1}{m^k}f(m^kx)\right\|_A &\le \sum_{i=k}^{n}\left\|\frac{1}{m^{i+1}}f(m^{i+1}x)-\frac{1}{m^i}f(m^ix)\right\|_A \\ &\le \sum_{i=k+1}^{n+1}\frac{1}{m^i}\varphi(0,...,\underbrace{m^ix}_{j\,th},...,0)\end{aligned} \tag{3.9}$$

for all $x \in A_0$ and all non-negative integers n and k with $n \ge k$. Therefore, we conclude from (3.3) and (3.9) that the sequence $\{\frac{1}{m^n}f(m^nx)\}_n$ is a Cauchy sequence in A for all $x \in A_0$. Since A is complete, the sequence $\{\frac{1}{m^n}f(m^nx)\}_n$ converges in A for all $x \in A_0$. So one can define the mapping $\delta : A_0 \to A$ by

$$\delta(x) := \lim_{n\to\infty}\frac{1}{m^n}f(m^nx) \tag{3.10}$$

for all $x \in A_0$. Letting $k = 0$ and passing the limit $n \to \infty$ in (3.9), we get (3.6). Now, we show that δ is a $\mathbb{C}$-linear mapping. It follows from (3.1), (3.4) and (3.10) that

$$\begin{aligned}\|D_1\delta(x_1,...,x_m)\|_A &= \lim_{n\to\infty}\frac{1}{m^n}\left\|D_1f(m^nx_1,...,m^nx_m)\right\|_A \\ &\le \lim_{n\to\infty}\frac{1}{m^n}\varphi(m^nx_1,...,m^nx_m)=0\end{aligned}$$

for all $x_1,\cdots,x_m \in A_0$. So the mapping δ satisfies (1.1). By Theorem 2.1, the mapping δ is additive.

Letting $x_j = mx$ and $x_i = 0$ for all $1 \le i \le m$ with $i \ne j$ in (3.4), we get

$$\|mf(\mu x)-\mu f(mx)\|_A \le \varphi(0,...,\underbrace{mx}_{j\,th},...,0) \tag{3.11}$$

for all $x \in A_0$. Replacing x by m^nx in (3.11) and dividing both sides of (3.11) by m^{n+1}, we get

$$\begin{aligned}&\left\|\frac{1}{m^n}f(\mu m^nx)-\frac{\mu}{m^{n+1}}f(m^{n+1}x)\right\|_A \\ &\le \frac{1}{m^{n+1}}\varphi(0,...,\underbrace{m^{n+1}x}_{j\,th},...,0)\end{aligned} \tag{3.12}$$

for all $x \in A_0$ and all non-negative integers n. Passing the limit $n \to \infty$ in (3.12) and using (3.1) and (3.10), we get

$$\delta(\mu x) = \mu\delta(x)$$

for all $\mu \in \mathbb{T}^1$ and for all $x \in A_0$. So by Lemma 3.1, we infer that the mapping $\delta : A_0 \to A$ is $\mathbb{C}$-linear. To prove the uniqueness of δ, let $\delta' : A_0 \to A$ be another additive mapping satisfying (3.6). It follows from (3.6) and (3.10) that

$$\begin{aligned}\|\delta(x)-\delta'(x)\|_A &= \lim_{n\to\infty}\frac{1}{m^n}\left\|f(m^nx)-\delta'(m^nx)\right\|_A \\ &\le \lim_{n\to\infty}\frac{1}{m^n}\widetilde{\varphi_j}(m^nx)=0\end{aligned}$$

for all $x \in A_0$. So $\delta = \delta'$.

It follows from (3.2), (3.5) and (3.10) that

$$\begin{aligned}&\big\|\delta([x_1, x_2]) - [\delta(x_1), x_2] - [x_1, \delta(x_2)]\big\|_A \\ &= \lim_{n\to\infty} \frac{1}{m^{2n}} \big\|f(m^{2n}[x_1, x_2]) - [f(m^n x_1), m^n x_2] - [m^n x_1, f(m^n x_2)]\big\|_A \\ &\le \lim_{n\to\infty} \frac{1}{m^{2n}} \psi(m^n x_1, m^n x_2) = 0\end{aligned}$$

for all $x_1, x_2 \in A_0$. So

$$\delta([x_1, x_2]) = [\delta(x_1), x_2] + [x_1, \delta(x_2)]$$

for all $x_1, x_2 \in A_0$. Hence the mapping $\delta : A_0 \to A$ is a unique Lie derivation satisfying (3.6).

Corollary 3.3. *Let $\delta, \alpha_1, \alpha_2, s_1, s_2, \{\theta_i\}_{i=1}^m$ and $\{r_i\}_{i=1}^m$ be non-negative real numbers such that $0 < s_1, s_2 < 2$, and $0 < r_i < 1$ for all $1 \le i \le m$. Suppose that $f : A_0 \to A$ is a mapping such that*

$$\|D_\mu f(x_1, ..., x_m)\|_A \le \delta + \sum_{i=1}^m \theta_i \|x_i\|_{A_0}^{r_i},$$

$$\|f([x_1, x_2]) - [f(x_1), x_2] - [x_1, f(x_2)]\|_A \le \delta + \alpha_1 \|x_1\|_{A_0}^{s_1} + \alpha_2 \|x_2\|_{A_0}^{s_2},$$

for all $x_1, \cdots, x_m \in A_0$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique Lie derivation $\delta : A_0 \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \frac{\delta}{m-1} + \gamma(x)$$

for all $x \in A_0$, where

$$\gamma(x) := \min_{1\le i\le m} \Big\{ \frac{\theta_i m^{r_i}}{m - m^{r_i}} \|x\|_{A_0}^{r_i} \Big\}.$$

Corollary 3.4. *Let $\delta, \alpha_1, \alpha_2, \alpha_3, s_1, s_2$ and $\{r_i\}_{i=1}^m$ be non-negative real numbers such that $s_1 + s_2 < 2$ and $0 < \sum_{i=1}^m r_i < 1$ for all $1 \le i \le m$. Suppose that $f : A_0 \to A$ is a mapping such that*

$$\|D_\mu f(x_1, ..., x_m)\|_A \le \delta + \sum_{i=1}^m \|x_i\|_{A_0}^{r_i} + \prod_{i=1}^m \|x_i\|_{A_0}^{r_i},$$

$$\|f([x_1, x_2]) - [f(x_1), x_2] - [x_1, f(x_2)]\|_A \le \delta + \alpha_1 \|x_1\|_{A_0}^{s_1} + \alpha_2 \|x_2\|_{A_0}^{s_2} + \alpha_3 \|x_1\|_{A_0}^{s_1} \|x_2\|_{A_0}^{s_2},$$

for all $x_1, \cdots, x_m \in A_0$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique Lie derivation $\delta : A_0 \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \frac{\delta}{m-1} + \tau(x)$$

for all $x \in A_0$, *where*

$$\tau(x) := \min_{1\le i\le m} \Big\{ \frac{m^{r_i}}{m - m^{r_i}} \|x\|_{A_0}^{r_i} \Big\}.$$

Note that the mixed "product-sum" function was introduced by J. M. Rassias in 2008-09 ([8, 9, 16, 17, 40, 41]).

Theorem 3.5. *Let* $\Phi : \underbrace{A_0 \times A_0 \times ... \times A_0}_{m-times} \to [0,\infty)$ *and* $\Psi : A_0 \times A_0 \to [0,\infty)$ *be mappings such that*

$$\lim_{n\to\infty} m^n \Phi\left(\frac{x_1}{m^n}, ..., \frac{x_m}{m^n}\right) = 0,$$

$$\lim_{n\to\infty} m^{2n} \Psi\left(\frac{x_1}{m^n}, \frac{x_2}{m^n}\right) = 0, \tag{3.13}$$

$$\widetilde{\Phi_j}(x) := \sum_{i=0}^{\infty} m^i \Phi(0, ..., \underbrace{\frac{x}{m^i}}_{j\,th}, ..., 0) < \infty$$

for all $x, x_1, \cdots, x_m \in A_0$. *Suppose that* $f : A_0 \to A$ *is a mapping such that*

$$\|D_\mu f(x_1, ..., x_m)\|_A \le \Phi(x_1, ..., x_m),$$

$$\big\|f([x_1, x_2]) - [f(x_1), x_2] - [x_1, f(x_2)]\big\|_A \le \Psi(x_1, x_2)$$

for all $x_1, \cdots, x_m \in A_0$ *and all* $\mu \in \mathbb{T}^1$. *Then there exists a unique Lie derivation* $\delta : A_0 \to A$ *such that*

$$\|f(x) - \delta(x)\|_A \le \widetilde{\Phi_j}(x) \tag{3.14}$$

for all $x \in A_0$.

Similarly to the proof of Theorem 3.2, we have

$$\|f(mx) - mf(x)\|_A \le \Phi(0, ..., \underbrace{mx}_{j\,th}, ...0) \tag{3.15}$$

for all $x \in A_0$. Replacing x by $\frac{x}{m^{n+1}}$ in (3.15) and multiplying both sides of (3.15) to m^n, we get

$$\Big\|m^{n+1} f\left(\frac{x}{m^{n+1}}\right) - m^n f\left(\frac{x}{m^n}\right)\Big\|_A \le m^n \Phi\big(0, ..., \underbrace{\frac{x}{m^n}}_{j\,th}, ...0)$$

for all $x \in A_0$ and all non-negative integers n. Hence

$$\begin{aligned} \Big\|m^{n+1} f\left(\frac{x}{m^{n+1}}\right) - m^k f\big(\frac{x}{m^k}\big)\Big\|_A &\le \sum_{i=k}^{n} \Big\|m^{i+1} f\left(\frac{x}{m^{i+1}}\right) - m^i f\big(\frac{x}{m^i}\big)\Big\|_A \\ &\le \sum_{i=k}^{n} m^i \Phi\big(0, ..., \underbrace{\frac{x}{m^i}}_{j\,th}, ...0) \end{aligned} \tag{3.16}$$

for all $x \in A_0$ and all non-negative integers n and k with $n \ge k$. Therefore the sequence $\{m^n f\big(\frac{x}{m^n}\big)\}$ is a Cauchy sequence in A for all $x \in A_0$. Since A is complete,

the sequence $\{m^n f\big(\frac{x}{m^n}\big)\}$ converges in A for all $x \in A_0$. So one can define the mapping $\delta : A_0 \to A$ by
$$\delta(x) := \lim_{n\to\infty} m^n f\big(\frac{x}{m^n}\big)$$
for all $x \in A_0$. Letting $k = 0$ and passing the limit $n \to \infty$ in (3.16), we get (3.14).

The rest of the proof is similar to the proof of Theorem 3.2.

Corollary 3.6. *Let $\alpha_1, \alpha_2, s_1, s_2, \{\theta_i\}_{i=1}^m$ and $\{r_i\}_{i=1}^m$ be non-negative real numbers such that $s_1, s_2 > 2$ and $r_i > 1$ for all $1 \le i \le m$. Suppose that $f : A_0 \to A$ is a mapping such that*
$$\|D_\mu f(x_1, ..., x_m)\|_A \le \sum_{i=1}^m \theta_i \|x_i\|_{A_0}^{r_i},$$
$$\|f([x_1, x_2]) - [f(x_1), x_2] - [x_1, f(x_2)]\|_A \le \alpha_1 \|x_1\|_{A_0}^{s_1} + \alpha_2 \|x_2\|_{A_0}^{s_2},$$
for all $x_1, \cdots, x_m \in A_0$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique Lie derivation $\delta : A_0 \to A$ such that
$$\|f(x) - \delta(x)\|_A \le \Gamma(x)$$
for all $x \in A_0$, where
$$\Gamma(x) := \min_{1\le i\le m} \Big\{ \frac{\theta_i m^{r_i}}{m^{r_i} - 1} \|x\|_{A_0}^{r_i} \Big\}.$$

Corollary 3.7. *Let $\alpha_1, \alpha_2, \alpha_3, s_1, s_2$ and $\{r_i\}_{i=1}^m$ be non-negative real numbers such that $s_1, s_2 > 2$ and $r_i > 1$ for all $1 \le i \le m$. Suppose that $f : A_0 \to A$ is a mapping such that*
$$\|D_\mu f(x_1, ..., x_m)\|_A \le \sum_{i=1}^m \|x_i\|_{A_0}^{r_i} + \prod_{i=1}^m \|x_i\|_{A_0}^{r_i},$$
$$\|f([x_1, x_2]) - [f(x_1), x_2] - [x_1, f(x_2)]\|_A \le \alpha_1 \|x_1\|_{A_0}^{s_1} + \alpha_2 \|x_2\|_{A_0}^{s_2} + \alpha_3 \|x_1\|_{A_0}^{s_1} \|x_2\|_{A_0}^{s_2},$$
for all $x_1, \cdots, x_m \in A_0$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique Lie derivation $\delta : A_0 \to A$ such that
$$\|f(x) - \delta(x)\|_A \le \Delta(x)$$
for all $x \in A_0$, where
$$\Delta(x) := \min_{1\le i\le m} \Big\{ \frac{m^{r_i}}{m^{r_i} - m} \|x\|_{A_0}^{r_i} \Big\}.$$

4. SUBADDITIVE MAPPING AND STABILITY OF EQ. (1.1)

Next, using some idea of [35], we are going to establish other theorems about the stability of Eq. (1.1)

We call that a subadditive mapping is a mapping $\varphi : A \to B$, having a domain A and a codomain $(B, \le)$ that are both closed under addition, with the following property:
$$\varphi(x + y) \le \varphi(x) + \varphi(y)$$

for all $x, y \in X$. Now we say that a mapping $\varphi : X \to Y$ is contractively subadditive if there exists a constant L with $0 < L < 1$ such that

$$\varphi(x+y) \leq L[\varphi(x) + \varphi(y)]$$

for all $x, y \in X$. Therefore φ satisfies the following properties $\varphi(mx) \leq mL\varphi(x)$ and so $\varphi(m^n x) \leq (mL)^n \varphi(x)$, for all $x \in X$ and all positive integer $m \geq 2$.
Similarly, we say that a mapping $\varphi : A \to B$ is expansively superadditive if there exists a constant L with $0 < L < 1$ such that

$$\varphi(x+y) \geq \frac{1}{L}[\varphi(x) + \varphi(y)]$$

for all $x, y \in X$. Therefor φ satisfies the following properties $\varphi(x) \leq \frac{L}{m}\varphi(mx)$ and so $\varphi(\frac{x}{m^n}) \leq (\frac{L}{m})^n \varphi(x)$, for all $x \in X$ and all positive integer $m \geq 2$.

Theorem 4.1. *Let* $\varphi : \underbrace{A_0 \times A_0 \times ... \times A_0}_{m-times} \to [0, \infty)$ *be a contractively subadditive with the constant* L *and* $\psi : A_0 \times A_0 \to [0, \infty)$ *be a mapping such that*

$$\lim_{n\to\infty} \frac{1}{m^{2n}} \psi(m^n x_1, m^n x_2) = 0, \tag{4.1}$$

for all $x_1, x_2 \in A_0$. *Suppose that* $f : A_0 \to A$ *is a mapping such that*

$$\|D_\mu f(x_1, ..., x_m)\|_A \leq \varphi(x_1, ..., x_m), \tag{4.2}$$

$$\big\|f([x_1, x_2]) - [f(x_1), x_2] - [x_1, f_1(x_2)]\big\|_A \leq \psi(x_1, x_2) \tag{4.3}$$

for all $x_1, \cdots, x_m \in A_0$ *and all* $\mu \in \mathbb{T}^1$. *Then there exists a unique Lie derivation* $\delta : A_0 \to A$ *such that*

$$\|f(x) - \delta(x)\|_A \leq \frac{L}{1-L}\varphi(0, ..., \underbrace{x}_{j\,th}, ...0) \tag{4.4}$$

for all $x \in X$.

Letting $\mu = 1$, $x_j = mx$ and $x_i = 0$ for all $1 \leq i \leq m$ with $i \neq j$ in (4.2) , we get

$$\|f(mx) - mf(x)\|_A \leq \varphi(0, ..., \underbrace{mx}_{j\,th}, ...0) \tag{4.5}$$

for all $x \in A_0$.

Replacing x by $m^n x$ in (4.5) and dividing both sides of (4.5) by m^{n+1}, we get

$$\begin{aligned} \Big\|\frac{1}{m^{n+1}} f(m^{n+1}x) - \frac{1}{m^n} f(m^n x)\Big\|_A &\leq \frac{1}{m^{n+1}}\varphi(0, ..., \underbrace{m^{n+1}x}_{j\,th}, ..., 0) \\ &\leq \frac{(mL)^{n+1}}{m^{n+1}}\varphi(0, ..., \underbrace{x}_{j\,th}, ..., 0) \\ &\leq L^{n+1}\varphi(0, ..., \underbrace{x}_{j\,th}, ..., 0) \end{aligned} \tag{4.6}$$

for all $x \in A_0$ and all non-negative integers n. Hence

$$\begin{aligned}\Big\|\frac{1}{m^{n+1}}f(m^{n+1}x)-\frac{1}{m^k}f(m^kx)\Big\|_A &\le \sum_{i=k}^{n}\Big\|\frac{1}{m^{i+1}}f(m^{i+1}x)-\frac{1}{m^i}f(m^ix)\Big\|_A \\ &\le \sum_{i=k+1}^{n+1} L^i\varphi(0,...,\underbrace{x}_{j\,th},...,0)\end{aligned} \tag{4.7}$$

for all $x \in A_0$ and all non-negative integers n and k with $n \ge k$. Therefore, we conclude from and (4.7) that the sequence $\{\frac{1}{m^n}f(m^nx)\}$ is a Cauchy sequence in A for all $x \in A_0$. Since A is complete, the sequence $\{\frac{1}{m^n}f(m^nx)\}$ converges in A for all $x \in A_0$. So one can define the mapping $\delta : A_0 \to A$ by

$$\delta(x) := \lim_{n\to\infty}\frac{1}{m^n}f(m^nx) \tag{4.8}$$

for all $x \in A_0$. Letting $k = 0$ and passing the limit $n \to \infty$ in (4.7), we get (4.4). Now, we show that δ is a $\mathbb{C}$-linear mapping. It follows from (4.8) that

$$\begin{aligned}\|D_1\delta(x_1,...,x_m)\|_A &= \lim_{n\to\infty}\frac{1}{m^n}\big\|D_1f(m^nx_1,...,m^nx_m)\big\|_A \\ &\le \lim_{n\to\infty}\frac{1}{m^n}\varphi(m^nx_1,...,m^nx_m) \\ &\le \lim_{n\to\infty}L^n\varphi(x_1,...,x_m) = 0\end{aligned}$$

for all $x_1, \cdots, x_m \in A_0$. So the mapping δ satisfies (1.1). By Lemma 2.1, the mapping δ is additive.

Letting $x_j = mx$ and $x_i = 0$ for all $1 \le i \le m$ with $i \ne j$ in (4.2), we get

$$\|mf(\mu x)-\mu f(mx)\|_A \le \varphi(0,...,\underbrace{mx}_{j\,th},...,0) \tag{4.9}$$

for all $x \in A_0$. Replacing x by m^nx in (4.9) and dividing both sides of (4.9) by m^{n+1}, we get

$$\begin{aligned}&\Big\|\frac{1}{m^n}f(\mu m^nx)-\frac{\mu}{m^{n+1}}f(m^{n+1}x)\Big\|_A \\ &\le \frac{1}{m^{n+1}}\varphi(0,...,\underbrace{m^{n+1}x}_{j\,th},...,0)\end{aligned} \tag{4.10}$$

for all $x \in A_0$ and all non-negative integers n. Passing the limit $n \to \infty$ in (4.10) and using (4.8), we get

$$\delta(\mu x) = \mu\delta(x)$$

for all $\mu \in \mathbb{T}^1$ and for all $x \in A_0$. So by Lemma 3.1, we infer that the mapping $\delta : A_0 \to A$ is $\mathbb{C}$-linear. To prove the uniqueness of δ, let $\delta' : A_0 \to A$ be another additive mapping satisfying (4.4). It follows from (4.8) that

$$\begin{aligned}\|\delta(x)-\delta'(x)\|_A &= \lim_{n\to\infty}\frac{1}{m^n}\left\|f(m^nx)-\delta'(m^nx)\right\|_A\\ &\le \lim_{n\to\infty}\frac{L^{n+1}}{1-L}\varphi(0,...,\underbrace{x}_{j\,th},...0)=0\end{aligned}$$

for all $x \in A_0$. So $\delta = \delta'$.

The rest of the proof is similar to the proof of Theorem 3.2.

Corollary 4.2. *Let θ be non-negative real number and $f : A_0 \to A$ be a mapping for which*

$$\|D_\mu f(x_1,...,x_m)\|_A \le \theta$$

$$\left\|f([x_1,x_2])-[f(x_1),x_2]-[x_1,f_1(x_2)]\right\|_A \le \theta$$

for all $x_1,...,x_m \in A_0$. Then there exists a unique Lie derivation $\delta : A_0 \to A$ such that

$$\|f(x)-\delta(x)\|_A \le \theta \tag{4.11}$$

for all $x \in A_0$.

The proof follows from Theorem 4.1 by taking

$$\varphi(x_1,...,x_m) := \theta$$

for all $x_1,...,x_m \in A_0$.

Replacing contractively subadditive by expansively superadditive in Theorem 4.1, one can obtain the following theorem:

Theorem 4.3. *Let $\varphi : \underbrace{A_0 \times A_0 \times ... \times A_0}_{m-times} \to [0,\infty)$ be a expansively superadditive with the constant L and $\psi : A_0 \times A_0 \to [0,\infty)$ be a mapping such that*

$$\lim_{n\to\infty} m^{2n}\psi\left(\frac{x_1}{m^n},\frac{x_2}{m^n}\right)=0, \tag{4.12}$$

for all $x_1,x_2 \in A_0$. Suppose that $f : A_0 \to A$ is a mapping such that

$$\|D_\mu f(x_1,...,x_m)\|_A \le \varphi(x_1,...,x_m), \tag{4.13}$$

$$\left\|f([x_1,x_2])-[f(x_1),x_2]-[x_1,f_1(x_2)]\right\|_A \le \psi(x_1,x_2) \tag{4.14}$$

for all $x_1,\cdots,x_m \in A_0$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique Lie derivation $\delta : A_0 \to A$ such that

$$\|f(x)-\delta(x)\|_A \le \frac{1}{1-L}\varphi(0,...,\underbrace{x}_{j\,th},...0) \tag{4.15}$$

for all $x \in X$.

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ON PRINCIPALLY QUASI-BAER MODULES

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Abstract. Let R be an arbitrary ring with identity and M a right R-module with $S = \mathrm{End}_R(M)$. In this paper, we introduce a class of modules that is a generalization of principally quasi-Baer rings and Baer modules. The module $_SM$ is called *principally quasi-Baer* if for any $m \in M$, $l_S(Sm) = Se$ for some $e^2 = e \in S$. It is proved that (1) if $_SM$ is regular and semicommutative module or (2) if M_R is principally semisimple and $_SM$ is abelian, then $_SM$ is a principally quasi-Baer module. The connection between a principally quasi-Baer module $_SM$ and polynomial extension, power series extension, Laurent polynomial extension, Laurent power series extension of $_SM$ is investigated.

1. Introduction

Throughout this paper R denotes an associative ring with identity, and modules will be unitary right R-modules. For a module M, $S = \mathrm{End}_R(M)$ denotes the ring of right R-module endomorphisms of M. Then M is a left S-module, right R-module and (S, R)-bimodule. In this work, for any rings S and R and any (S, R)-bimodule M, $r_R(.)$ and $l_M(.)$ denote the right annihilator of a subset of M in R

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and the left annihilator of a subset of R in M, respectively. Similarly, $l_S(.)$ and $r_M(.)$ will be the left annihilator of a subset of M in S and the right annihilator of a subset of S in M, respectively. A ring R is *reduced* if it has no nonzero nilpotent elements. Recently the reduced ring concept was extended to modules by Lee and Zhou in [9], that is, a module M is called *reduced* if for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. A ring R is called *semicommutative* if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. The module ${}_SM$ is called *semicommutative* if for any $f \in S$ and $m \in M$, $fm = 0$ implies $fSm = 0$ (see [3] for details). *Baer rings* [7] are introduced as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. A ring R is said to be *right quasi-Baer* [5] if the right annihilator of each right ideal of R is generated (as a right ideal) by an idempotent. A ring R is called *right principally quasi-Baer* [4] if the right annihilator of a principal right ideal of R is generated by an idempotent. An R-module ${}_SM$ is called *Baer* [12] if for all R-submodules N of M, $l_S(N) = Se$ with $e^2 = e \in S$. The module ${}_SM$ is said to be *quasi-Baer* if for all fully invariant R-submodules N of M, $l_S(N) = Se$ with $e^2 = e \in S$. A ring R is called *abelian* if every idempotent element is central, that is, $ae = ea$ for any $e^2 = e$, $a \in R$. Abelian modules are introduced in the context by Roos in [14] and studied by Goodearl and Boyle [6], Rizvi and Roman [13]. A module ${}_SM$ is called *abelian* if for any $f \in S$, $e^2 = e \in S$, $m \in M$, we have $fem = efm$. Note that ${}_SM$ is an abelian module if and only if S is an abelian ring. In what follows, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$ and $\mathbb{Z}/n\mathbb{Z}$ we denote integers, rational numbers, the ring of integers modulo n and the $\mathbb{Z}$-module of integers modulo n, respectively.

2. Principally Quasi-Baer Modules

Some properties of R-modules do not characterize the ring R, namely there are reduced R-modules but R need not be reduced and there are abelian R-modules but R is not an abelian ring. Because of that the investigation of some classes of modules in terms of their endomorphism rings are done by the present authors (see [2] for details). In this section we introduce a class of modules that is a generalization of principally quasi-Baer rings and Baer modules. We prove that some results of principally quasi-Baer rings can be extended to this general setting.

Definition 2.1. Let M be an R-module with $S = \mathrm{End}_R(M)$. The module ${}_SM$ is called *principally quasi-Baer* if for any $m \in M$, $l_S(Sm) = Se$ for some $e^2 = e \in S$.

It is straightforward that all Baer, quasi-Baer, semisimple modules are principally quasi-Baer. But a submodule of principally quasi-Baer module may not be principally quasi-Baer. If e is an idempotent element in the ring R and $ere = re$ ($ere = er$) for all $r \in R$, then e is called *left (right) semicentral*. In the following proposition we prove that idempotents in the definition of principally quasi-Baer modules are right semicentral.

Proposition 2.2. *Let M be an R-module with $S = End_R(M)$. If ${}_SM$ is a principally quasi-Baer module, then there exists a right semicentral idempotent $e \in S$ such that $l_S(Sm) = Se$ for each $m \in M$.*

Proof. Let $m \in M$ and ${}_SM$ be a principally quasi-Baer module. By hypothesis, there exists $e^2 = e \in S$ with $l_S(Sm) = Se$. Since $SefSm \subseteq SeSm = 0$, we have $SefSm = 0$ for all $f \in S$. Hence, $Sef \subseteq l_S(Sm) = Se$. Thus, $ef = efe$ for all $f \in S$. $\square$

Theorem 2.3. *Let M be an R-module with $S = End_R(M)$. The following are equivalent.*

(1) *${}_SM$ is principally quasi-Baer.*

(2) *The left annihilator of every finitely generated S-submodule of ${}_SM$ in S is generated (as a left ideal) by an idempotent.*

Proof. (1) $\Rightarrow$ (2) Let $N = \sum_{i=1}^{n} Sm_i$ ($n \in \mathbb{N}$) be a finitely generated S-submodule of M. Then, $l_S(N) = \bigcap_{i=1}^{n} l_S(Sm_i)$. Since M is principally quasi-Baer, there exist $e_i^2 = e_i \in S$ such that $l_S(Sm_i) = Se_i$ for $i = 1, 2, \ldots, n$. So $l_S(N) = \bigcap_{i=1}^{n} Se_i$ with each e_i a right semicentral idempotent of S by Proposition 2.2. Now we show that $Se_1 \cap Se_2 = Se_1e_2$. Since $Se_1e_2 = Se_1e_2e_1$, then $Se_1e_2 \subseteq Se_1 \cap Se_2$. In order to see other inclusion, let $f = f_1e_1 = f_2e_2 \in Se_1 \cap Se_2$ for some $f_1, f_2 \in S$. Then, $fe_2 = f_1e_1e_2 = f_2e_2 = f \in Se_1e_2$. Thus, $Se_1 \cap Se_2 \subseteq Se_1e_2$. On the other hand $(e_1e_2)^2 = e_1e_2$, because e_1 is right semicentral. In a similar way, we have $l_S(N) = \bigcap_{i=1}^{n} Se_i = S(e_1e_2 \ldots e_n)$ with $(e_1e_2 \ldots e_n)^2 = e_1e_2 \ldots e_n$.

(2) $\Rightarrow$ (1) It is obvious from (2) since every cyclic S-submodule of ${}_SM$ is finitely generated. □

Corollary 2.4. *Let M be an R-module with $S = End_R(M)$. If ${}_SM$ is a finitely generated module and S is a principal ideal domain (or a Noetherian ring), then the following are equivalent.*

(1) *${}_SM$ is Baer.*

(2) *${}_SM$ is quasi-Baer.*

(3) *${}_SM$ is principally quasi-Baer.*

Proposition 2.5. *Let M be an R-module with $S = End_R(M)$. If ${}_SM$ is a principally quasi-Baer module and N a direct summand of M, then ${}_TN$ is principally quasi-Baer, where $T = End_R(N)$.*

Proof. Let N be a direct summand of M. There exists $e^2 = e \in S$ such that $N = eM$. So the endomorphism ring T of N is eSe. Let $n \in N$. Since ${}_SM$ is a principally quasi-Baer module, there exists a right semicentral idempotent f in S such that $l_S(Sn) = Sf$. Hence, efe is an idempotent of eSe. We claim that $l_{eSe}(Tn) = (eSe)(efe)$. For any $g \in S$, $egefeTn = 0$, and so $(eSe)(efe) \leq l_{eSe}(Tn)$. On the other hand, let $x \in Sf \cap eSe$. Then, $xTn = xeSen = xeSn \leq xSn = 0$. Hence we have $x \in l_{eSe}(Tn)$. This implies that $Sf \cap eSe \leq l_{eSe}(Tn)$. Now let $eye \in l_{eSe}(Tn)$ with $y \in S$. Since $eyeTn = eyeSen = eyeSn = 0$, we have $eye \in Sf$. It follows that $l_{eSe}(Tn) \leq Sf \cap eSe$. Thus, $l_{eSe}(Tn) = Sf \cap eSe$. In order to see $l_{eSe}(Tn) \leq (eSe)(efe)$, let $x \in l_{eSe}(Tn)$. Then, $x = s_1f = es_2e$ for some $s_1, s_2 \in S$. Notice that $x = xf = s_1f = es_2ef$ and $x = xe = s_1fe = es_2e$. Hence, $x = xe = xfe = s_1fe = es_2efe \in (eSe)(efe)$. Thus, $l_{eSe}(Tn) \leq (eSe)(efe)$. This completes the proof. □

The direct sum of principally quasi-Baer modules is not principally quasi-Baer as the following example shows.

Example 2.6. Consider $M = \mathbb{Z} \oplus \mathbb{Z}_2$ as a $\mathbb{Z}$-module. Since $\mathbb{Z}$ is a domain and $\mathbb{Z}_2$ is simple, $\mathbb{Z}$ and $\mathbb{Z}_2$ are Baer and so principally quasi-Baer $\mathbb{Z}$-modules. It can be easily determined that $S = \mathrm{End}_{\mathbb{Z}}(M)$ is $\begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{bmatrix}$. For $m = (2, \overline{1}) \in M$, $l_S(Sm) = \begin{bmatrix} 0 & 0 \\ \mathbb{Z}_2 & 0 \end{bmatrix}$ and $l_S(Sm)$ is not a direct summand of S. This implies that ${}_SM$ is not principally quasi-Baer.

Theorem 2.7. *Let $M = M_1 \oplus M_2$ be an R-module with $S = End_R(M)$. If ${}_{S_1}M_1$ and ${}_{S_2}M_2$ are principally quasi-Baer, where $S_1 = End_R(M_1)$, $S_2 = End_R(M_2)$ and $Hom(M_i, M_j) = 0$ for $i \neq j$, $i = j = 1, 2$, then ${}_SM$ is also principally quasi-Baer.*

Proof. By hypothesis, $Hom(M_i, M_j) = 0$ for $i \neq j$, $i = j = 1, 2$, we have $S = S_1 \oplus S_2$. Let $m = (m_1, m_2) \in M$ for some $m_1 \in M_1$ and $m_2 \in M_2$. Since ${}_{S_i}M_i$ is principally quasi-Baer, there exists an idempotent $e_i \in S_i$ with $l_{S_i}(S_i m_i) = S_i e_i$ for $i = 1, 2$. On the other hand, we have $l_S(Sm) = l_{S_1}(S_1 m_1) \oplus l_{S_2}(S_2 m_2)$, and so $l_S(Sm)$ is a direct summand of S. □

Let M be an R-module with $S = \mathrm{End}_R(M)$. Recall that the submodule N of M is called *fully invariant* if $f(N) \leq N$ for all $f \in S$.

Proposition 2.8. *Let M be an R-module with $S = End_R(M)$. If ${}_SM$ is a principally quasi-Baer module, then every principal fully invariant submodule of M is not essential in M.*

Proof. Let mR be a fully invariant submodule of M. Since ${}_SM$ is a principally quasi-Baer module, there exists $e^2 = e \in S$ with $l_S(Sm) = Se$. Then we have $Sm \subseteq r_M(l_S(Sm)) = r_M(Se) = (1-e)M$. Hence, mR is not essential in M. □

A module M is said to be *principally semisimple* if every principal submodule is a direct summand of M.

Proposition 2.9. *Let M be an R-module with $S = End_R(M)$. If M_R is principally semisimple and ${}_SM$ is abelian, then ${}_SM$ is a principally quasi-Baer module.*

Proof. If $m \in M$, then by hypothesis $M = mR \oplus K$ for some submodule K of M. Let e denote the projection of M onto mR. It is routine to show that $l_S(Sm) \leq S(1-e)$. Since $m = em$ and ${}_SM$ is abelian, we have $S(1-e)Sm = S(1-e)Sem = S(1-e)eSm = 0$. Thus, $S(1-e) \leq l_S(Sm)$. This completes the proof. □

A left T-module M is called *regular* (in the sense Zelmanowitz [15]) if for any $m \in M$ there exists a left T-homomorphism $M \xrightarrow{\phi} T$ such that $m = \phi(m)m$.

Proposition 2.10. *Let M be an R-module with $S = End_R(M)$. If ${}_SM$ is regular and semicommutative, then ${}_SM$ is a principally quasi-Baer module.*

Proof. If $m \in M$, then by hypothesis there exists a left S-homomorphism $M \xrightarrow{\phi} S$ such that $m = \phi(m)m$. Note that $\phi(m)$ is an idempotent of S. We prove $l_S(Sm) = S(1-\phi(m))$. Since $(1-\phi(m))m = 0$ and ${}_SM$ is semicommutative, we have $(1-\phi(m))Sm = 0$. Then, $S(1-\phi(m)) \leq l_S(Sm)$. Now let $f \in l_S(Sm)$. Hence, $fm = 0$ and so $\phi(fm) = f\phi(m) = 0$. Thus, $f = f - f\phi(m) = f(1-\phi(m)) \in S(1-\phi(m))$. Therefore, $l_S(Sm) \leq S(1-\phi(m))$, and this completes the proof. □

Lemma 2.11. *Let M be an R-module with $S = End_R(M)$. If ${}_SM$ is a semicommutative module, then $l_S(Sm) = l_S(m)$ for any $m \in M$.*

Proof. We always have $l_S(Sm) \subseteq l_S(m)$. Conversely, let $f \in l_S(m)$. Since ${}_SM$ is a semicommutative module, $fm = 0$ implies $f \in l_S(Sm)$. □

According to Lambek, a ring R is called *symmetric* [8] if whenever $a, b, c \in R$ satisfy $abc = 0$ implies $cab = 0$. The module M_R is called *symmetric* ([8] and [10]) if whenever $a, b \in R$, $m \in M$ satisfy $mab = 0$, we have $mba = 0$. Symmetric modules are also studied by the present authors in [1] and [11]. In our case, we have the following.

Definition 2.12. Let M be an R-module with $S = \mathrm{End}_R(M)$. The module ${}_SM$ is called *symmetric* if for any $m \in M$ and $f, g \in S$, $fgm = 0$ implies $gfm = 0$.

Example 2.13. Let M be a finitely generated torsion $\mathbb{Z}$-module. Then M is isomorphic to the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p_1^{n_1}) \oplus (\mathbb{Z}/\mathbb{Z}p_2^{n_2}) \oplus ... \oplus (\mathbb{Z}/\mathbb{Z}p_t^{n_t})$ where p_i $(i = 1, ..., t)$ are distinct prime numbers and n_i $(i = 1, ..., t)$ are positive integers. $\mathrm{End}_{\mathbb{Z}}(M)$ is isomorphic to the commutative ring $(\mathbb{Z}_{p_1^{n_1}}) \oplus (\mathbb{Z}_{p_2^{n_2}}) \oplus ... \oplus (\mathbb{Z}_{p_t^{n_t}})$. So ${}_SM$ is a symmetric module.

Lemma 2.14. *Let M be an R-module with $S = End_R(M)$. If ${}_SM$ is symmetric, then ${}_SM$ is semicommutative. Converse is true if ${}_SM$ is a principally quasi-Baer module.*

Proof. Let $f \in S$ and $m \in M$ with $fm = 0$. Then for all $g \in S$, $gfm = 0$ implies $fgm = 0$. So $fSm = 0$. Conversely, let $f, g \in S$ and $m \in M$ with $fgm = 0$. By Lemma 2.11, $f \in l_S(gm) = l_S(Sgm) = Se$ for some $e^2 = e \in S$. So $f = fe$ and $egm = 0$. Since ${}_SM$ is semicommutative, $egSm = 0$. Therefore, $gfm = gfem = gefm = egfm = 0$ because e is central. □

The proof of Proposition 2.15 is straightforward.

Proposition 2.15. *Let M be an R-module with $S = End_R(M)$. Consider the following conditions for $f \in S$.*
(1) *$SKerf \cap Imf = 0$.*
(2) *Whenever $m \in M$, $fm = 0$ if and only if $Imf \cap Sm = 0$.*
Then (1) $\Rightarrow$ (2). *If ${}_SM$ is a semicommutative module, then* (2) $\Rightarrow$ (1).

A module ${}_SM$ is called *reduced* if condition (2) of Proposition 2.15 holds for each $f \in S$.

Example 2.16. Let p be any prime integer and M the $\mathbb{Z}$-module $(\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Q}$. Then $S = \mathrm{End}_R(M)$ is isomorphic to the matrix ring $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a \in \mathbb{Z}_p, b \in \mathbb{Q} \right\}$. It is evident that ${}_SM$ is a reduced module.

Proposition 2.17. *Let M be an R-module with $S = End_R(M)$. Then the following are equivalent.*
(1) *${}_SM$ is a reduced module.*
(2) *For any $f \in S$ and $m \in M$, $f^2m = 0$ implies $fSm = 0$.*

Proof. It follows from [9, Lemma 1.2]. □

Lemma 2.18. *Let M be an R-module with $S = End_R(M)$. If ${}_SM$ is a reduced module, then ${}_SM$ is symmetric. The converse holds if ${}_SM$ is a principally quasi-Baer module.*

Proof. For any $f, g \in S$ and $m \in M$ suppose that $fgm = 0$. Then, $(fg)^2(m) = 0$ and by hypothesis $fgSm = 0$. So $fgfm = 0$ and $(gf)^2m = 0$. Then, $gfSm = 0$ implies $gfm = 0$. Therefore, ${}_SM$ is symmetric. Conversely, let $f \in S$ and $m \in M$ with $f^2m = 0$. By Lemma 2.14, ${}_SM$ is semicommutative and from Lemma 2.11, $f \in l_S(fm) = l_S(Sfm) = Se$ for some $e^2 = e \in S$. So $f = fe$ and $efm = 0$. Since ${}_SM$ is semicommutative, $efSm = 0$. Then, $fgm = fegm = efgm = 0$ for any $g \in S$. Therefore, $fSm = 0$ and so ${}_SM$ is a reduced module. □

Next example shows that the reverse implication of the first statement in Lemma 2.18 is not true in general, i.e., there exists a symmetric module which is neither reduced nor principally quasi-Baer.

Example 2.19. Consider the ring

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$$

and the right R-module

$$M = \left\{ \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}.$$

Let $f \in S$ and $f \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c \\ c & d \end{bmatrix}$. Multiplying the latter by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we have $f \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$. For any $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$, $f \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & ac \\ ac & ad + bc \end{bmatrix}$. Similarly, let $g \in S$ and $g \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c' \\ c' & d' \end{bmatrix}$. Then, $g \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & c' \end{bmatrix}$. For any $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$, $g \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & ac' \\ ac' & ad' + bc' \end{bmatrix}$. Then it is easy to check that for any $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$,

$$fg \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = f \begin{bmatrix} 0 & ac' \\ ac' & ad' + bc' \end{bmatrix} = \begin{bmatrix} 0 & ac'c \\ ac'c & ad'c + adc' + bc'c \end{bmatrix}$$

and

$$gf \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = g \begin{bmatrix} 0 & ac \\ ac & ad + bc \end{bmatrix} = \begin{bmatrix} 0 & acc' \\ acc' & acd' + ac'd + bcc' \end{bmatrix}$$

Hence, $fg = gf$ for all f, $g \in S$. Therefore, S is commutative and so ${}_SM$ is symmetric. Define $f \in S$ by $f \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$ where $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$. Then, $f \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $f^2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = 0$. Hence, ${}_SM$ is not reduced. Let $m = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. By Lemma 2.14, ${}_SM$ is semicommutative and so by Lemma 2.11, $l_S(Sm) = l_S(m) \neq 0$ since the endomorphism f defined preceding belongs to the $l_S(m)$. The module M is indecomposable as a right R-module, therefore S does not have any idempotents other than zero and identity. Hence, $l_S(Sm)$ can not be generated by an idempotent as a left ideal of S.

We can summarize the relations between reduced modules, symmetric modules and semicommutative modules by using principally quasi-Baer modules.

Theorem 2.20. *Let M be an R-module with $S = End_R(M)$. If ${}_SM$ is a principally quasi-Baer module, then the following conditions are equivalent.*
(1) *${}_SM$ is a reduced module.*
(2) *${}_SM$ is a symmetric module.*
(3) *${}_SM$ is a semicommutative module.*

Proof. It follows from Lemma 2.18 and Lemma 2.14. □

In the sequel we investigate extensions of principally quasi-Baer modules. We show that there is a strong connection between principally quasi-Baer modules and polynomial extension, power series extension, Laurent polynomial extension, Laurent power series extension of M.

Let $R[x], R[[x]], R[x, x^{-1}]$ and $R[[x, x^{-1}]]$ be the polynomial ring, the power series ring, the Laurent polynomial ring and the Laurent power series ring over R, respectively and M an R-module with $S = \mathrm{End}_R(M)$. Lee and Zhou [9] introduced the following notations. Consider

$$M[x] = \left\{ \sum_{i=0}^{s} m_i x^i \ : \ s \geq 0, m_i \in M \right\},$$
$$M[[x]] = \left\{ \sum_{i=0}^{\infty} m_i x^i \ : m_i \in M \right\},$$
$$M[x, x^{-1}] = \left\{ \sum_{i=-s}^{t} m_i x^i \ : \ s \geq 0, t \geq 0, m_i \in M \right\},$$
$$M[[x, x^{-1}]] = \left\{ \sum_{i=-s}^{\infty} m_i x^i \ : \ s \geq 0, m_i \in M \right\}.$$

Each of these is an abelian group under an obvious addition operation. For a module M, $M[x]$ is a left $S[x]$-module by the scalar product:

$$m(x) = \sum_{j=0}^{s} m_j x^j \ \in M[x] \quad , \quad \alpha(x) = \sum_{i=0}^{t} f_i x^i \ \in S[x]$$

$$\alpha(x)m(x) = \sum_{k=0}^{s+t} \left(\sum_{i+j=k} f_i m_j \right) x^k.$$

With a similar scalar product, $M[[x]]$, $M[x, x^{-1}]$ and $M[[x, x^{-1}]]$ become left modules over $S[[x]]$, $S[x, x^{-1}]$ and $S[[x, x^{-1}]]$, respectively. The modules $M[x]$, $M[[x]]$, $M[x, x^{-1}]$ and $M[[x, x^{-1}]]$ are called the *polynomial extension*, the *power series extension*, *Laurent polynomial extension* and the *Laurent power series extension of* M, respectively. The module $M[x]$ is called a *principally quasi-Baer* if for any $m(x) \in M[x]$, there exists $e^2 = e \in S[x]$ such that $l_{S[x]}(S[x]m(x)) = S[x]e$. Also $M[[x]]$, $M[x, x^{-1}]$ and $M[[x, x^{-1}]]$ may be defined in a similar way.

Theorem 2.21. *Let M be an R-module with $S = End_R(M)$. Then*
(1) *$M[x]$ is a principally quasi-Baer module if and only if ${}_SM$ is a principally quasi-Baer module.*
(2) *If $M[[x]]$ is a principally quasi-Baer module, then ${}_SM$ is a principally quasi-Baer module.*

(3) *If $M[x,x^{-1}]$ is a principally quasi-Baer module, then ${}_SM$ is a principally quasi-Baer module.*
(4) *If $M[[x,x^{-1}]]$ is a principally quasi-Baer module, then ${}_SM$ is a principally quasi-Baer module.*

Proof. (1) Assume that $M[x]$ is a principally quasi-Baer module and $m \in M$. There exists $e(x)^2 = e(x) \in S[x]$ such that $l_{S[x]}(S[x]m) = S[x]e(x)$. Thus, $S[x]e(x) \subseteq l_{S[x]}(Sm) = l_S(Sm)[x]$. For $f(x) = \sum_{i=0}^{n} f_i x^i \in l_S(Sm)[x]$, $f_iSm = 0$ for all $i \geq 0$. For any $g(x) = \sum_{j=0}^{k} g_j x^j \in S[x]m, f(x)g(x) = \sum_i \sum_j f_i g_j x^{i+j} = 0$. So $f(x) \in l_{S[x]}(S[x]m)$. Thus, $l_S(Sm)[x] = S[x]e(x)$. Write $e(x) = \sum_{i=0}^{t} e_i x^i$, where all $e_i \in l_S(Sm)$. Then for any $h \in l_S(Sm), h = h_1(x)e(x)$ for some $h_1(x) \in S[x]$. So $he(x) = h_1(x)e(x)e(x) = h_1(x)e(x) = h$. It follows that $h = he_0$ for all $h \in l_S(Sm)$. Thus, $l_S(Sm) = Se_0$ with $e_0^2 = e_0$. It means that ${}_SM$ is principally quasi-Baer. Conversely, assume ${}_SM$ is a principally quasi-Baer module. Let $m(x) = m_0 + m_1x + ... + m_nx^n \in M[x]$. Then, $l_S(Sm_i) = Se_i$ where e_i's are right semicentral idempotents for all $i = 0, 1, ..., n$. Let $e = e_0e_1...e_n$. Then e is also a right semicentral in S and $Se = \bigcap_{i=0}^{n} l_S(Sm_i)$. Hence, $S[x]e \subseteq l_{S[x]}(S[x]m(x))$. Note that $l_{S[x]}(S[x]m(x)) = l_{S[x]}(Sm(x))$. So, $S[x]e \subseteq l_{S[x]}(Sm(x))$. Now, let $h(x) = h_0 + h_1x + ... + h_kx^k \in l_{S[x]}(Sm(x))$. Then, $(h_0 + h_1x + ... + h_kx^k)S(m_0 + m_1x + ... + m_nx^n) = 0$. Hence for any $\alpha \in S$, we have

$$\begin{aligned} h_0\alpha m_0 &= 0 && (1)\\ h_0\alpha m_1 + h_1\alpha m_0 &= 0 && (2)\\ h_0\alpha m_2 + h_1\alpha m_1 + h_2\alpha m_0 &= 0 && (3)\\ \cdots & && \cdots \end{aligned}$$

By the first equation, $h_0 \in l_S(Sm_0) = Se_0$. It means that $h_0 = h_0e_0$ and $Se_0Sm_0 = 0$. For $f \in S$ consider e_0f instead of α in (2). Then, $h_0e_0fm_1 + h_1e_0fm_0 = h_0e_0fm_1 = h_0fm_1 = 0$. So $h_0 \in l_S(Sm_1) = Se_1$. Thus, $h_0 \in Se_0e_1$. Since $h_0Sm_1 = 0$, (2) yields $h_1Sm_0 = 0$. Hence, $h_1 \in l_S(Sm_0) = Se_0$. Now we take $\alpha = e_0e_1f \in S$ and apply in (3). Then, $h_0e_0e_1fm_2 + h_1e_0e_1fm_1 + h_2e_0e_1fm_0 = 0$. But $h_1e_0e_1fm_1 = h_2e_0e_1fm_0 = 0$. Hence, $h_0e_0e_1fm_2 = h_0fm_2 = 0$. So $h_0 \in l_S(\bigcap_{i=0}^{2} l_S(Sm_i)) = Se_0e_1e_2$. By (3), we have $h_1Sm_1 + h_2Sm_0 = 0$. Then we have $h_1e_0fm_1 + h_2e_0fm_0 = 0$. But $h_2e_0fm_0 = 0$, so $h_1e_0fm_1 = h_1fm_1 = 0$. Thus, $h_1 \in l_S(\bigcap_{i=0}^{1} l_S(Sm_i)) = Se_0e_1$ and $h_2Sm_0 = 0$. Hence, $h_2 \in l_S(Sm_0) = Se_0$. Continuing this procedure, yields $h_i \in Se$ for all $i = 1, 2, ..., k$. Hence, $h(x) \in S[x]e$. Consequently $S[x]e = l_{S[x]}(S[x]m(x))$.

(2), (3) and (4) are proved similarly. □

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QUANTUM CODES FROM SUPERELLIPTIC CURVES

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Abstract. Let $\mathcal{X}$ be an algebraic curve of genus $g \geq 2$ defined over a field $\mathbb{F}_q$ of characteristic $p > 0$. From $\mathcal{X}$, under certain conditions, we can construct an algebraic geometry code $C_\mathcal{X}$. When this code (or its dual) is self-orthogonal under the symplectic product, a quantum algebraic geometry code $Q_\mathcal{X}$ is constructed. In this paper we study the construction of such codes from curves with automorphisms and the relation between the automorphism group of the curve $\mathcal{X}$ and the codes $C_\mathcal{X}$ and $Q_\mathcal{X}$.

1. Introduction

In recent years there is an increased interest on the use of algebraic geometry in the theory of quantum cryptography and quantum coding.

Let $\mathcal{X}$ denote a genus g irreducible, algebraic curve defined over a finite field $\mathbb{F}_q$. Under certain conditions, starting with $\mathcal{X}$ one can construct an algebraic geometry code which we denote by $C_\mathcal{X}$. If $C_\mathcal{X}$ (or its dual) is self-orthogonal under an appropriate symplectic form, then from $C_\mathcal{X}$ we can construct a quantum code $Q_\mathcal{X}$, which will be called a QAG-code. In classical coding theory, AG-codes with a large group of automorphisms have good error-correcting properties. Under certain conditions the automorphism group of the curve is embedded in the automorphism group of the corresponding code. Hence, AG-codes which come from algebraic

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curves with a large group of automorphisms are interesting. Very little is known how the automorphism group of the quantum code $Q_{\mathcal{X}}$ relates to the automorphism group of $\mathcal{X}$ and $C_{\mathcal{X}}$.

In this paper we explain

(a) how to construct quantum codes from algebraic geometry codes (quantum algebraic geometry codes), and

(b) the relations between the automorphism group of the algebraic curve, the automorphism group of the AG-code, and the automorphism group of the corresponding quantum code.

It is interesting to note that our method of constructing QAG-codes is based on the existence of an automorphism of the curve $\mathcal{X}$. We focus on the algebraic curves with cyclic automorphism group, but other curves may be used as well. Hence, curves with non-trivial automorphism groups are interesting in this construction. In the last section we give a complete table of groups which occur as automorphism groups of curves of genus 3 and 4 over a field of characteristic 2.

Notation: Throughout this paper $\mathbb{F}_q$ denotes a finite field of q elements where q is a prime power. The notation $[n, k, d]$ denotes a classical code of length n, dimension k, and minimum distance d. $[[n, k, d]]$ will denote a quantum code of the same parameters. A cyclic group of order n is denoted by C_n. In general, given a genus $g \geq 2$ algebraic curve $\mathcal{X}$ defined over $\mathbb{F}$, the automorphism group of $\mathcal{X}$ is denoted by Aut $(\mathcal{X})$ and is defined to be the group of automorphisms of $\mathcal{X}$ defined over the algebraic closure of $\mathbb{F}$. The group of automorphisms defined over $\mathbb{F}$ is denoted by Aut $_{\mathbb{F}}(\mathcal{X})$.

The permutation automorphism group of the code $C \subseteq \mathbb{F}_q^n$ is the subgroup of S_n (acting on $\mathbb{F}_q^n$ by coordinate permutation) which preserves C. We denote such group by PAut (C). The set of monomial matrices that map C to itself forms the monomial automorphism group, denoted by MAut (C). Every monomial matrix M can be written as $M = DP$ where D is a diagonal matrix and P a permutation matrix. Let γ be a field automorphism of $\mathbb{F}_q$ and M a monomial matrix. Denote by M_γ the map $M_\gamma : C \to C$ such that $\forall \mathrm{x} \in \mathrm{C}$ we have $M_\gamma(\mathrm{x}) = \gamma(\mathrm{Mx})$. The set of all maps M_γ forms the automorphism group of C, denoted by ΓAut (C).

2. Linear Codes, Algebraic Geometry Codes and Quantum Stabilizer Codes

2.1. **Linear Codes.** A linear code C of length n and dimension k over a finite field $\mathbb{F}_q$ is a k-dimensional subspace of $V = \mathbb{F}_q^n$. A k-bit codeword of C is encoded into an n-bit word of V to protect the information and recover errors during transmission. The Hamming distance d of the linear code C is the minimum of the weights of the vectors in C. Here the weight of a vector is the number of nonzero coordinates. Such a linear code is denoted by $[n, k, d]$. It detects up to $d-1$ errors and corrects up to $(d-1)/2$ errors.

2.2. **Algebraic geometry codes.** Let $\mathcal{X}$ be a genus g algebraic curve defined over a finite field $\mathbb{F}_q$ with characteristic $p > 0$ and $\mathbb{F} = \mathbb{F}_q(\mathcal{X})$ its function field. Let $P_1, \dots, P_n$ be places of degree one, $D = P_1 + \cdots + P_n$ and G a divisor with $\mathrm{supp}(G) \cap \mathrm{supp}(D) = \emptyset$. The following two algebraic geometry codes have been introduced by Goppa:

A) $C_{\mathcal{L}}(D,G) := \{(f(P_1),\dots,f(P_n)) \mid f \in \mathcal{L}(G)\} \subseteq \mathbb{F}_q^n$. To compute its parameters consider the following **evaluation map**

$$\begin{aligned} \varphi:\ \mathcal{L}(G) &\to \mathbb{F}_q^n \\ f &\mapsto (f(P_1),\dots,f(P_n)), \end{aligned} \tag{1}$$

Since $P_1,\dots,P_n$ are places of degree one and $\operatorname{supp}(G)\cap\operatorname{supp}(D)=\emptyset$, φ is a well-defined map with kernel is $\mathcal{L}(G-D)$. Clearly

$$C_{\mathcal{L}}(D,G) = \varphi(\mathcal{L}(G)),$$

a linear $[n,\,k,\,d]$ code with parameters

$$k = \dim G - \dim(G-D), \quad d \geq n - \deg G.$$

One can easily see that

(1) If we assume $\deg G < n = \deg D$ then $\deg(G-D)<0$ hence $\dim(G-D) = 0$. It follows that $\varphi : \mathcal{L}(G) \to C_{\mathcal{L}}(D,G)$ is injective and $C_{\mathcal{L}}(D,G)$ is an $[n,k,d]$ code with

$$\begin{aligned} k &= \dim G \geq \deg G + 1 - g \\ d &\geq n - \deg G. \end{aligned}$$

(2) If in addition $2g-2 < \deg G < n$, then

$$k = \deg G + 1 - g.$$

(3) If $(f_1,\dots,f_k)$ is a basis of $\mathcal{L}(G)$, then

$$M = \begin{pmatrix} f_1(P_1) & \cdots & f_1(P_n) \\ \vdots & & \vdots \\ f_k(P_1) & \cdots & f_k(P_n) \end{pmatrix}$$

is a *generator matrix* for $C_{\mathcal{L}}(D,G)$.

(4) Let $\operatorname{Aut}_{D,G}(\mathcal{X}) := \{\sigma \in \operatorname{Aut}(\mathcal{X}) | \sigma(D) = D \text{ and } \sigma(G) = G\}$. This group acts on $C_{\mathcal{L}}(D,G)$ via

$$\sigma(f(P_1)\dots f(P_n)) = (f(\sigma(P_1))\dots f(\sigma(P_n))).$$

If $n > 2g+2$, then this action is faithful.

B) The second algebraic geometry code $C_\Omega(D,G)$ is defined by

$$C_\Omega(D,G) := \{(\operatorname{res}_{P_1}(\omega),\dots,\operatorname{res}_{P_n}(\omega)) \mid \omega \in \Omega_F(G-D)\} \subseteq \mathbb{F}_q^n.$$

The following result is well known:

Lemma 1. *If D and G are as above then:*

(1) *$C_{\mathcal{L}}(D,G)^\perp = C_\Omega(D,G)$ under the standard pairing in $\mathbb{F}_q^n$.*

(2) *Let η be any differential with $v_{P_i}(\eta) = -1$ for $i=1,\dots,n$. (Notice that by the approximation theorem such differentials exist.) Let $H = D - G + (\eta)$. Then $C_\Omega(D,G) = a\cdot C_{\mathcal{L}}(D,H)$ where $a = (\operatorname{res}_{P_1}(\eta),\dots,\operatorname{res}_{P_n}(\eta))$.*

(3) *$C_{\mathcal{L}}(D,G)^\perp = a \cdot C_{\mathcal{L}}(D,H)$.*

2.3. **One point codes and their automorphism groups.** Let D be a divisor as above, $P \notin \sup(D)$, and m an integer. The **one point codes of level** m are defined to be algebraic geometry codes of the form

$$C_{\mathcal{L}}(D, mP)$$

Definition 1. A genus $g \geq 1$ curve $\mathcal{X}/F_q$ is called **admissible** if it satisfies:

i) there exists a rational point P_∞ and two functions $x,\ y \in F(\mathcal{X})$ such that $(x)_\infty = kP_\infty,\ (y)_\infty = lP_\infty$, and $k,\ l \geq 1$;

ii) for $m \geq 0$, the elements $x^i y^j$ with $0 \leq i, 0 \leq j \leq k-1$, and $ki + lj \leq m$ form a basis of the space $\mathcal{L}(mP_\infty)$.

Lemma 2. *Let $\mathcal{X}/F_q$ be an admissible curve over F_q of genus g where $l > k$. Assume that $m \geq l$. If*

$$n > max\{2g+2, 2m, k(l + \frac{k-1}{\beta}), lk(1 + \frac{k-1}{m-k+1})\},$$

where $n = |J|,\ \beta = min\{k-1,\ r | y^r \in \mathcal{L}(mP_\infty)\}$ *then*

$$Aut\ (C_{\mathcal{L}}(D, mP_\infty)) \cong Aut\ _{D, mP_\infty}(\mathcal{X}).$$

Proof. See [16] for details □

2.4. **Quantum codes, stabilizer codes and connection with classical codes.** Let $q = p^m$ be a prime power and V_q a q-dimensional complex vector space. **A** q**-ary quantum code of length** n **and dimension** k is a k-dimensional subspace Q of $V := V_q^{\otimes n}$. V_q is the quantum counterpart of $\mathbb{F}_q$. Its elements are called **quantum states**. A codeword of k quantum states in Q is encoded into a word of n quantum states in V for protection and error correction.

A general quantum error of a q-ary quantum system is a linear transformation of the space V_q. Let $e_1, e_2, \dots, e_{q^2}$ be a basis for the space of quantum errors of such a system, where e_1 is the identity linear transformation. *A quantum error of an n q-ary system* is a linear transformation of V. A basis for the space of general quantum errors is formed by $E := \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_n$ with $\sigma_i \in \{e_1, e_2, \dots, e_{q^2}\}$. Define the *weight* of such E as

$$wt(W) = |\{\sigma_i \neq e_1\}|.$$

Recall that if a code can correct a set of errors then it can correct their linear span.

For a quantum code Q, we let P be the projection $P : V \to Q$. It can be shown that a quantum error E is *detectable* iff

$$PEP = c_E P$$

for some scalar c_E. *The minimum distance* of a quantum code Q is the largest integer d such that every error of weight $d-1$ can be detected by Q.

Stabilizer codes are a general class of quantum codes that can be constructed from classical codes. We review briefly this construction here, see [1] for more details.

First, we construct a special error basis. Let T, R be $p \times p$ matrices defined by $T_{i,j} = \delta_{i,j-1\ (mod\ p)}$ and $R_{ij} = \omega^i \delta_{i,j}$ where ω is a p-th root of unity. For $a \in F_q$, let $(a_1, \dots, a_m)$ be the coordinates of a in some fixed basis of F_q as an F_p-vector space. Define

$$T_a := T^{a_1} \otimes \cdots \otimes T^{a_m},\ R_a := R^{a_1} \otimes \cdots \otimes R^{a_m}.$$

Finally, for $a = (a^1, a^2, \dots, a^n) \in F_q^n$ and $b = (b^1, b^2, \dots, b^n) \in F_q^n$, let

$$E_{a,b} := T_{a^1} R_{b^1} \otimes \cdots \otimes T_{a^n} R_{b^n}.$$

It is shown in [1] that $E_{a,b}$ form a basis for quantum errors of an n q-ary system.

Let S be an abelian subgroup of $\Omega = \{\omega^i E_{a,b}, i = 0, 1, \dots, p-1; a, b \in F_q^n\}$ and μ an S linear character that satisfies $\mu(\omega I) = \omega$. The eigenspace $Q_{S,\mu}$ of S associated with μ, i.e.

$$Q_{S,\mu} := \{v \in V \ : \ E(v) = \mu(E)v, \forall E \in S\},$$

is called *a quantum stabilizer code*. If the order of S is p^{r+1}, then the dimension of $Q_{S,\mu}$ is p^{mn-r}.

Quantum stabilizer codes constructed above are related to classical codes in F_q^{2n}. First some preliminaries. Let tr: $F_q \to F_p$ the trace map. Elements of F_q^{2n} will be denoted by $v = (a, b)$ where $a = (a_1, a_2, \dots, a_n), b = (b_a, b_2, \dots, b_n) \in F_q^n$. *The (symplectic) weight* of such an element is

$$wt(v) = |\{i \ : \ (a_i, b_i) \neq (0, 0)\}|.$$

For $x = (x_1, \dots, x_n); y = (y_1, \dots, y_n)$ we let

$$\langle x, y\rangle := x_1 y_1 + \cdots + x_n y_n$$

be the standard pairing in F_q^n . Define two symplectic products on F_q^{2n} as follows:

$$\langle (a, b), (a', b')\rangle_s := \langle a, b'\rangle - \langle a', b\rangle$$

and

$$tr_s((a, b), (a', b')) := tr(\langle a, b'\rangle - \langle a', b\rangle).$$

The quantum stabilizer code $Q_{S,\mu}$ yields a classical code $C := \{(a, b) \ : \ E_{a,b} \in S\} \subset F_q^{2n}$. It is an F_p-linear code of length $2n$ and size p^r. The commutativity of S implies that the code C is self-orthogonal relative to the symplectic product tr_s, i.e. $C \subset C^{\perp_s}$ where $C^{\perp_s}$ be the dual of the code C. The minimum distance of the quantum stabilizer code $Q_{S,\mu}$ is related to the classical minimum distance of $C^{\perp_s} \backslash C$. In fact, if $S^{\perp_s}$ denotes the centralizer of S, it is shown in [1] that an error E is detectable by $Q_{S,\mu}$ iff $E \in S^{\perp_s} \backslash S$. It follows that the minimum distance of $Q_{S,\mu}$ is the $\min\{wt(v) \ v \in C^{\perp_s} \backslash C\}$. It is clear that this process is reversible, i.e. a tr_s self-orthogonal code C in F_q^{2n} with p^r codewords determines a quantum stabilizer code Q of dimension p^{mn-r}.

Remark: We note that if C is an F_q-linear code then the duals of C relative to the two symplectic forms are the same (this not true, however, for general F_p-linear codes.) From now on, we will be interested only in F_q-linear codes and the symplectic form $\langle (a, b), (a', b')\rangle_s$. The dual of a linear code C relative to this symplectic form will be denoted by $C^{\perp_s}$.

The following proposition will be used in constructing AG-quantum stabilizers codes from AG-classical self-orthogonal codes.

Proposition 1. *Let $C \subset F_q^{2n}$ be a $(n+k)$-dimensional F_q-subspace such that such that $C^{\perp_s} \subset C$ (i.e. $C^{\perp_s}$ is self orthogonal). Then, there exist a quantum code $Q \subset V$ of dimension q^k and minimum distance $d = \min\left\{wt(x) \mid x \in C \backslash C^{\perp_s}\right\}$.*

Proof. The F_q-code $C^{\perp_s}$ is self-orthogonal and has dimension $n-k$, so it has p^r codewords with $r = m(n-k)$. From the above construction we get a quantum stabilizer code of dimension $p^{mn-r} = p^{mn-m(n-k)} = p^{mk} = q^k$.

□

Hence, in order to construct quantum AG-codes we need to construct self-orthogonal AG-codes.

3. Quantum algebraic geometry codes from algebraic curves with automorphisms

We continue with the notation of the previous session; $\mathcal{X}$ is a genus g curve defined over a finite field $\mathbb{F}_q$ and F is its function field. The following lemma is cited from [15, Prop. VII.1.2]. It allows for the construction of differentials with special properties that help to construct a self-orthogonal code.

Lemma 3. *Let x and y be elements of F such that $v_{P_i}(y) = 1$, $v_{P_i}(x) = 0$ and $x(P_i) = 1$ for $i = 1, \dots, n$. Then the differential $\eta := x \cdot \frac{dy}{y}$ satisfies $v_{P_i}(\eta) = -1$ and $\operatorname{res}_{P_i}(\eta) = 1$ for $i = 1, \dots, n$.*

A quantum stabilizer code can be obtained from an algebraic geometric construction related to curves with an involution.

Theorem 1. *Let $\mathcal{X}$ be a genus g irreducible algebraic curve defined over $\mathbb{F}_q$ and $P_1, \dots, P_n$ degree one rational points on $\mathcal{X}$. Let $\sigma \in Aut_{\mathbb{F}}(\mathcal{X})$ be an involution such that $\sigma P_i \neq P_j$, $\forall\, i, j = 1, \dots, n$. Further assume that we have a divisor G such that $\sigma G = G$, $v_{P_i}(G) = v_{\sigma P_i}(G) = 0$ for all i. Then, there exists a quantum code $Q_{\mathcal{X}} = [[n, k, d]]$ such that*

$$k = \dim G - \dim\left(G - P_1 - \cdots P_n - \sigma(P_1) - \cdots \sigma(P_n)\right) - n, \quad d \geq n - \left\lfloor \frac{deg\ G}{2} \right\rfloor$$

Proof. Let Let $D = P_1 + \cdots + P_n + \sigma P_1 + \cdots + \sigma P_n$. By the strong approximation theorem, there is a differential η such that

$$\left\{ \begin{array}{l} v_{P_i}(\eta) = v_{\sigma P_i}(\eta) = -1, \\ res_{P_i}(\eta) = 1, \\ res_{\sigma P_i}(\eta) = 1. \end{array} \right. \tag{2}$$

It follows that $H := D - G + (\eta) \leq G$. Consider the algebraic geometry code

$$C_{\mathcal{L}}(D, G) = \{ (f(P_1), \dots, f(P_n), f(\sigma P_1), \dots, f(\sigma P_n)) \mid f \in \mathcal{L}(G) \} \subseteq \mathbb{F}_q^{2n}.$$

One can easily see that

$$(x_1, \dots, x_n, x_{n+1}, \dots, x_{2n}) \in C_{\mathcal{L}}(D, G) \Leftrightarrow (x_{n+1}, \dots, x_{2n}, x_1, \dots, x_n) \in C_{\mathcal{L}}(D, G),$$

which implies that $C_{\mathcal{L}}(D, G)^{\perp_s} = C_{\mathcal{L}}(D, H)$.

Notice that $\mathcal{L}(H) \subset \mathcal{L}(G)$ since $H \leq G$. It follows that $C(D, G)^{\perp_s} = C(D, H) \subset C(D, G)$. Now apply the last proposition with $k = \dim C(D, G) - n$.

To show the inequality of the distance, let $f \in \mathcal{L}(G)$ such that

$$\text{wt}\ \ (f(P_1), \dots, f(\sigma(P_n)) = \delta \neq 0.$$

Hence, there exists a set of $n-\delta$ pairs $(f(P_{i_j}), f(\sigma P_{i_j})), j = 1, 2, \ldots, n-\delta$ which are all zero. Thus, we have $f \in \mathcal{L}\left(G - \sum_{j=1}^{n-\delta}(P_{i_j} + \sigma P_{i_j})\right)$. Hence

$$\dim \mathcal{L}\left(G - \sum_{j=1}^{n-\delta}(P_{i_j} + \sigma P_{i_j})\right) > 0,$$

which implies the result.

□

From this theorem we see that curves with an involution and many rational points are useful in generating quantum codes. These are also the typical curves used in constructing AG-codes. Clearly hyperelliptic curves have an involution but there are others. It is however unknown how the choice of the involution affects the parameters of the code - this remains an open question.

Curves which have a cyclic group embedded in their automorphism group may also be used to produce quantum stabilizer codes. We deal with them in the next section.

3.1. **Quantum codes from superelliptic curves.** A genus $g \geq 2$ superelliptic curve of level n is given by an equation of the form $y^n = f(x)$ for some degree $d > 4$ polynomial $f(x)$ and $n \geq 2$. Let us assume that

$$y^n = f(x) = \prod_{i=1}^{s}(x - \alpha_i)^{d_i}, \quad 0 < d_i < d.$$

Then, $\sum_{i=1}^{s} d_i = d$. We call this the **standard form** of the superelliptic curve. Superelliptic curves of level n are studied in detail in [2]. They are interesting from the point of view of this article because they have extra automorphisms and we can determine their automorphism groups and their equations; see [7, 8]

Let k be an algebraic closed field of characteristic $p \geq 0$. Let $F_0 = k(x)$ be the function field of the projective line $\mathbb{P}^1(k)$ and $F := k(x, y)$. If $d := \sum_{i=1}^{s} d_i \equiv 0 \mod n$ then the place at infinity does not ramify at the above extension. The only places at F_0 that are ramified are the places P_i that correspond to the points $x = \alpha_i$ and the corresponding ramification indices are given by

$$e_i = \frac{n}{(n, d_i)}.$$

Moreover if $(n, d_i) = 1$ then the places P_i are ramified completely and the Riemann-Hurwitz formula implies that the function field F has genus

$$g = \frac{(n-1)(s-2)}{2}.$$

Notice that the condition $g \geq 2$ is equivalent to $s \geq 2\frac{n+1}{n-1}$. In particular, $s > 2$.

For the proof of the following Lemmas see [2].

Lemma 4. *Let $G = Aut(F)$. Suppose that a cyclic extension F/F_0 of the rational function field F_0 is ramified completely at s places and $n := |Gal(F/F_0)|$. If $2n < s$ then $Gal(F/F_0) \triangleleft G$.*

Lemma 5. *Suppose that τ is an extra automorphism of F, and let s be the number of ramified places at the extension F/F_0 and let d be the degree of the defining polynomial. Then $\delta | s, \delta | d$ and the defining equation of F can be written as*

$$y^n = \sum_{i-0}^{d/\delta} a_i x^{\delta \cdot i},$$

where $a_0 = 1$.

We will say that the superelliptic curve is in **normal form** if and only if it is given by an equation:

$$y^n = x^s + \sum_{i-1}^{\frac{d}{\delta}} a_i x^{\delta \cdot i} + 1.$$

Parametrizing superelliptic curves that admit an extra automorphism of order δ, is the set of coefficients $\{a_{s/\delta-1}, \cdots, a_1\}$ of a normal form up to a change of coordinate in x. The condition $\tau(x) = \zeta x$, implies that $\bar{\tau}$ fixes the places $0, \infty$. Moreover we can change the defining equation by a morphism $\gamma \in PGL(2,k)$ of the form $\gamma : x \to mx$ or $\gamma : x \to \frac{m}{x}$ so that the new equation is again in normal form. Substituting $a_0 = (-1)^{d/s} \prod_{i=1}^{d/s} \beta_i^s$ we have

$$(-1)^{s/\delta} \prod_{i=1}^{s/\delta} \gamma(\beta_i)^\delta = 1$$

and this gives $m^s = (-1)^{s/\delta}$. Then, x is determined up to a coordinate change by the subgroup $D_{s/\delta}$ generated by

$$\tau_1 : x \to \epsilon x, \; \tau_2 : x \to \frac{1}{x}$$

where ϵ is a primitive s/δ-root of one, see [2] for details.

The action of $D_{s/\delta}$ on the parameter space $k(a_1, \ldots, a_{s/\delta})$ is given by

$$\begin{aligned} \tau_1 &: a_i \to \epsilon^{\delta i} a_i, \; \textit{for } i = 1, \ldots s/\delta \\ \tau_2 &: a_i \to a_{d/\delta - i}, \; \textit{for } i = 1, \ldots [s/\delta] \end{aligned}$$

Notice that if $s/\delta = 1$ then the above actions are trivial, therefore the normal form determines the equivalence class. If $s/\delta = 2$ then

$$\tau_1(a_1) = -a_1, \tau_1(a_2) = a_2, \tau_2 = 1$$

and the action is not dihedral but cyclic on the first vector.

Lemma 6. *Let $r := s/\delta > 2$ The elements*

$$\mathfrak{s}_i := a_1^{r-i} a_1 + a_{r-1}^{r-i} a_{r-i}, \; \textit{for } i = 1, \ldots, r$$

are invariants under the action of the group $D_{s/\delta}$ defined as above.

See [2] for details. The elements $\mathfrak{s}_i$ are called the **dihedral invariants** or **$\mathfrak{s}$-invariants** of $D_{s/\delta}$. Two superelliptic curves are isomorphic if and only if they have the same $\mathfrak{s}$-invariants.

Let $\mathcal{X}$ be a genus g superelliptic curve of level $r > 2$ and σ the automorphism of order r of $\mathcal{X}$. The corresponding projection $\psi_\sigma : \mathcal{X} \to \mathbb{P}^1$ has d branch points. Let $\mathcal{B}$ be the branch set. For a given rational point $P \in \mathcal{X}$ we define $\mathcal{O}rb_\sigma(P) = \{\sigma(P) \in X\}$. If $\psi(P) \notin \mathcal{B}$ then $|\mathcal{O}rb_\sigma(P)| = r$.

Let $P_1, \dots, P_n$ rational points on $\mathcal{X}$ such that $\psi(P_i) \notin \mathcal{B}$ for all $i = 1, \dots, n$ and let

$$D = \sum_{i=1}^{n} \left(P_i + \sigma(P_i) + \cdots + \sigma^{r-1}(P_i) \right) = \sum_{i=1}^{n} \mathcal{O}rb(P_i)$$

Then $\deg D = rn$. For some $P \in \mathcal{X}$ such that $\psi(P) \in \mathcal{B}$ we define $G = mP$ for some integer m. Then $\sigma(G) = G$. We can take infinity to be one of the branch points in $\mathcal{B}$. In that case the point P in the fiber is denoted by P_∞. It is common in coding theory to take G to be mP_∞.

Again we work with $C_\mathcal{X} = C_\mathcal{L}(G, D)$. The proof of the following theorem should go through like in Thm. 1.

Theorem 2. *Let $\mathcal{X}$ be an algebraic curve defined over a field $\mathbb{F}_q$ of characteristic $p > 0$ such that $C_r = \langle \sigma \rangle \hookrightarrow Aut\ (\mathcal{X})$. Let $P_1, \dots, P_n$ rational points on $\mathcal{X}$ such that $|\mathcal{O}rb_\sigma(P_i)| = r$ and $\mathcal{O}rb_\sigma(P_i) \cap \mathcal{O}rb_\sigma(P_j) = \emptyset$ for all i, j. Further assume that we have a divisor G such that $\sigma G = G$, $v_{P_i}(G) = v_{\sigma P_i}(G) = 0$ for all i. Then, there exists a quantum code $Q_\mathcal{X} = [[nr, k, d]]$ such that*

$$k = \dim G - \dim(G - D) - n\, r, \quad d \geq n\, r - \left\lfloor \frac{deg\ G}{2} \right\rfloor$$

As in the case of curves with involutions, it is unclear how the cyclic group or the automorphism group of the curve affects the parameters of the quantum code. This remains an open question.

Example 1. Let $\mathcal{X}$ be the curve

$$y^3 - y = x^4$$

defined over $\mathbb{F}_q$. For characteristic $p > 7$, Aut $(\mathcal{X})$ is a group of order 96 with Gap identity $(96, 64)$. Denote the set of affine rational points of $\mathcal{X}$ over $\mathbb{F}_q$ by $\{P_1, \dots, P_n\}$. Let $C = C_\mathcal{L}(D, G)$, where $n + 1$ is the number of rational points of $\mathcal{X}$ and

$$G = mP_\infty, \quad D = P_1 + \cdots P_n$$

The permutation automorphism group PAut (C) is as follows:

i) If $0 \leq m < 3$ or $m > n + 4$ then PAut $(C) \cong S_n$.

ii) If $n > 24$ and $4 \leq m < n/2$ then PAut $(C) \cong Aut_{D, mP_\infty}(\mathcal{X})$.

For a proof of the above statements see [11].

Let $\mathcal{X}$ be defined over F_4. Take $m = 6$. By computation using GAP, we find that $C_\mathcal{L}(D, G)$ is a $[4, 4, 1]$ code with a generator matrix

$$\begin{pmatrix} \alpha & \alpha^2 & 0 & 0 \\ \alpha^2 & \alpha & 0 & 0 \\ \alpha & \alpha^2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

where α is a primitive element of F_4. The permutation automorphism group is isomorphic to the group with GAP identity $[24, 12]$. In this case

$$\text{PAut } (C) \hookrightarrow \text{Aut } (\mathcal{X}).$$

This code is clearly an MDS code. The automorphism group ΓAut (C) has Gap identity $(1944, 3876)$. One can check that this code is self-orthogonal with respect to the inner product, hence there is a quantum code Q which has parameters $[[4,4]]$. Its automorphism group has order 31104 and is a degree 2 extension of $\Gamma\mathrm{Aut}\ (C)$. $\square$

In the next section we see how the hyperelliptic involution can be used to construct quantum algebraic geometry codes. However, other involutions can be used as well.

4. Hyperelliptic quantum codes

The goal of this section is to construct quantum stabilizer codes starting with AG-codes which come from hyperelliptic curves. We focus on odd characteristic. Let $\mathcal{X}_g$ a genus g hyperelliptic curve given by the equation $y^2 = f(x)$, $F := K(x, y)$ its function field and σ the hyperelliptic involution of $\mathcal{X}_g$. Then F has a set of rational places which are not fixed by the hyperelliptic involution. Choose a set of distinct places in F

$$\{P_1, \dots, P_n, \sigma(P_1), \dots, \sigma(P_n)\},$$

such that $\pi(P_i) = \alpha_i$, where π is the hyperelliptic projection.

Let P_∞ denote the place at infinity and $D, G \in \mathrm{Jac}\ (\mathcal{X}_g)$ be as follows

$$D := \sum_{i=1}^{n} P_i \ + \ \sum_{i=1}^{n} \sigma(P_i) \quad \textit{and} \qquad G := (n + g - 1 - r)P_\infty,$$

where $0 \leq r \leq n - g$. Then D has degree $2n$. By the Riemann's theorem there exists $\eta \in F$ such that

$$\eta = \frac{1}{y\ \prod_{i=1}^{n}(x - \alpha_i)} \mathrm{d}x$$

Hence, $(\eta) = (2n + 2g - 2)P_\infty - D$. We denote

$$W := (\eta) = (2n + 2g - 2)P_\infty - D, \ \textit{and} \quad H := D - G + W.$$

Then W is a canonical divisor. and the residues of η at the places $P_1, \dots, P_n$, $\sigma(P_1), \dots, \sigma(P_n)$ satisfy

$$a_i := \mathrm{res}_{P_i}(\eta) = -\mathrm{res}_{\sigma(P_i)}(\eta)\,.$$

for $i = 1, \dots, n$.

Now we can construct algebraic geometry codes $C_\mathcal{L}(D, G)$ and $C_\mathcal{L}(D, H)$. The weighted symplectic inner product is defined as below

$$\langle x, y\rangle_s^a = \sum_{i=0}^{C} 4na_i\left(x_i y_{n+i} - x_{n+i} y_i\right).$$

for all $x, y \in C$ and all $a_i \neq 0$.

Lemma 7. *Let $C_\mathcal{L}(D, G)$ and $C_\mathcal{L}(D, H)$ be as above. Then*

$$C_\mathcal{L}(D, G)^{\perp_s} = C_\mathcal{L}(D, H) \cdot \mathrm{diag}(a_1, \dots, a_n, 1, \dots, 1)$$

Moreover, $C_\mathcal{L}(D, G) \subseteq C_\mathcal{L}(D, G)^{\perp_s^a}$ with respect to the symplectic inner product. $\langle\ ,\ \rangle_s^a$.

We transform $C_{\mathcal{L}}(D,G)$ to a code $C'_{\mathcal{L}}(D,G)$ which has the same parameters as $C_{\mathcal{L}}(G,D)$ and is self-orthogonal with respect to the standard symplectic inner product by multiplying each component x_i of every codeword by the corresponding a_i, for $1 \leq i \leq n$.

Then, we have the following:

Proposition 2. *$C'_{\mathcal{L}}(D,G)$ yields a stabilizer code with parameters $[[n, k, d]]$, where $k = g + r - 1$ and $d \geq \frac{n-k}{2}$.*

4.1. **Explicit construction of quantum AG-codes.** Here is an algorithm which would create a hyperelliptic quantum code.

Algorithm 1. Hyperelliptic quantum codes

Input: A genus g hyperelliptic curve over a finite field $\mathbb{F}_q$.
Output: A quantum code Q

i) *Find all rational places of degree 1 of $\mathcal{X}_g$ which are not fixed by the hyperelliptic involution, say $S = \{P_1, \dots, P_n, \sigma(P_1), \dots, \sigma(P_n)\}$.*

ii) *Let*

$$\begin{aligned} D &:= \sum_{P \in S} (P + \sigma(P)) \\ G &:= (n + g - 1 - r)P_\infty \\ (\eta) &:= -D + (2n + 2g - 2)P_\infty \end{aligned}$$

iii) *Create a list $A = [a_1, \dots, a_n]$, where*

$$a_i := res_{P_i}(\eta) = -res_{\sigma(P_i)}(\eta)$$

iv) *Construct the AG code $C = \mathcal{L}(D,G)$ and let the generator matrix of C, to be $\mathcal{G}$.*

v) Transform C to a self-orthogonal symplectic code Q by multiplying each coordinate x_i by a_i,

$$(\dots, x_i, \dots) \to (\dots, a_i x_i, \dots)$$

vi) Return Q.

5. Automorphism groups

In this section we give a brief survey of automorphism groups of curves over finite fields, automorphism groups of codes, and automorphism groups of quantum codes.

5.1. **Automorphism groups of curves.** It has been known since Hurwitz (1892) that a Riemann surface of genus $g > 1$ has at most $84(g-1)$ automorphisms. This estimate is optimal; there are Riemann surfaces of arbitrarily high genus with $84(g-1)$ automorphisms (Hurwitz' bound in characteristic 0), the Klein curve most notable of them. The Hurwitz estimate is not valid in prime characteristic. Roquette (1970) found that the estimate

$$|G| \leq 84(g-1),$$

on the order of the automorphism group G, holds under the additional assumption $p > g+1$, with one exception: the function field $F = K(x, y)$ with $y^p - y = x^2$ has genus $g = \frac{1}{2}(p-1)$ and $8g(g+1)(2g+1)$ automorphisms.

Stichtenoth (1973) gives a general estimate for the number of automorphisms of a smooth projective curve in characteristic $p > 0$. He proves the inequality

$$|G| < 16 \cdot g^4,$$

but also with one series of exceptions: the function field $F = K(x, y)$ with

$$y^{p^n} + y = x^{p^{n+1}}$$

has genus $g = \frac{1}{2}p^n(p^n - 1)$ and $|G| = p^{3n}(p^{3n}+1)(p^{2n}-1)$ automorphisms, so $|G|$ is in this case slightly larger than $16g^4$.

Let X denote a smooth, genus g algebraic curve defined over k, char $k = p > 0$. A theorem of Blichfeld on invariants (in char 0) of subgroups of $PGL_3(k)$ implies that the genus g curve lifts to characteristic 0 for $p > 2g+1$; see [4, pg. 236-254]. Hence, for large enough p (i.e., $p > 2g+1$) methods described in [7] can be used to determine such groups. Thus, to determine the list of groups that occur as automorphism groups of genus g curves we have to classify the groups that occur for all primes $p \leq 2g+1$.

5.1.1. *Automorphism groups of superelliptic curves.* The automorphism groups of superelliptic curves for every characteristic $p \neq 2$ were determined in [8]. For genus 3 and 4 the following theorem are simply an application of Sanjeewa's results.

Lemma 8. *Let $\mathcal{X}_g$ be a genus 3 superelliptic curve defined over a field of characteristic p. Then the automorphism groups of $\mathcal{X}_g$ are as follows.*

- **i):** $p = 0:$ (2, 1), (4, 2), (3, 1), (4, 1), (8, 2), (8, 3), (7, 1), (21, 1), (14, 2), (6, 2), (12, 2), (9, 1), (8, 1), (8, 5), (16, 11), (16, 10), (32, 9), (30, 2), (42, 3), (12, 4), (16, 7), (24, 5), (18, 3), (16, 8), (48, 33), (48, 48).
- **ii):** $p = 3:$ (2, 1), (4, 2), (3, 1), (4, 1), (8, 2), (8, 3), (7, 1), (14, 2), (6, 2), (8, 1), (8, 5), (16, 11), (16, 10), (32, 9), (30, 2), (16, 7), (16, 8), (6, 2).
- **iii):** $p = 5:$ (2, 1), (4, 2), (3, 1), (4, 1), (8, 2), (8, 3), (7, 1), (21, 1), (14, 2), (6, 2), (12, 2), (9, 1), (8, 1), (8, 5), (16, 11), (16, 10), (32, 9), (42, 3), (12, 4), (16, 7), (24, 5), (18, 3), (16, 8), (48, 33), (48, 48).
- **iv):** $p = 7:$ (2, 1), (4, 2), (3, 1), (4, 1), (8, 2), (8, 3), (7, 1), (21, 1), (6, 2), (12, 2), (9, 1), (8, 1), (8, 5), (16, 11), (16, 10), (32, 9), (30, 2), (42, 3), (12, 4), (16, 7), (24, 5), (18, 3), (16, 8), (48, 33), (48, 48).
- **v):** $p > 7:$ (2, 1), (4, 2), (3, 1), (4, 1), (8, 2), (8, 3), (7, 1), (21, 1), (14, 2), (6, 2), (12, 2), (9, 1), (8, 1), (8, 5), (16, 11), (16, 10), (32, 9), (30, 2), (42, 3), (12, 4), (16, 7), (24, 5), (18, 3), (16, 8), (48, 33), (48, 48).

We obtain the following groups as automorphism groups of a genus 4 cyclic curve defined over algebraically closed field of characteristic 0,3,5,7 and bigger than 7. We listed GAP group ID of those groups in following theorem.

Lemma 9. *Let $\mathcal{X}_g$ be a genus 4 superelliptic curve defined over a field of characteristic p. Then the automorphism groups of $\mathcal{X}_g$ are as follows.*

i): $p=0$: $(2,1)$, $(4,2)$, $(3,1)$, $(6,2)$, $(9,2)$, $(5,1)$, $(10,2)$, $(20,1)$, $(9,1)$, $(27,4)$, $(18,2)$, $(15,1)$, $(4,1)$, $(20,4)$, $(18,3)$, $(8,3)$, $(40,8)$, $(12,5)$, $(36,12)$, $(54,4)$, $(16,7)$, $(20,5)$, $(32,19)$, $(24,10)$, $(8,4)$, $(60,9)$, $(36,11)$, $(24,3)$, $(72,42)$.

ii): $p=3$: $(2,1)$, $(4,2)$, $(3,1)$, $(6,2)$, $(5,1)$, $(10,2)$, $(20,1)$, $(9,1)$, $(18,2)$, $(15,1)$, $(4,1)$, $(20,4)$, $(8,3)$, $(40,8)$, $(12,5)$, $(16,7)$, $(20,5)$, $(32,19)$, $(24,10)$, $(8,4)$, $(9,2)$, $(18,5)$.

iii): $p=5$: $(2,1)$, $(4,2)$, $(3,1)$, $(6,2)$, $(9,2)$, $(5,1)$, $(10,2)$, $(20,1)$, $(9,1)$, $(27,4)$, $(18,2)$, $(4,1)$, $(18,3)$, $(8,3)$, $(12,5)$, $(36,12)$, $(54,4)$, $(16,7)$, $(20,5)$, $(32,19)$, $(24,10)$, $(8,4)$, $(60,9)$, $(36,11)$, $(24,3)$, $(72,42)$, $(10,2)$, $(18,5)$.

iv): $p=7$: $(2,1)$, $(4,2)$, $(3,1)$, $(6,2)$, $(9,2)$, $(5,1)$, $(10,2)$, $(20,1)$, $(9,1)$, $(27,4)$, $(18,2)$, $(15,1)$, $(4,1)$, $(20,4)$, $(18,3)$, $(8,3)$, $(40,8)$, $(12,5)$, $(36,12)$, $(54,4)$, $(16,7)$, $(20,5)$, $(32,19)$, $(24,10)$, $(8,4)$, $(60,9)$, $(36,11)$, $(24,3)$, $(72,42)$.

v): $p>7$: $(2,1)$, $(4,2)$, $(3,1)$, $(6,2)$, $(9,2)$, $(5,1)$, $(10,2)$, $(20,1)$, $(9,1)$, $(27,4)$, $(18,2)$, $(15,1)$, $(4,1)$, $(20,4)$, $(18,3)$, $(8,3)$, $(40,8)$, $(12,5)$, $(36,12)$, $(54,4)$, $(16,7)$, $(20,5)$, $(32,19)$, $(24,10)$, $(8,4)$, $(60,9)$, $(36,11)$, $(24,3)$, $(72,42)$.

Remark 1. Note that these lists contain all groups and not just the full automorphism groups.

5.1.2. *Automorphisms groups over finite fields of characteristic 2.* Let C be a hyperelliptic curve of genus g over an algebraically closed field K of characteristic 2. We use an Artin-Schreier generation $y^2+y=g(x)$ such that $g(x)\in K(x)$. We can find a rational function $h(x)\in K(x)$ such that the rational function $g(x)+h(x)+h(x)^2$ has no poles of even order. Let $f(x):=g(x)+h(x)+h(x)^2$ and use the normalized form $y^2+y=f(x)$. Then, y is unique up to transformations of the form $y\longmapsto y+B(x)$, where $B(x)$ is a rational function of x.

Let $\Sigma n_a(a)$ be the polar divisor of $f(x)$ on the projective line, $\mathbf{P}^1$. C is ramified at each a and if P_a is the unique point of C over a then the curve $y^2+y=f(x)$ has the different

$$Diff(C/\mathbf{P}^1)=\Sigma(n_a+1)P_a$$

where the n_a are odd ([14], Prop III.7.8)

$$2g-2=-2[F:K(x)]+deg(Diff(C/\mathbf{P}^1)) \implies deg(Diff(C/\mathbf{P}^1))=2g+2$$

Take two hyperelliptic curves, $C: y^2+y=f(x)$ and $C': y^2+y=h(x)$. Then there are finite morphisms $f_1: C\longmapsto \mathbf{P}^1$, and $f_2: C'\longmapsto \mathbf{P}^1$ of degree 2, and there exists a unique automorphism σ of $\mathbf{P}^1$ such that $f_2=\sigma\circ f_1$. Any isomorphism between these curves has the form

$$(x,y)\longmapsto (\frac{ax+b}{cx+d}, y+B(x))$$

for some $B(x)\in K(x)$. Hence, these curves are isomorphic if and only if

$$h(x)=f(\frac{ax+b}{cx+d})+s(x)+s(x)^2$$

for some $s(x)\in K(x)$. The ramification types determine the isomorphism classes of the hyperelliptic curves. The solutions of the equation $\Sigma(n_a+1)=2g+2$ in the unknown odd positive integers give us the following ramification types:

$$
\begin{aligned}
&(1,1,1,1),(3,1,1),(3,3),(5,1),(7) \quad for\ genus\ 3\\
&(1,1,1,1,1),(3,1,1,1),(3,3,1),(5,1,1),(5,3),(7,1),(9) \quad for\ genus\ 4
\end{aligned} \tag{3}
$$

Therefore we get the following normal forms for genus 3 and 4 respectively.

$$
\begin{aligned}
y^2+y &= \begin{cases} \alpha_1 x + \alpha_2 x^{-1} + \alpha_3 (x-1)^{-1} + \alpha_4 (x-\lambda)^{-1}\\ x^3+\alpha x + \beta x^{-1} + \gamma (x-1)^{-1}\\ x^3 + \alpha x + \beta x^{-3} + \gamma x^{-1}\\ x^5 + \alpha x^3 + \beta x^{-1}\\ x^7 + \alpha x^5 + \beta x^3 \end{cases}\\
y^2+y &= \begin{cases} \alpha_1 x + \alpha_2 x^{-1} + \alpha_3 (x-1)^{-1} + \alpha_4 (x-\lambda)^{-1} + \alpha_5 (x-\mu)^{-1}\\ x^3+\alpha x + \beta_1 x^{-1} + \beta_2 (x-1)^{-1} + \beta_3 (x-\lambda)^{-1}\\ x^3 + \alpha x + \beta x^{-3} + \gamma x^{-1} + \sigma (x-1)^{-1}\\ x^5 + \alpha x^3 + \beta x^{-1} + \gamma (x-1)^{-1}\\ x^5 + \alpha x^3 + \beta x^{-3} + \gamma x^{-1}\\ x^7 + \alpha x^5 + \beta x^3 + \gamma x^{-1}\\ x^9 + \alpha_1 x^7 + \alpha_2 x^5 + \alpha_3 x^3 \end{cases}
\end{aligned} \tag{4}
$$

These are plane curves given in inhomogeneous form, birational to the given nonsingular curves(i.e. the function fields are isomorphic). We will use the above normal forms to determine $\bar{G}$, the reduced group of automorphisms, namely the quotient of the group of automorphisms, G by $\langle \iota \rangle$ which is contained in the center of G. And then we will compute G.

Proposition 3. *Let C be a genus g hyperelliptic curve defined over an algebraically closed field K of characteristic 2.*

i) If $g = 3$ then the automorphism group of C is one of the following: C_2, C_4, V_4, $C_2 \times C_2 \times C_2$, C_6, C_{14}, D_{12}.

ii) If $g = 4$ then the automorphism group of C is one of the following: C_2, V_4, C_4, $C_2 \times C_2 \times C_2$, C_6, C_{18}, D_{20}.

Proof. See [3] for details. □

Furthermore, the parametric equation of the curve in each case is given by equation in Table 1. Determining complete lists of full automorphism groups for a given genus $g > 3$ is still an open problem with many applications in theoretical mathematics, computer science, and electrical engineering.

5.2. **Automorphism groups of codes.** The **permutation automorphism group** of the code $C \subseteq \mathbb{F}_q^n$ is the subgroup of S_n (acting on $\mathbb{F}_q^n$ by coordinate permutation) which preserves C. We denote such group by PAut (C). The set of monomial matrices that map C to itself forms the **monomial automorphism group**, denoted by MAut (C). Every monomial matrix M can be written as $M = DP$ where D is a diagonal matrix and P a permutation matrix. Let γ be a field automorphism of $\mathbb{F}_q$ and M a monomial matrix. Denote by M_γ the map $M_\gamma : C \to C$ such that $\forall \mathrm{x} \in \mathrm{C}$ we have $M_\gamma(\mathrm{x}) = \gamma(\mathrm{Mx})$. The set of all maps M_γ forms the **automorphism group** of C, denoted by ΓAut (C). It is well known that

$$\mathrm{PAut}\ (C) \leq \mathrm{MAut}\ (C) \leq \Gamma\mathrm{Aut}\ (C)$$

Curve	Condition	G
g=3		
$y^2+y=\alpha_1 x+\alpha_2 x^{-1}$ $+\alpha_3(x-1)^{-1}+\alpha_4(x-\lambda)^{-1}$	$\alpha_1=\alpha_2\lambda^{-1}, \alpha_3=\alpha_4\lambda^{-1}, \alpha_1\neq\alpha_3\lambda$	V_4
	$\alpha_1=\alpha_3\lambda,\ \alpha_2=\alpha_4\lambda,\ \alpha_1\neq\alpha_2\lambda^{-1}$	V_4
	$\alpha_1=\alpha_4,\ \alpha_2=\alpha_3\lambda^2,\ \alpha_1\neq\alpha_2\lambda^{-1}$	V_4
	$\alpha_1=\alpha_2\lambda^{-1}, \alpha_3=\alpha_4\lambda^{-1}, \alpha_1=\alpha_3\lambda$	${C_2}^3$
$y^2+y=x^3+\alpha x$ $+\beta x^{-1}+\gamma(x-1)^{-1}$	$\alpha\neq 0$, or $\beta\neq\gamma$	C_2
	$\alpha=0$, and $\beta=\gamma$	V_4
$y^2+y=x^3+\alpha x$ $+x^{-3}+\beta x^{-1}$	none	C_2
	$\beta\neq 1, \alpha=\gamma=0$	C_6
	$\beta=1, \alpha=\gamma\neq 0$	V_4
	$\beta=1, \alpha=\gamma\zeta\neq 0$	V_4
	$\beta=1, \alpha=\gamma\zeta^2\neq 0$	V_4
	$\beta=1, \alpha=\gamma=0$	D_{12}
$y^2+y=x^5+\alpha x^3+\beta x^{-1}$	none	C_2
$y^2+y=x^7+\alpha x^5+\beta x^3$	$\alpha=\beta=0$	C_{14}
	$\alpha=0, \beta\neq 0$	C_2
	$\alpha\neq 0, \beta=c_3=0$	C_2
	$\alpha\neq 0, \beta=0, c_3\neq 0$	C_{14}
	$\alpha\neq 0, \beta\neq 0, c_3=0$	C_2
	$\alpha\neq 0, \beta\neq 0, c_3\neq 0$	C_{14}
g =4		
$y^2+y=\alpha_1 x+\alpha_2 x^{-1}+\alpha_3(x-1)^{-1}$ $+\alpha_4(x-\lambda_1)^{-1}+\alpha_5(x-\lambda_2)^{-1}$	$\alpha_2=\alpha_3, \alpha_4=\alpha_5, \alpha_2\neq\alpha_4$	V_4
	$\alpha_2=\alpha_4, \alpha_3=\alpha_5, \alpha_2\neq\alpha_3$	V_4
	$\alpha_2=\alpha_5, \alpha_3=\alpha_4, \alpha_2\neq\alpha_3$	V_4
	$\alpha_2=\alpha_3=\alpha_4=\alpha_5$	${C_2}^3$
	$\alpha_1=\alpha_2=\alpha_3\lambda=\alpha_4\lambda=\alpha_5$	D_{20}
	$\alpha_1=\alpha_2=\alpha_3\lambda^{-1}=\alpha_4=\alpha_5\lambda^{-1}$	D_{20}
	$\dots$	$\dots$
	$\dots$	$\dots$
$y^2+y=x^3+\alpha x+\beta_1 x^{-1}$ $+\beta_2(x-1)^{-1}+\beta_3(x-\lambda)^{-1}$	$\alpha\neq 0$	C_2
	$\alpha=0, \beta=\gamma\lambda, \gamma=\sigma\lambda, \sigma=\beta\lambda$	C_6
$y^2+y=x^3+\alpha x$ $+x^{-3}+\beta x^{-1}+\gamma(x-1)^{-1}$	none	C_2
	$\beta=1, \alpha=\gamma$	V_4
$y^2+y=x^5+\alpha x^3$ $+\beta x^{-1}+\gamma(x-1)^{-1}$	$\alpha\neq 0, or 1$	C_2
	$\alpha=0$	V_4
	$\alpha=1$	C_4
$y^2+y=x^5+\alpha x^3+x^{-3}+\beta x^{-1}$	none	C_2
$y^2+y=x^7+\alpha x^5+\beta x^3+\gamma x^{-1}$	none	C_2
$y^2+y=x^9+\alpha_1 x^7+\alpha_2 x^5+\alpha_3 x^3$	$\alpha_1\neq 0$	C_2
	$\alpha_1=0$	C_{18}

TABLE 1. Automorphism groups of hyperelliptic curves of genus 3 and 4 over fields of characteristic 2

Recall that for binary codes PAut (C) = MAut (C) = ΓAut (C), which we simply denote by Aut (C). If the code C is defined over a prime field then MAut (C) = ΓAut (C). Two codes C and C' are called **permutation equivalent**, **monomially equivalent**, or **equivalent** if there is an element σ in the respective automorphism group such that $\sigma(C) = C'$. In classical coding theory these automorphism groups of codes play an important role in classifying codes. There is a weight preserving linear transformation between $[n, k]$ codes C and C' over $\mathbb{F}_q$ if and only if $C$and C' are monomially equivalent. Furthermore, the linear transformation agrees with the associated monomial transformation on every codeword in C; see [**?**, Thm. 7.9.4].

If $\mathcal{X}$ is a genus $g \geq 2$ algebraic curve defined over $\mathbb{F}_q$ then Aut $(\mathcal{X})$ the group of automorphisms of $\mathcal{X}$ over the algebraic closure of $\mathbb{F}_q$. There have been many papers studying the relation between the automorphism group of the algebraic curve $\mathcal{X}$ and the automorphism groups as defined above of the corresponding AG-code $C_\mathcal{X}$; see [11] among others. Let us assume that $C_\mathcal{X}$ is a self-orthogonal code such that we can construct a quantum code $Q_\mathcal{X}$ as in the previous section. If Q is a symplectic quantum code then the group of equivalences of the code is the complex Clifford group.

5.3. **Some computational remarks on the automorphism groups of codes.** In this section we want to make a few remarks on the efficiency of computing the automorphism group of a given code. There are several open questions related to automorphism groups of algebraic curves, AG-codes, and naturally quantum codes. We suggest some problems and point some inefficiencies on some existing programs.

Problem 1. Let $\mathcal{X}$ be a genus g curve defined over a finite field F_q. Determine the list of groups that occur as full groups of automorphisms of $\mathcal{X}$ over the algebraic closure of $\mathbb{F}_q$.

Problem 2. Let $\mathcal{X}$ be a genus g curve defined over a finite field $\mathbb{F}_q$. Design and implement a program that computes the automorphism group of $\mathcal{X}$ over $\mathbb{F}_q$.

Let $C_\mathcal{X}$ and $Q_\mathcal{X}$ be the codes constructed as in sections 2 and 3. In GAP, the package GUAVA which is specifically written for coding theory, creates such codes (with some simple implementations of our algorithms) and computes groups of such codes using an algorithm of Leon. Similar capabilities are available also in Magma. Both MAGMA and GAP come short when it comes to computing the automorphism group of a code over a relatively large size field $\mathbb{F}_q$. Magma only computes automorphism groups of codes over a field $\mathbb{F}_q$ where $q = p$ or p^2.

Problem 3. Design and implement an algorithm which computes the automorphism groups PAut (C), MAut (C), ΓAut (C) of a given code C (including quantum codes) over any field $\mathbb{F}_q$.

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ASYMPTOTICALLY OPTIMAL TESTS WHEN PARAMETERS ARE ESTIMATED

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Abstract. The main purpose of this paper is to provide an asymptotically optimal test. The proposed statistic is of Neyman-Pearson-type when the parameters are estimated with a particular kind of estimators. It is shown that the proposed estimators enables us to achieve this end. Two particular cases, $AR(1)$ and $ARCH$ models were studied and the asymptotic power function was derived.

Introduction

Local asymptotic normality LAN for the log likelihood ratio was studied for a several classes of nonlinear time series model, from a LAN the contiguity property follows, for more details the interested reader may refer to [2], [11], and [4]. Applying the contiguity property, we construct a statistic for testing a null hypothesis H_0 against the alternative hypothesis $H_1^{(n)}$, often a various classical test statistics depends on the central sequence which appears in the expression of the log likelihood ratio. In the case when the parameter of the time series model is known we obtain good properties of the test, precisely, the optimality, see for instance [8, Theorem 3].

However, in a general case, particularly in practice, the parameter is unspecified, in the expression of the estimated central sequence appears an additional term which is non degenerate asymptotically. The latter, alters the power function of the constructed test.

In order to solve this very problem, and on a basis of an estimator of the unknown parameter, we introduce and define another estimator which does not effects asymptotically the power function of the test, more precisely the additional term is absorbed. The principle of this construction is to modify one component of the

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first estimator in order to avoid the additional term, the details of this method are expanded further in the section (1).
The main purpose of this paper is to investigate the problem of testing two hypothesis corresponding to a stochastic model which is described in the following way. Let $\{(Y_i, X_i)\}$ be a sequence of stationary and ergodic random vectors with finite second moment such that for all $i \in \mathbb{Z}$, where Y_i is a univariate random variable and X_i is a d-variate random vector.
We consider the class of stochastic models

$$Y_i = T(Z_i) + V(Z_i)\,\epsilon_i, \quad i \in \mathbb{Z}, \tag{0.1}$$

where, for given non negative integers q and s, the random vectors Z_i is equal to $(Y_{i-1}, Y_{i-2}, \ldots, Y_{i-s}, X_i, X_{i-1}, \ldots, X_{i-q})$, the ϵ_i's are centered i.i.d. random variables with unit variance and density function $f(\cdot)$, such that for each $i \in \mathbb{Z}$, ϵ_i is independent of the filtration $\mathcal{F}_i = \sigma(Z_j, j \le i)$, the real-valued functions $T(\cdot)$ and $V(\cdot)$ are assumed to be unknown. We consider the problem of testing whether the bivariate vector of functions $(T(\cdot), V(\cdot))$ belongs to a given class of parametric functions or not. More precisely, let

$$\mathcal{M} = \{(m(\rho, \cdot), \sigma(\theta, \cdot))\,,\ (\rho', \theta')' \in \Theta_1 \times \Theta_2\}\,,$$

$\Theta_1 \times \Theta_2 \subset \mathbb{R}^\ell \times \mathbb{R}^p$, $\mathring{\Theta}_1 \neq \emptyset$, $\mathring{\Theta}_2 \neq \emptyset$, where for all set A, $\mathring{A}$ denotes the interior of the set A and the script " $'$ " denotes the transpose, ℓ and p are two positive integers, and each one of the two functions $m(\rho, \cdot)$ and $\sigma(\theta, \cdot)$ has a known form such that $\sigma(\theta, \cdot) > 0$. For a sample of size n, we derive a test of

$$H_0 : [(T(\cdot), V(\cdot)) \in \mathcal{M}] \text{ against } H_1 : [(T(\cdot), V(\cdot)) \notin \mathcal{M}]\,. \tag{0.2}$$

It is easy to see that the null hypothesis H_0 is equivalent to

$$H_0 : [(T(\cdot), V(\cdot)] \quad = \quad \Big(m(\rho_0, \cdot), \sigma(\theta_0, \cdot)\Big), \tag{0.3}$$

while the alternative hypothesis H_1 is equivalent to

$$H_1 : [(T(\cdot), V(\cdot)] \quad \neq \quad \Big(m(\rho_0, \cdot), \sigma(\theta_0, \cdot)\Big),$$

for some $(\rho_0', \theta_0')' \in \Theta_1 \times \Theta_2$.
In the sequel, our study will be focused on the following alternative hypotheses. For all integers $n \ge 1$, the alternative hypothesis $H_1^{(n)}$ is defined by the following equation

$$H_1^{(n)} : [(T(\cdot), V(\cdot)] \quad = \quad \Big(m(\rho_0, \cdot) + n^{-\frac{1}{2}} G(\cdot), \sigma(\theta_0, \cdot) + n^{-\frac{1}{2}} S(\cdot)\Big), \tag{0.4}$$

where $G(\cdot)$ and $S(\cdot)$ are two specified real functions. The situation is different in the case when the used statistic is the Neyman-Pearson test which is based on the log-likelihood ratio Λ_n defined as follows

$$\Lambda_n = \log\left(\frac{f_n}{f_{n,0}}\right) = \sum_{i=1}^{n} \log(g_{n,i}), \tag{0.5}$$

where $f_{n,0}(\cdot)$ and $f_n(\cdot)$ denote the probability densities of the random vector $(Y_1, \ldots, Y_n)$ corresponding to the null hypothesis and the alternative hypothesis, respectively.

The use of the Neyman-Pearson statistics needs to resort to the following conditions:
Under the hypothesis H_0, there exists a random variable $\mathcal{W}_n$ such that

$$\mathcal{W}_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tau^2),$$

where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution and some constant $\tau > 0$ depending on the parameter $\phi_0 = (\rho_0', \theta_0')'$, such that

$$\Lambda_n = \mathcal{W}_n(\phi_0) - \frac{\tau^2(\phi_0)}{2} + o_P(1). \tag{0.6}$$

The equality (0.6) is a modified version of the LAN given by [8, Theorem 1]. We mention that there exist other versions of the LAN, we may refer to [7], [10] and the references therein. On the basis of the LAN, an efficient test of linearity based on Neyman-Pearson-type statistics was obtained in a class of nonlinear time series models contiguous to a first-order autoregressive process $AR(1)$ and its asymptotic power function is derived, see for instance
([8, Theorem 1 and Theorem 3]). Note that this proposed test is given by this equality:

$$T_n \quad = \quad I\left\{\frac{\mathcal{W}_n(\rho_0)}{\tau(\rho_0)} \geq Z(\alpha)\right\}, \tag{0.7}$$

where $Z(\alpha)$ is the (1 -α)-quantile of a standard normal distribution $\Phi(\cdot)$.
The expression of the obtained test depends on the central sequence $\mathcal{W}_n(\phi_0)$ which itself depends on the parameter ϕ_0. In a general case the parameter ϕ_0 is unspecified, so, in order to estimate it, we need to present of a coherent methodology for the estimation of the parameters of mathematical models. An experimental data will be examined in the end.
More precisely, under some assumptions, we define and introduce an estimator preserving, asymptotically, the power on Neyman-Pearson test when we replace, in the expression of the statistics, the parameter ϕ_0 by an appropriate estimator, $\bar{\phi}_n$. Say, this estimator will be constructed on the tangent space with the direction of the partial derivatives of the central sequences in $\hat{\phi}_n$, where $\hat{\phi}_n$ is a $\sqrt{n}$-consistent estimator of ϕ_0. In the sequel, $\bar{\phi}_n$ will be called a *modified estimator* M.E..
This paper describes a method to estimate parametric models and consists of two parts essentially:
The first part corresponds to the introducing of a new estimator (Modified estimator) of the unknown parameter of the time series model.More precisely, if $\hat{\phi}_n$ is a consistency estimator of unknown parameter ϕ, and $\mathcal{W}_n$ a real random variable defined on the set $\Theta_1 \times \Theta_2$ such that the following condition is satisfied:

$$\mathcal{W}_n(\hat{\phi_n}) \quad = \quad \mathcal{W}_n(\phi_0) - D_n + o_P(1),$$

where D_n is a specified bounded random function. Then, we shall construct another estimator $\bar{\phi_n}$ of the parameter ϕ_0 which absorb the error corresponding to the function D_n.The proprieties of this estimator are expanded further in the section (1).
The second part corresponds to the applying of this new estimator (Modified estimator) in the problem of the test. In this case the random variable $\mathcal{W}_n$ is equal to the central sequence $\mathcal{V}_n$ which appears in the LAN version defined by the equality (0.6). Under some assumptions, the optimality of the constructed test is obtained

and its asymptotic power function is derived.
This paper is organized as follows:
Section (1) is devoted for the estimation. In Subsection (1.1), we describe the methodology used to construct the M.E.. In Subsection (1.2), we give the asymptotic properties of the proposed estimator.
In Section (2), we applied the modified estimator in the problem of testing. Section (3) treats specially the problem of testing in $AR(1)$ in two cases.
In Section (4), we conduct a simulation in order to evaluate the power of the proposed test in $AR(1)$ model.
All mathematical developments are relegated to Section (5).

1. Estimation

Large sample theory of estimation is developed. Attention was confined to parametric model. Other problems having a link with the considered problem depend on the used estimator. More precisely, random functions based on the unspecified parameter of the model appear. The replacing of this unknown parameter by its estimator induces an additional term. This latter is not asymptotically degenerate, the performance of the considered study problem is effected.
For instance, if we consider the problem of testing, the most of the classical statistics tests are based on the central sequence which appears in the expression of the established local asymptotic normality of the log-likelihood ratio. In this case the random function corresponds to the central sequence. By replacing the unknown parameter by its estimator, an additional term appears in the expression of the estimated central sequence. Therefore, the power of the constructed test is effected.
Our goal in this paper is to treat this problem in a general case. In order to avoid this additional term, we develop under some assumptions a method for constructing another estimator. The principle of this construction is to absorb this additional term asymptotically.
Next we discuss estimation and testing for $AR(1)$ with an extension to $ARCH$ models and note that these models lead to some interesting and particular problems.

In Subsection (1.1), we give under some assumptions, the methodology of constructing this estimator.

In Subsection (1.2), we expand further the problem of the consistency of our constructed estimator. Under some assumptions, the consistency of the modified estimator is established.
Throughout, $\hat{\phi}_n = (\hat{\rho}_n', \hat{\theta}_n')'$ a $\sqrt{n}$-consistent estimator of the parameter $\phi_0 = (\rho_0', \theta_0')'$, where

$$\begin{aligned} \hat{\rho}_n' &= \Big(\hat{\rho}_{n,1}, \ldots, \hat{\rho}_{n,\ell}\Big), \quad \hat{\theta}_n' = \Big(\hat{\theta}_{n,1}, \ldots, \hat{\theta}_{n,p}\Big), \\ {\rho_0}' &= \Big(\rho_1, \ldots, \rho_\ell\Big) \quad \text{and} \quad {\theta_0}' = \Big(\theta_1, \ldots, \theta_p\Big). \end{aligned}$$

D_n and $\mathcal{W}_n$ are the additional term and a real random function respectively.

1.1. Estimation with modifying one component.

Our purpose is to construct another estimator $\bar{\phi}_n'$ of the parameter $(\rho_0'; \theta_0')'$, such that the following fundamental equality is fulfilled

$$\mathcal{W}_n(\bar{\phi}_n) - \mathcal{W}_n(\hat{\phi}_n) = D_n, \tag{1.1}$$

where D_n is a specified bounded random function.
Our goal, is to find an estimator $\bar{\phi}_n$ satisfying (1.1) pertaining to the tangent space Γ_n, such that, for $(X', Y')' \in \mathbb{R}^\ell \times \mathbb{R}^p$, the following equation holds

$$\Gamma_n : \mathcal{W}_n((X,Y)) - \mathcal{W}_n(\hat{\phi}_n) \quad = \quad \partial \mathcal{W}'_n(\hat{\phi}_n).\Big((X-\hat{\rho}_n)', (Y-\hat{\theta}_n)'\Big)',$$

where

$$\partial \mathcal{W}_n(\hat{\phi}_n)' \quad = \quad \Big(\frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_1}, \ldots, \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_\ell}, \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \theta_1}, \ldots, \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \theta_p}\Big),$$

and the script " $\cdot$ " denotes the inner product.
With the connection with the equality (1.1), the new estimator is then given by imposing that the value $(X', Y')'$ satisfied the following identity

$$D_n \quad = \quad \partial \mathcal{W}_n(\hat{\phi}_n)'.\Big((X-\hat{\rho}_n)', (Y-\hat{\theta}_n)'\Big)'. \tag{1.2}$$

Clearly, the equation (1.2) has $\ell+p$ unknown values, so it has an infinity of solutions, after modification of the j_n-th component of the first estimator $\hat{\rho}_n$, we shall propose an element in tangent space Γ_n which satisfies the equality (1.2). We obtain then a new estimator $\bar{\phi}'_n = \phi_n^{(1,j_n)'} = (\bar{\rho}'_n, \hat{\theta}'_n)'$ of the unknown parameter ϕ_0, where

$$\bar{\rho_n}' \quad = \quad \Big(\bar{\rho}_{n,1}, \ldots, \bar{\rho}_{n,\ell}\Big),$$

and such that: for s $\in \{1, \ldots, \ell\}$, $\bar{\rho}_{n,s} = \hat{\rho}_{n,s}$ if $s \neq j_n$ and $\bar{\rho}_{n,j_n} \neq \hat{\rho}_{n,j_n}$.
The use of the notation $\phi_n^{(1,j_n)}$ explains that we obtain the new estimator $\bar{\phi}_n$ of the parameter ϕ_0 when we change in the expression of the estimator $\hat{\phi}_n$ the j_n component with respect to the first estimator $\hat{\rho}_n$ corresponding to the step n of the estimation. It follows from the equality (1.1) combined with the constraint (1.2) that

$$\begin{aligned}
\mathcal{W}_n(\phi_n^{(1,j_n)}) - \mathcal{W}_n(\hat{\phi}_n) \quad &= \quad \sum_{s=1}^{\ell} \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_s}(\bar{\rho}_{n,s} - \hat{\rho}_{n,s}) + \sum_{t=1}^{p} \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \theta_t}(\bar{\theta}_{n,t} - \hat{\theta}_{n,t}), \\
&= \quad \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}(\bar{\rho}_{n,j_n} - \hat{\rho}_{n,j_n}). \qquad (1.3)
\end{aligned}$$

By imposing the following condition

$$\frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}} \quad \neq \quad 0, \tag{1.4}$$

and with the use of the equality (1.2) combined with (1.4), we deduce that

$$\bar{\rho}_{n,j_n} \quad = \quad \frac{D_n}{\frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}} + \hat{\rho}_{n,j_n}. \tag{1.5}$$

In summary, we define the modified estimator by

$$\bar{\phi}'_n = \phi_n^{(1,j_n)'} = \Big(\hat{\rho}_{n,1}, \ldots, \hat{\rho}_{n,j_n-1}, \bar{\rho}_{n,j_n}, \hat{\rho}_{n,j_n+1}, \ldots, \hat{\rho}_{n,\ell}, \hat{\theta}_{n,1}, \ldots, \hat{\theta}_{n,p}\Big)'.$$

With a same reasoning as the previous case and after modifying the k_n-th component with respect to the second estimator, we shall define a new estimator

$$\bar{\phi_n}' = \phi_n^{(2,k_n)'} = (\hat{\rho}'_n, \bar{\theta}'_n)',$$

such that for $t \in \{1,\dots,p\}$, $\bar{\theta}_{n,t} = \hat{\theta}_{n,t}$ if $t \neq k_n$ and $\bar{\theta}_{n,k_n} \neq \hat{\theta}_{n,k_n}$. We obtain

$$\mathcal{W}_n(\phi_n^{(2,k_n)}) - \mathcal{W}_n(\hat{\phi}_n) = \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \theta_{k_n}}(\bar{\theta}_{n,k_n} - \hat{\theta}_{n,k_n}). \tag{1.6}$$

Under the following condition

$$\frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \theta_{k_n}} \neq 0, \tag{1.7}$$

it follows from the equality (1.2) combined with (1.7), that

$$\bar{\theta}_{n,k_n} = \frac{D_n}{\frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \theta_{k_n}}} + \hat{\theta}_{n,k_n}. \tag{1.8}$$

In summary, we obtain the modified estimator

$$\bar{\phi}_n' = \phi_n^{(2,k_n)\prime} = \Big(\hat{\rho}_{n,1},\dots,\hat{\rho}_{n,\ell},\hat{\theta}_{n,1},\dots,\hat{\theta}_{n,k_n-1},\bar{\theta}_{n,k_n},\hat{\theta}_{n,k_n+1},\dots,\hat{\theta}_{n,p}\Big)'.$$

The estimator $\phi_n^{(1,j_n)}$ (respectively, $\phi_n^{(2,k_n)}$) is called a modified estimator in j_n-th component with respect to the first estimator (respectively, in k_n-th component with respect to second estimator). We denote this estimator by M.E..

Remark 1.1. For each step n of the estimation corresponding a value of the position j_n or k_n of the component where the estimator was modified.

1.2. **Consistency.** Throughout, $\hat{\phi}_n$ is a $\sqrt{n}$-consistent estimator of the unknown parameter ϕ_0. The conditions (1.4) and (1.7) are not sufficient to get the consistency of the modified estimator M.E.. In order to get its consistency, we need to resort to one of the following additional conditions :

(C.1):

$$\frac{1}{\sqrt{n}}\frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}} \xrightarrow{P} c_1 \quad \text{as} \quad n \to \infty,$$

(C.2):

$$\frac{1}{\sqrt{n}}\frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \theta_{k_n}} \xrightarrow{P} c_2 \quad \text{as} \quad n \to \infty,$$

where c_1 and c_2 are two constants, such that $c_1 \neq 0$ and $c_2 \neq 0$.

Note that $\xrightarrow{P}$ denotes the convergence in probability.

Our first result concerning the consistency of the proposed estimator is summarized in the following proposition.

Proposition 1.1. *Under (1.4) and $(C.1)$ (or (1.7) and $(C.2)$), the estimator $\phi_n^{(1,j_n)}$ (or $\phi_n^{(2,k_n)}$,) is a $\sqrt{n}$-consistent estimator of the unknown parameter ϕ_0.*

Remark 1.2. In practice, it is not easy to verify the condition $(C.1)$ (or $(C.2)$). In the case when the unknown parameter ϕ_0 is univariate, a sufficient condition will be stated in Lemma (1.3). In this case, we need the following assumption:

(C.3): : For all real sequence $(\eta_n)_{n\geq 1}$ with values in the interval $[0,1]$, we have:

$$\frac{1}{\sqrt{n}}\ddot{\mathcal{W}}_n(\eta_n\phi_0 + (1-\eta_n)\hat{\phi}_n)) = O_P(1),$$

where $\ddot{\mathcal{W}}_n$ is a second derivative of $\mathcal{W}_n$.

Now, we may state the sufficient condition which implies assumptions $(C.1)$ corresponding to the case when the parameter of the time series model is univariate.

Lemma 1.3. *Let $\hat{\phi}_n$ be a $\sqrt{n}$-consistent estimator of the parameter ϕ_0. Let c_1 be a constant, such that $c_1 \neq 0$, then we have:*

(i) *Under $(C.3)$, if $\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\phi_0) \xrightarrow{P} c_1$, as $n \to \infty$, then $\forall A > 0$,*

$$P\left(\left|\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\hat{\phi}_n) - c_1\right| > A\right) \to 0, \text{ as } n \to \infty.$$

Remark 1.4. Consequently, with the applying of the modified estimator and in the case when the error between two central sequences is bounded. Under some assumptions, it is possible to absorb this error. This result is stated and proved in the following proposition.

Proposition 1.2. *Let $\hat{\phi}_n'$ be an estimator ($\sqrt{n}$ consitency) of the unknown parameter $(\rho', \theta')'$. We assume that there exists a known bounded function D_n, such that*

$$\mathcal{W}_n(\hat{\phi}_n) = \mathcal{W}_n(\phi_0) - D_n + o_P(1). \tag{1.9}$$

Then, there exists an estimator $\bar{\phi_n}'$ of $(\rho', \theta')'$ such that

$$\mathcal{W}_n(\bar{\phi_n}) = \mathcal{W}_n(\phi_0) + o_P(1).$$

Remark 1.5. These previous results will be used in the problem of testing. If we consider the problem of testing of the null hypothesis H_0 against the alternative hypothesis H_1^n corresponding to the equalities (0.3) and (0.4), and when the local asymptotic normality of log likelihood ratio corresponding to the equality (0.6) is established. It is possible to construct an optimal test.

2. Testing in parametrical model

The literature on specification testing in parametric model is vast. The goal is to obtain a test that is consistent. This paper provides a general framework for constructing specification tests for parametric models.
Now, we are ready to apply the results obtained in the section (1) in the testing problem. More precisely, under some assumptions, it is possible to preserve asymptotically the power function of the constructed test when we replace the unknown parameter by the modified estimator. Consequently the optimality of the test is proved and the power function is derived.
Consider again the problem of testing the hypothesis H_0 and H_1^n corresponding to the equalities (0.3) and (0.4) respectively. In the sequel the functions $(\rho, \cdot) \longrightarrow m(\rho, \cdot)$ and $(\theta, \cdot) \longrightarrow \sigma(\theta, \cdot)$ are assumed to be twice differentiable on the sets Θ_1 and Θ_2 respectively.
Throughout, we assume that the function $f(\cdot)$ is positive with a third derivative, we denote by $\dot{f}(\cdot)$, $\ddot{f}(\cdot)$ and $f^{(3)}(\cdot)$ the first, the second and the third derivative respectively. For all $x \in \mathbb{R}$, let

$$M_f(x) = \frac{\dot{f}(x)}{f(x)}.$$

For the considered problem of testing, we use the classical large sample of Neyman-Pearson. On the basis on the results of the section (1), under some assumptions, we shall prove that, asymtotically the power function of the constructed test is no effected with the replacing of the unknown parameter by the modified estimator. This latter is stated in the next theorem:

2.1. **Optimality of the proposed test.** Throughout, $\bar{T}_n$ and $\bar{\tau}$ are the statistics test and the constant respectively obtained with the subsisting of the unspecified parameter ϕ_0 by its modified estimator $\bar{\phi}_n$ in the expression of the test (0.7) and the constant τ appearing in the expression of the log likelihood ratio (0.6) respectively. We assume in the problem of testing the two hypothesis H_0 against $H_1^{(n)}$ that the LAN of the the model (0.1) is established, in order to prove the optimality of the proposed test. To this end, we need the following assumption:

(E.1): There exists a $\sqrt{n}$-estimator $\hat{\phi}_n$ of the unknown parameter ϕ_0 and a random bounded function D_n, such that

$$\mathcal{V}_n(\hat{\phi}_n) \quad = \quad \mathcal{V}_n(\phi_0) - D_n + o_P(1).$$

Note in this case that, the random variable $\mathcal{V}_n$ corresponds to the central sequence $\mathcal{W}_n$ which appears in the expression of the log likelihood ratio (0.6).

It is now obvious from the previous definitions that we can state the following theorem:

Theorem 2.1. *Under LAN and the conditions (1.4) (respectively, (1.7)),* $(C.1)$ *(*$(C.2)$*, respectively) and* $(E.1)$ *the asymptotic power of* $\bar{T}_n$ *under* H_1^n *is equal to to* $1 - \Phi(Z(\alpha) - \bar{\tau}^2)$. *Furthermore,* $\bar{T}_n$ *is asymptotically optimal.*

Remark 2.2. In practice, the use of the condition $(E.1)$ requires the specification of the random variable D_n. In this way, we specify this random variable in the problem of testing corresponding to the $AR(1)$ model. This specially case is expanded further in the next section.

3. Testing in $AR(1)$ model

In this work, we treat specially the problem of testing for the $AR(1)$ model in two cases: firstly, we study in the Subsection (3.1) the case when the sequence of nonlinear model is contiguous to $AR(1)$, and secondly, we discuss in the Subsection(3.2) the extension to autoregressive conditionally heteroscedastic contiguous alternative models to $AR(1)$.
In the aim to achieve this, we need, firstly to establish the local asymptotic normality for the log likelihood ratio, secondly, to specify the random function D_n, and thirdly to construct a modified estimator and an optimal test. Therefore, we require some results and assumptions for these two problems of testing.
Throughout, the scripts "$\| \cdot \|_{\ell+p}$ " , "$\| \cdot \|_{\ell}$ " and "$\| \cdot \|_{p}$ " denote the euclidian norms in $\mathbb{R}^{\ell+p}$, $\mathbb{R}^{\ell}$ and $\mathbb{R}^{p}$ respectively.

3.1. **Nonlinear time series contiguous to $AR(1)$ processes.** Consider the s-th order (nonlinear) time series

$$Y_i = \rho_0 Y_{i-1} + \alpha\, G(Y(i-1)) + \epsilon_i, \quad |\rho_0| < 1. \tag{3.1}$$

In this case and with the comparison to the equality (0.1), we have

$$Z_i = Y_i \quad , \quad T(Z_i) = \rho_0 Y_{i-1} + \alpha\, G(Y(i-1)) \quad \text{and} \quad V(Z_i) = 1.$$

In the sequel, it will be assumed that the model is a stationary and ergodic time series with finite second moment.
Consider the problem of testing the null hypothesis $H_0 : \alpha = 0$ against the alternative hypothesis $H_1^{(n)} : \alpha = n^{-\frac{1}{2}}$. With the comparison to (0.3) and (0.4), we have the following equalities:

$$\begin{aligned}\Big(m(\rho_0, Y_{i-1}), \sigma(\theta_0, Y_{i-1})\Big)' &= \Big(\rho_0\, Y_{i-1}, 1\Big)', \quad \mathcal{M} = \{m(\rho, \cdot), \rho \in \Theta_1\}\,, \\ Z_i{}' &= \Big(Y_{i-1}, \ldots, Y_{i-s}\Big) \quad \text{and} \quad S(\cdot) = 0.\end{aligned}$$

Note that this problem of testing is equivalent to test the linearity of the s-th $AR(1)$ time series model ($\alpha = 0$) against the nonlinearity of the s-th $AR(1)$ time series model ($\alpha = n^{-\frac{1}{2}}$).
In order to study this problem, we require some assumptions and results. We suppose that the following conditions are satisfied:

- (A.1): There exists positive constants η and c such that for all u with $\|u\|_{\ell+p} > \eta$, $G(u) \leq c\|u\|_{\ell+p}$.
- (A.2): for a location family $\{f(\epsilon_i - c),\ -\infty < c < -\infty\}$, there exist a square integrable functions Ψ_1, Ψ_2 and a constant δ such that for all ϵ_i and $|c| < \delta$, such that :

$$\Big|\frac{d^k f(\epsilon_i - c)}{f(\epsilon_i)\, dc^k}\Big| \leq \Psi_k(\epsilon_i), \quad \text{for} \quad k = 1, 2.$$

We begin by processing the propriety of the local asymptotic normality, then we have:

3.1.1. *Local asymptotic normality.* To aim to establish the local asymptotic normality for the the local asymptotic normality LAN and according to the equality (0.5), we require that the following conditions are satisfied under H_0:

- (L.1): $\max_{1\leq i\leq n} |g_{n,i} - 1| = o_P(1)$,
- (L.2): there exists a positive constante τ^2 such that $\sum_{i=1}^n (g_{n,i} - 1)^2 = \tau^2 + o_P(1)$,
- (L.3): there exists a-$\mathcal{F}_n$ mesurable $\mathcal{V}_n$ satisfying $\sum_{i=1}^n (g_{n,i} - 1) = \mathcal{V}_n + o_P(1)$, where $\mathcal{V}_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tau^2)$.

(L.1), (L.2) and (L.1) imply under H_0, the local asymptotic normality LAN for the log likelihood ratio corresponding to this problem of testing is established . This version of LAN is given by the following equality:

$$\Lambda_n = \mathcal{V}_n(\phi_0) - \frac{\tau^2(\phi_0)}{2} + o_P(1), \quad \text{with} \quad \mathcal{V}_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tau^2).$$

Fore more details, refer to ([8, Theorem 1]).
One consequence of the applying of the ([8, Theorem 1]), that, under H_0, (A.1) and (A.2) imply the local asymptotic normality LAN for the log likelihood. More precisely, we have:

$$\Lambda_n = \mathcal{V}_n(\phi_0) - \frac{\tau^2(\phi_0)}{2} + o_P(1), \quad \text{with} \quad \mathcal{V}_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tau^2),$$

$$\mathcal{V}_n(\rho_0) = -\frac{1}{\sqrt{n}} \sum_{i=1}^n M_f(\epsilon_i) G(Y(i-1)), \quad \text{and} \quad \tau^2 = \mathbf{E}(M_f^2(\epsilon_0))\mathbf{E}(G^2(Y(0))).$$

For more details, see ([8, Theorem 2]).

3.1.2. *The considering test.* According to the notation and results of the previous Subsection, under the conditions (A.1) and (A.2), the proposed test T_n is the Neyman-Pearson statistic which is given by the following equality

$$T_n \quad = \quad I\left\{\frac{\mathcal{V}_n(\rho_0)}{\tau(\rho_0)} \geq Z(\alpha)\right\}.$$

The asymptotic power of the test is derived and equal to $1 - \Phi(Z(\alpha) - \tau^2)$. Recall that when ρ_0 is known, we obtain an efficiency test, for more details see [8, Theorem 3].
To achieve this problem of testing, it remains to specify the random function D_n, the method is developed in the next subsection.

3.1.3. *Specification of the random variable D_n.* Our aim is to specify the form of the function D_n which is defined in (1.9).
In the sequel, the parameter ρ_0 is estimated by the $\sqrt{n}$-consistent estimator $\hat{\rho}_n$ and the residual ϵ_i is estimated by $\hat{\epsilon}_{i,n} = Y_i - Y_{i-1}\hat{\rho}_n$. We have the following statement:

Proposition 3.1. *Assume that, under H_0, the conditions (A.1) and (A.2) hold and ϵ_i's are centered i.i.d. and $\epsilon_0 \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$. We have*

$$\mathcal{V}(\hat{\rho}_n) = \mathcal{V}_n(\rho_0) - D_n + o_P(1), \tag{3.2}$$

where

$$D_n = -c_1\sqrt{n}(\hat{\rho}_n - \rho_0), \text{ and } c_1 = -\mathbb{E}\Big[Y_0 G(Y(0))\Big]. \tag{3.3}$$

3.1.4. *Modified estimator and optimal test.* Under the conditions of the Proposition (3.1), the modified estimator and an optimal test are given by the following proposition

Proposition 3.2. *The modified estimator is given by the equality*

$$\bar{\rho_n} = \frac{D_n}{\dot{\mathcal{V}}_n(\phi_n)} + \hat{\rho}_n,$$

and the statistic test is given by

$$\bar{T_n} \quad = \quad I\left\{\frac{\mathcal{V}_n(\bar{\rho_n})}{\bar{\tau}} \geq Z(\alpha)\right\}.$$

Remark 3.1.
- The use of the ergodicity of the model imposes to require the condition $\mathbb{E}\Big[Y_{-1}G(Y_0)\Big] < \infty$, therefore we choose the function $G(\cdot)$ in order to get this condition. For instance, we shall choose $G(Y(i-1)) = \frac{2a}{1+Y_{i-1}^2}$, where $a \neq 0$.
- With this choice of the function G, the condition $(A.1)$ remains satisfied, in fact, we can remark that $|G(u)| \leq 2|a|$, then for all u with $\|u\|_{\ell+p} \geq \eta$ we have $G(u) \leq 2a \times \|u\|_{\ell+p} \times \frac{1}{\|u\|_{\ell+p}} \leq \frac{2a}{\eta} \times \|u\|_{\ell+p}$, therefore, we shall choose $c = \frac{2a}{\eta}$.

3.2. **An extension to $ARCH$ processes.** Consider the following time series model with conditional heteroscedasticity

$$(3.4)\qquad Y_i = \rho_0 Y_{i-1} + \alpha\, G(Y(i-1)) + \sqrt{1+\beta B(Y(i-1))}\,\epsilon_i, \quad i \in \mathbb{Z}.$$

We consider the problem of testing the null hypothesis H_0 against the alternative hypothesis $H_1^{(n)}$ such that

$$\begin{aligned} H_0 \quad &: \quad m(\rho, Z_i) = \rho_0 Y_{i-1} \quad \text{and} \quad \sigma(\theta_0, \cdot) = 1, \\ H_1^{(n)} \quad &: \quad m(\rho, Z_i) = \rho_0 Y_{i-1} + n^{-\frac{1}{2}} G(Y(i-1)) \text{ and } \sigma(\theta_0, Z_i) = \sqrt{1 + n^{-\frac{1}{2}} B(Y(i-1))}. \end{aligned}$$

Remark that H_0, $H_1^{(n)}$ correspond to $\alpha = \beta = 0$ (linearity of (3.4)) and $\alpha = \beta = n^{-\frac{1}{2}}$ (non linearity of (3.4)) with the comparison to the equality (0.1), we have

$$Z_i = Y_i, \quad T(Z_i) = \rho_0 Y_{i-1} + \alpha\, G(Y(i-1)) \text{ and } V(Z_i) = \sqrt{1 + \beta B(Y(i-1))}.$$

Note that when n is large, we have

$$\sigma(\theta_0, Z_i) = \sqrt{1 + n^{-\frac{1}{2}} B(Y(i-1))} \sim 1 + \frac{n^{-\frac{1}{2}}}{2} B(Y(i-1)) = 1 + n^{-\frac{1}{2}} S(Y(i-1)).$$

It is assumed that the model (3.4) is ergodic and stationary. It will be assumed that the conditions (B.1), (B.2) and (B.3) are satisfied, where

- **:** (B.1): The fourth order moment of the stationary distributions of (3.4) exists.
- **:** (B.2): There exists a positive constants η and c such that for all u with $\|u\|_{\ell+p} > \eta$, $B(u) \leq c\|u\|^2_{\ell+p}$.
- **:** (B.3): for a location family $\{b^{-1} f(\frac{\epsilon_i - a}{b}),\ -\infty < a < -\infty,\ b > 0\}$, there exists a square integrable function $\varphi(\cdot)$, and a strictly positive real ς, where $\varsigma > \max(|a|, |b-1|)$, such that,

$$\left| \frac{\partial^2 b^{-1} f\left(\frac{\epsilon_i - a}{b}\right)}{f(\epsilon_i)\, \partial a^j\, \partial b^k} \right| \leq \varphi(\epsilon_i),$$

  where j and k are two positive integers such that $j + k = 2$.

3.2.1. *Local asymptotic normality.* Under the conditions (A.1), (B.1), (B.2) and (B.3), the local asymptotic normality for the LAN corresponding to this problem of testing is established. In this case we have, under H_0:

$$\Lambda_n = \mathcal{V}_n(\phi_0) - \frac{\tau^2(\phi_0)}{2} + o_P(1),$$

$$\mathcal{V}_n(\rho_0) = -\frac{1}{\sqrt{n}} \left\{ \sum_{i=1}^n M_f(\epsilon_i) G(Y(i-1)) + \sum_{i=1}^n (1 + \epsilon_i M_f(\epsilon_i)) B(Y(i-1)) \right\},$$

$$\text{with} \quad \mathcal{V}_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tau^2),$$

$$\text{and} \quad \tau^2 = I_0 \mathbf{E}\left(G(Y(0))\right)^2 + \frac{(I_2 - 1)}{4} \mathbf{E}\left(B(Y(0))\right)^2 + I_1 \mathbf{E}\left(G(Y(0)) B(Y(0))\right),$$

$$\text{where } I_j = \mathbf{E}\Big(\epsilon_0^j M_f^2(\epsilon_0)\Big) \quad \text{where} \quad j = 0, 1, 2.$$

For more details, see ([8, Theorem 4]).

3.2.2. *The considering test.* The proposed test is then given by

$$T_n \quad = \quad I\left\{\frac{\mathcal{V}_n(\rho_0)}{\tau(\rho_0)} \geq Z(\alpha)\right\}.$$

This test is asymptotically optimal with a power function equal asymptotically to $1 - \Phi(Z(\alpha) - \tau^2)$? for more details refer to ([8, Theorem 3]).

3.2.3. *Specification of the random variable D_n.* By the subsisting ρ_0 by its $\sqrt{n}$-consistent estimator $\hat{\rho}_n$ in the expression of the central sequence, we shall state the following proposition:

Proposition 3.3. *Suppose that, under H_0 the conditions $(A.1)$, $(B.1)$, $(B.2)$ and $(B.3)$ hold and ϵ_i's are centered i.i.d. and $\epsilon_0 \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$. We have*

$$\mathcal{V}(\hat{\rho}_n) \quad = \quad \mathcal{V}_n(\rho_0) - D_n + o_P(1), \tag{3.5}$$

where

$$D_n = -c_1\sqrt{n}(\hat{\rho}_n - \rho_0), \quad \text{and} \quad c_1 = -\mathbb{E}\Big[Y_0 G(Y(0))\Big]. \tag{3.6}$$

3.2.4. *Modified estimator and optimal test.* Under the conditions of the Proposition (3.1), the modified estimator and an optimal test are given by the following proposition

Proposition 3.4. *The modified estimator is given by the equality*

$$\bar{\rho_n} = \frac{D_n}{\dot{\mathcal{V}}_n(\phi_n)} + \hat{\rho}_n,$$

and the statistic test is given by

$$\bar{T}_n \quad = \quad I\left\{\frac{\mathcal{V}_n(\bar{\rho_n})}{\bar{\tau}} \geq Z(\alpha)\right\}.$$

Remark 3.2. We mention that the limiting distributions appearing in Proposition (3.1) and Proposition (3.3) depend on the unknown quantity $b_n = (\hat{\rho}_n - \rho_0)$, i.e., in practice ρ_0 is not specified, in general. To circumvent this difficulty, we use the Efron's Bootstrap in order to evaluate b_n, more precisely, the interested reader may refer to the following references : [6] for the description of the Bootstrap methods, [1], [9] for the Bootstrap methods in $AR(1)$ time series models and [Fryzlewicz *et al.*(2008)] for the $ARCH$ models.

We shall now apply the results of the Section(1) and theorem (2.1) in order to conduct simulations corresponding to the representation of the derived asymptotic power function. The concerned model is the Nonlinear time series contiguous to $AR(1)$ processes with an extension to $ARCH$ processes which are detailed in the Subsections (3.1) and (3.2) respectively.

4. Simulations

In this section we consider particular classes which results already figure in the Subsections (3.1) and (3.2). We illustrate these results by doing simulations. We represent simultaneously the power functions, with the true parameter, with the consistency estimator of this parameter and with the modified estimator of this parameter respectively. This representation is given in term of the value of a which appears in the expressions of the random functions G and S. When n is large, we

compare the power functions.
The first aim of the conducted simulation is to evaluate the performance of the modified estimator. The second aim is to obtain a best power by the use of the modified estimator. The considering problem of testing concerns the linearity against a contiguous (to $AR(1)$) sequence of no alternative nonlinear models. An extension to contiguous autoregressive conditional heteroscedastic model is treated.
Simulations are carried out with comments in Subsections (4.1) and (4.2) for these problems of testing.
Throughout, we suppose that ϵ_i's are centered i.i.d. where $\epsilon_0 \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$, in this case, we have:
$\mathbb{E}(\epsilon_i) = 0, \quad \mathbb{E}(\epsilon_i^2) = 1, \quad \text{and } \mathbb{E}(\epsilon_i^4) = 3.$

4.1. **Simulations: Nonlinear time series contiguous to $AR(1)$ processes.** In this subsection, current simulation are carried out. All this results and representations are detailed in the subsection (3.1).
We assume, under H_0 that the conditions (A.1) and (A.2) are satisfied. We treat the case when the unknown parameter $\phi_0 = \rho_0 \in \Theta_1 \subset \mathbb{R}$, under H_0, the considering time series model can also rewritten

$$Y_i = \rho_0 Y_{i-1} + \epsilon_i \text{ where } \quad |\rho_0| < 1. \tag{4.1}$$

In the case when the parameter ρ_0 is known, the test T_n is optimal and its power is asymptotically equal to $1 - \Phi(Z(\alpha) - \tau^2)$, for more details see
[8, Theorem 3]. In a general case, when the parameter ρ_0 is unspecified, firstly, we estimate it with the least square estimators L.S.E. $\hat{\rho}_n = \frac{\sum_{i=1}^n Y_i Y_{i-1}}{\sum_{i=1}^n Y_{i-1}^2}$,
secondly, under the conditions (1.4) and $(C.1)$, the modified estimator M.E. $\bar{\rho}_n$ exists and remains $\sqrt{n}$-consistent, making use of (1.5) in connection with the Proposition (3.2) it follows:

$$\bar{\rho}_n = \frac{D_n}{\dot{\mathcal{V}}_n(\hat{\rho}_n)} + \hat{\rho}_n = \frac{-c_1(\hat{\rho}_n - \rho_0)}{\frac{\dot{\mathcal{V}}_n(\hat{\rho}_n)}{\sqrt{n}}} + \hat{\rho}_n, \tag{4.2}$$

with the substitution of the parameter ρ_0 by its estimator $\bar{\rho}_n$ in (3.5), we obtain from Theorem (2.1), the following optimal statistics test

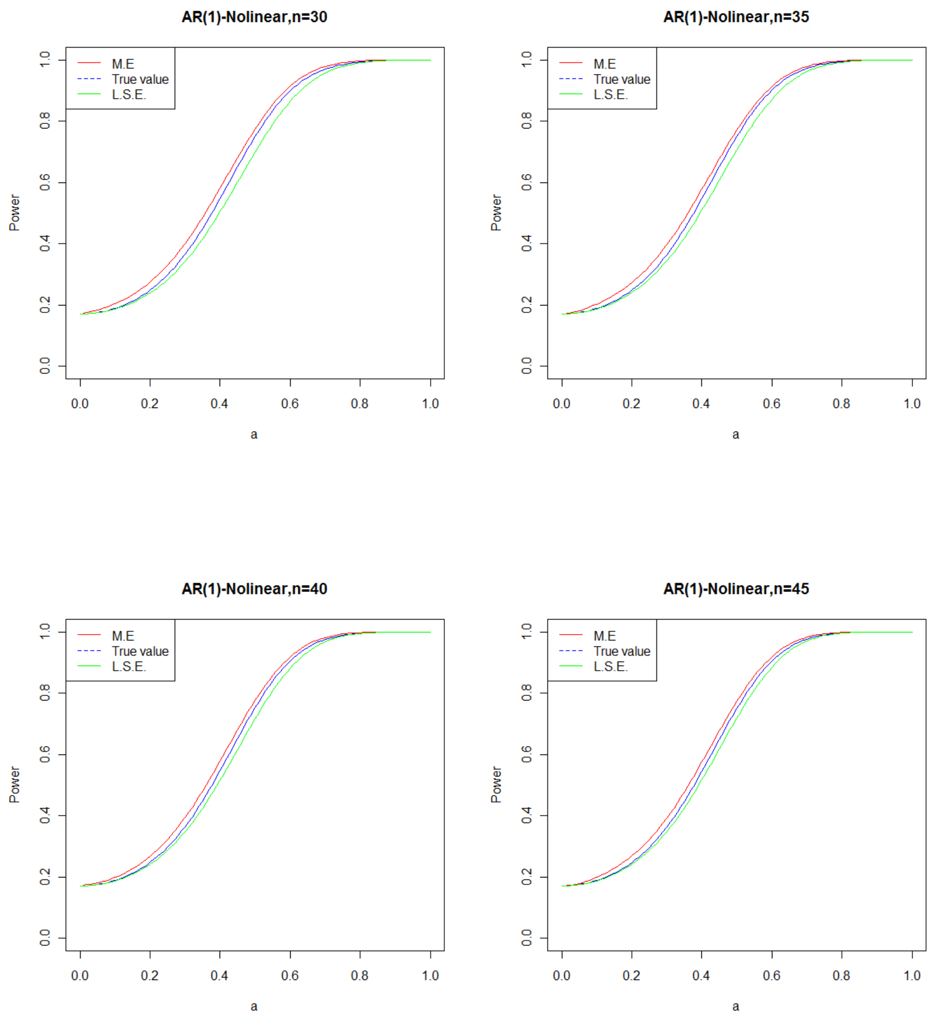
$$\bar{T}_n = \left\{ \frac{\mathcal{V}_n(\bar{\rho}_n)}{\tau(\bar{\rho}_n)} \geq Z(\alpha) \right\} \text{ where } \bar{\tau}^2 = \mathbf{E}(M_f^2(\bar{\epsilon}_{0,n}))\mathbf{E}(G^2(Y_0)),$$
$$\text{and } \bar{\epsilon}_{0,n} = Y_0 - Y_{-1}\bar{\rho}_n.$$

In this case, we have, We choose the function G like this

$$G : \Big(x_1, x_2, \cdots, x_s, x_{s+1}, x_{s+2}, \cdots, x_{s+q} \Big) \longrightarrow \frac{5a}{1 + x_1^2} \text{ where } a \neq 0.$$

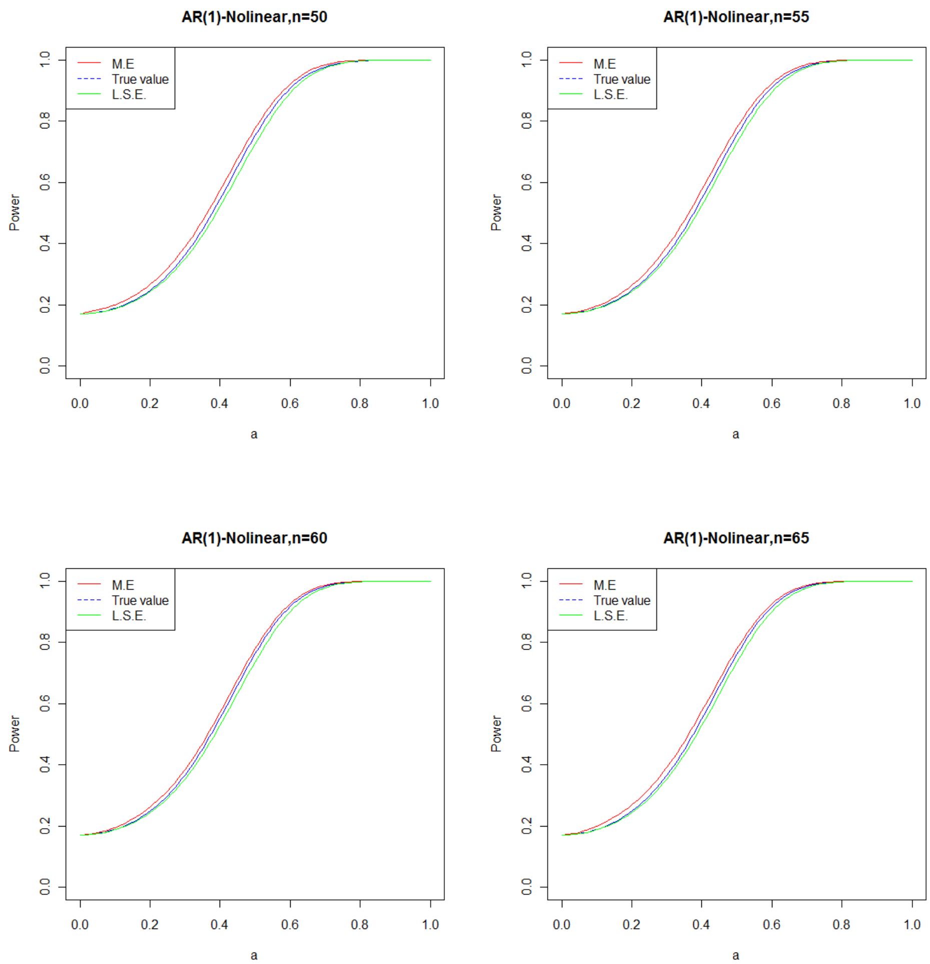
It follows from Theorem (2.1) that $\bar{T}_n$ is optimal with an asymptotic power function equal to $1 - \Phi(Z(\alpha) - \tau^2(\bar{\rho}_n))$.
In our simulations, the true value of the parameter ρ_0 is fixed at 0.1 and the sample sizes are fixed at $n = 30, 35, 40, 45, 50, 55, 60$ and 65. For a level
$\alpha = 0.05$, the power relative for each test estimated upon $m = 1000$ replicates. We represent simultaneously the power test with a true parameter ρ_0, the empirical power test which is obtained with the replacing the true value ρ_0 by its estimator M.E. $\bar{\rho}_n$ corresponding to the equality (4.2), and the empirical power test which is

obtained with the subsisting the true value ρ_0 by its least square estimator L.S.E. $\hat{\rho_n}$ (an estimator with no correction).
We observe that, the two representations with the true value and the modified estimator M.E. are close for large n and large a.



4.2. **Simulations: An extension to** $ARCH$ **processes.** These results concern the problem of testing which is described in the subsection (3.2).
In this case, we assume under H_0, that the conditions (A.1), (B.1), (B.2) and (B.3) are satisfied. On a basis of the results of the Propositions (3.3) and (3.4) and by following the same previous reasoning as the previous Subsection, it follows that:

$$\bar{T}_n = I\left\{\frac{\mathcal{V}_n(\bar{\rho}_n)}{\tau(\bar{\rho}_n)} \geq Z(\alpha)\right\},$$



such that

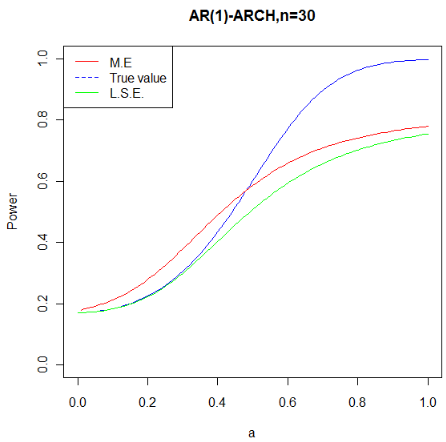
$$\begin{aligned}\bar{\tau}^2 &= \bar{I}_{0,n}\mathbf{E}\left(G(Y(0)\right)^2 + \frac{(\bar{I}_{2,n}-1)}{4}\mathbf{E}\left(B(Y(0)\right)^2 + \bar{I}_{1,n}\mathbf{E}\left(G(Y(0))B(Y(0)\right),\\ \bar{I}_{j,n} &= \mathbf{E}\Big(\bar{\epsilon}_{0,n}^{j} M_f^2(\bar{\epsilon}_{0,n})\Big), \quad j = 0,1,2, \text{ and } \bar{\epsilon}_{0,n} = Y_0 - Y_{-1}\bar{\rho}_n.\end{aligned}$$

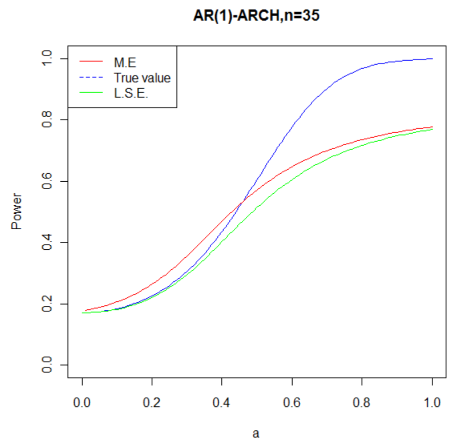
In this case, we choose the functions G and B like this

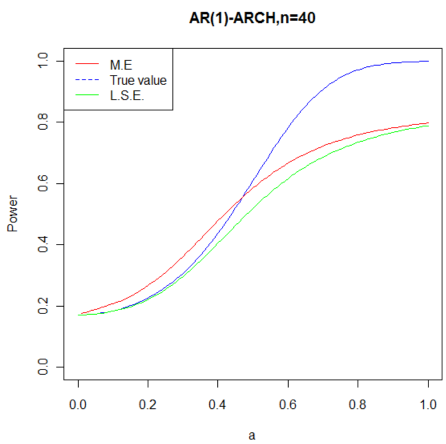
$$G = B : \Big(x_1, x_2, \cdots, x_s, x_{s+1}, x_{s+2}, \cdots, x_{s+q}\Big) \longrightarrow \frac{3.5a}{1+x_1^2} \text{ where } a \neq 0.$$

In our simulations, the true value of the parameter ρ_0 is fixed at 0.1 and the sample sizes are fixed at $n = 30, 35, 40, 45, 50, 55, 60$ and 100. For a level $\alpha = 0.05$, the power relative for each test estimated upon $m = 1000$ replicates.

We remark that, when n and a are large, we have a similar conclusion as the previous case.


AR(1)-ARCH,n=30
M.E
True value
L.S.E.
Power
a
0.0
0.2
0.4
0.6
0.8
1.0


AR(1)-ARCH,n=35
M.E
True value
L.S.E.
Power
a
0.0
0.2
0.4
0.6
0.8
1.0


AR(1)-ARCH,n=40
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True value
L.S.E.
Power
a
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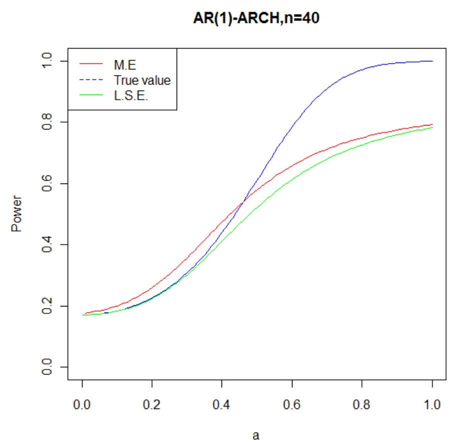
1.0


AR(1)-ARCH,n=40
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True value
L.S.E.
Power
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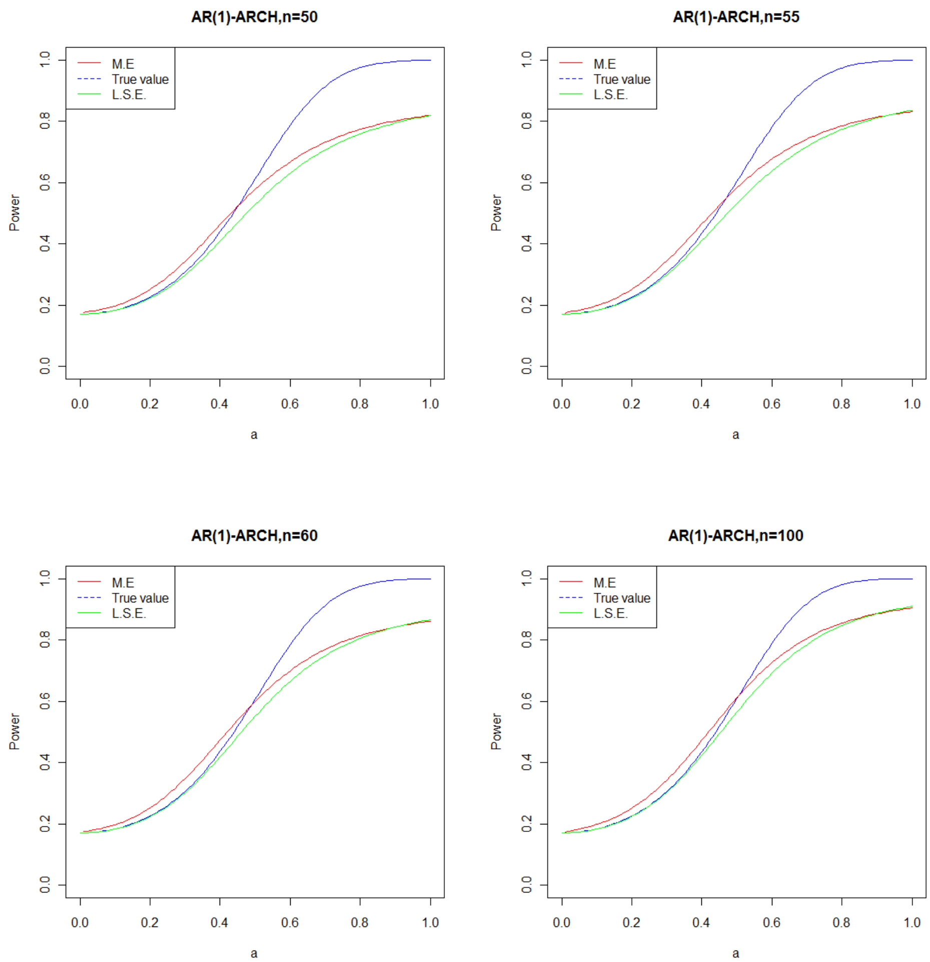

5. Proof of the results

Proof of the Proposition 1.1. Consider the following fundamental decomposition:

$$(\phi_n^{(1,j_n)})' = (\hat{\phi}_n)' + (O_{j_n})', \tag{5.1}$$

where

$$O'_{j_n} = (O_{j_n,i})'_{i\in\{1,\ldots,\ell+p\}}, \text{ such that } O_{j_n,i} = 0 \text{ when } i \neq j_n, \\ \text{and } O_{j_n,j_n} = \bar{\rho}_{n,j_n} - \hat{\rho}_{n,j_n}.$$

Firstly, we have $\hat{\phi}_n \xrightarrow{P} \phi_0$.
Secondly we can deduce from (1.5) that:

$$O_{j_n,j_n} = \frac{D_n}{\frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}} = \frac{1}{\sqrt{n}} D_n \frac{1}{\frac{1}{\sqrt{n}} \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}}. \tag{5.2}$$

Since D_n is bounded, we can remark that $\frac{1}{\sqrt{n}} D_n \xrightarrow{P} 0$.
From $(C.1)$, there exists some constant $c_1 \neq 0$, such that

$$\frac{1}{\sqrt{n}} \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}} \xrightarrow{P} c_1.$$

From (1.4) and since the function $x \to \frac{1}{x}$ is continuous on $\mathbb{R} - \{0\}$, it follows that the random variable $\frac{1}{\frac{1}{\sqrt{n}} \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}} \xrightarrow{P} \frac{1}{c_1}$, it results that the couple $\left(\frac{1}{\sqrt{n}} D_n; \frac{1}{\frac{1}{\sqrt{n}} \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}}\right)$ converges in probability to the couple $\left(0 \ ; \ \frac{1}{c_1}\right)$.
Since the function $(x, y) \to xy$ is continuous on $\mathbb{R} \times \mathbb{R}$, it result from (5.2), that the random variable $O_{j_n,j_n} \xrightarrow{P} \frac{0}{c_1} = 0$, therefore

$$O_{j_n}{}' = (0, \ldots 0, O_{j_n,j_n}, 0 \ldots 0)' \xrightarrow{P} (0, \ldots 0, 0, 0 \ldots 0)'. \tag{5.3}$$

Consider again the equality (5.1). Since the function $(x, y) \to x + y$ is continuous on $\mathbb{R}^{\ell+p} \times \mathbb{R}^{\ell+p}$, it results from (5.3) that $\phi_n^{(1,j_n)}$ converges in probability to ϕ_0 as $n \to \infty$.
Note that the last previous convergence in probability follows immediately with the use of the continuous mapping theorem, for more details, see for instance [3] or [12].
By following the same previous reasoning, we shall prove the consistency of the estimator $\phi_n^{(2,k_n)}$.
Note that $\phi_n^{(1,j_n)}$ is $\sqrt{n}$-consistent estimator of the parameter ϕ_0 and $\sqrt{n}(\phi_n^{(1,j_n)} - \phi_0) = O_P(1)$, where $O_P(1)$ is bounded in probability in $\mathbb{R}^{\ell+p}$.
In fact, it follows from (5.1) that:

$$\sqrt{n}(\phi_n^{(1,j_n)} - \phi_0) = \sqrt{n}(\hat{\phi}_n - \phi_0) + \sqrt{n} O_{j_n} = O_P(1) + \sqrt{n} O_{j_n}.$$

Since $\sqrt{n} O_{j_n,j_n} = D_n \frac{1}{\frac{1}{\sqrt{n}} \frac{\partial \mathcal{W}_n(\hat{\phi}_n)}{\partial \rho_{j_n}}}$ and under $(C.1)$, it results that $\sqrt{n} O_{j_n} = O_{P_1}(1)$, where $O_{P_1}(1)$ is bounded in probability in $\mathbb{R}$.
We deduce that:

$$\sqrt{n}(\phi_n^{(1,j_n)} - \phi_0) = O_P(1). \tag{5.4}$$

Note that with a similar argument and with changing $\phi_n^{(1,j_n)}$, $(C.1)$ and (1.4) by $\phi_n^{(2,k_n)}$, $(C.2)$ and (1.7) respectively, we obtain

$$\sqrt{n}(\phi_n^{(2,k_n)} - \phi_0) = O_P(1). \tag{5.5}$$

Proof of the Lemma 1.3. In this case $\phi_0 = \rho_0 \in \Theta_1 \subset \mathbb{R}$, we denote by $\hat{\rho}_n$ the $\sqrt{n}$-consistent estimator of ρ_0.
Let $A > 0$, from the triangle inequality combined with a classical inequality, we

obtain:

$$\begin{aligned} & P\left(\left|\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\hat{\rho}_n)-c_1\right|>A\right) \\ &= P\left(\left|\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\hat{\rho}_n)-\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\rho_0)\right|+\left|\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\rho_0)-c_1\right|>A\right) \\ &\leq P\left(\left|\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\hat{\rho}_n)-\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\rho_0)\right|>\frac{A}{2}\right)+P\left(\left|\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\rho_0)-c_1\right|>\frac{A}{2}\right). \end{aligned}$$

Firstly, we have

$$P\left(\left|\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\rho_0)-c_1\right|>\frac{A}{2}\right)\to 0 \text{ as } n\to\infty, \tag{5.6}$$

Secondly, we have

$$\left|\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\hat{\rho}_n)-\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\rho_0)\right| = \frac{1}{\sqrt{n}}\left|\ddot{\mathcal{W}}_n(\tilde{\rho}_n)\right|\left|\hat{\rho}_n-\rho_0\right| \tag{5.7}$$

$$= \frac{1}{\sqrt{n}}\left|\frac{1}{\sqrt{n}}\ddot{\mathcal{W}}_n(\tilde{\rho}_n)\right|\left|\sqrt{n}(\hat{\rho}_n-\rho_0)\right|, \tag{5.8}$$

where $\tilde{\rho}_n$ is a point between ρ_0 and $\hat{\rho}_n$, then there exists a sequence η_n with values in the interval $[0,1]$, such that $\tilde{\rho}_n = \eta_n\rho_0 + (1-\eta_n)\hat{\rho}_n$.

This implies that
$|\tilde{\rho}_n-\rho_0| \leq (1-\eta_n)|\hat{\rho}_n-\rho_0| \leq |\hat{\rho}_n-\rho_0|$.

This last inequality enables us to concluded that $\tilde{\rho}_n$ is $\sqrt{n}$-consistency estimator of ρ_0, it follows from $(C.3)$ applied on the equality (5.8) that

$$P\left(\left|\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\hat{\rho}_n)-\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\rho_0)\right|>\frac{A}{2}\right)\to 0 \text{ as } n\to 0. \tag{5.9}$$

Thus we obtain (i).

Proof of Proposition 1.2. It suffices to choose under (1.4) and $(C.1)$ the estimator $\bar{\phi_n} = \phi_n^{(1,j_n)}$, (or to choose under (1.7) and $(C.2)$ the estimator $\bar{\phi_n} = \phi_n^{(2,k_n)}$,).

Proof of Proposition 3.1. Since (A.1) and (A.2) hold, we deduce from ([8, Theorem 1]), that the local asymptotic normality LAN for the log likelihood ratio is established.

ϵ_i's are centered i.i.d. and $\epsilon_0 \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$, making use of the results of [8, Theorem 2], we have

$$\mathcal{W}_n(\rho_0) = -\frac{1}{\sqrt{n}}\sum_{i=1}^{n} M_f(\epsilon_i)G(Y(i-1)).$$

The estimated central sequence is

$$\mathcal{W}_n(\hat{\rho}_n) = -\frac{1}{\sqrt{n}}\sum_{i=1}^{n} M_f(\hat{\epsilon}_{i,n})G(Y(i-1)).$$

By Taylor expansion with order 2, we have :

$$\mathcal{W}_n(\hat{\rho}_n)-\mathcal{W}_n(\rho_0) = \dot{\mathcal{W}}_n(\hat{\rho}_n)(\hat{\rho}_n-\rho_0)+\frac{1}{2}\ddot{\mathcal{W}}_n(\tilde{\rho_n})(\hat{\rho}_n-\rho_0)^2,$$

where $\tilde{\rho_n}$ is a point between ρ_0 and $\hat{\rho}_n$ and

$$\dot{\mathcal{W}}_n(\tilde{\rho_n}) = \frac{-1}{\sqrt{n}}\sum_{i=1}^{n} Y_{i-1}G(Y(i-1)).$$

Note that

$$R_n = \frac{1}{2}\ddot{\mathcal{W}}_n(\tilde{\rho_n})(\hat{\rho}_n - \rho_0)^2 = \frac{1}{2\sqrt{n}}\frac{1}{\sqrt{n}}\ddot{\mathcal{W}}_n(\tilde{\rho_n})\Big(\sqrt{n}(\hat{\rho}_n - \rho_0)\Big)^2.$$

Since the estimator $\hat{\rho}_n$ is $\sqrt{n}$-consistent, it results that

$$\Big(\sqrt{n}(\hat{\rho}_n - \rho_0)\Big)^2 = O_P(1),$$

from the assumption $(C.3)$, it follows that

$$R_n = o_P(1),$$

finally we deduce that,

$$\mathcal{W}_n(\hat{\rho}_n) - \mathcal{W}_n(\rho_0) = \dot{\mathcal{W}}_n(\hat{\rho}_n)(\hat{\rho}_n - \rho_0) + o_P(1). \tag{5.10}$$

This implies that

$$\frac{\dot{\mathcal{W}}_n(\hat{\rho}_n)}{\sqrt{n}} - \frac{\dot{\mathcal{W}}_n(\rho_0)}{\sqrt{n}} = \frac{\ddot{\mathcal{W}}_n(\check{\rho}_n)}{\sqrt{n}}(\hat{\rho}_n - \rho_0) + o_P(1) = \frac{1}{\sqrt{n}}\frac{\ddot{\mathcal{W}}_n(\check{\rho}_n)}{\sqrt{n}}\sqrt{n}(\hat{\rho}_n - \rho_0) + o_P(1), \tag{5.11}$$

where $\check{\rho}_n$ is between $\hat{\rho}_n$ and ρ_0, and $\ddot{\mathcal{W}}_n$ is the second derivative of $\mathcal{W}_n$. From the assumption $(C.3)$, we have

$$\frac{1}{\sqrt{n}}\frac{\ddot{\mathcal{W}}_n(\check{\rho}_n)}{\sqrt{n}} = o_P(1),$$

since the estimator $\hat{\rho}_n$ is $\sqrt{n}$-consistent, it result that

$$\frac{\dot{\mathcal{W}}_n(\hat{\rho}_n)}{\sqrt{n}} - \frac{\dot{\mathcal{W}}_n(\rho_0)}{\sqrt{n}} = o_P(1),$$

this implies that

$$\frac{\dot{\mathcal{W}}_n(\hat{\rho}_n)}{\sqrt{n}} = \frac{\dot{\mathcal{W}}_n(\rho_0)}{\sqrt{n}} + o_P(1). \tag{5.12}$$

With the use of (5.12), the equality (5.10) can also rewritten

$$\begin{aligned}\mathcal{W}_n(\hat{\rho}_n) - \mathcal{W}_n(\rho_0) &= \frac{\dot{\mathcal{W}}_n(\hat{\rho}_n)}{\sqrt{n}}\sqrt{n}(\hat{\rho}_n - \rho_0) + o_P(1),\\ &= \frac{\dot{\mathcal{W}}_n(\rho_0)}{\sqrt{n}}\sqrt{n}(\hat{\rho}_n - \rho_0) + o_P(1). \end{aligned} \tag{5.13}$$

It follows from the assumption $(C.1)$ combined with the ergodicity and the stationarity of the model that, the random variable $\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\rho_0)$ converges in probability to the constant c_1, as $n \to +\infty$, where

$$c_1 = -\mathbb{E}\Big[Y_0G(Y(0))\Big],$$

therefore there exists a random variable X_n, $X_n \xrightarrow{P} 0$ such that

$$\frac{1}{\sqrt{n}}\dot{\mathcal{W}}_n(\rho_0) = c_1 + X_n.$$

We deduce from the equality (5.13) and the $\sqrt{n}$-consistence of the estimator $\hat{\rho_n}$, that

$$\mathcal{W}_n(\hat{\rho}_n) - \mathcal{W}_n(\rho_0) = c_1\sqrt{n}(\hat{\rho}_n - \rho_0) + o_P(1) = -D_n + o_P(1),$$

where $D_n = -c_1\sqrt{n}(\hat{\rho}_n - \rho_0)$.
Recall that the second derivative $\ddot{\mathcal{W}}_n$ is equal to 0, this implies that the assumption $(C.3)$ is satisfied.

Proof of Proposition 3.2. This proposition is one consequence of the results of Subsection (1.1). More precisely the direct application of the definition (1.5) (or (1.8)).

Proof of Proposition 3.3. The assumption $(C.1)$ remains satisfied. From ([8, Theorem 4]), assumptions $(A.1)$, $(B.1)$, $(B.2)$ and $(B.3)$ imply the local asymptotic normality LAN for the log likelihood ratio. The proof is similar as the proof of Proposition (3.1), in this case, for all $\rho \in \Theta_1$, we have

$$\ddot{\mathcal{W}}_n(\rho) = \frac{-1}{\sqrt{n}} \sum_{i=1}^{n} Y_{i-1}^2 B(Y(i-1)) 2\dot{M}_f(\rho).$$

By a simple calculus and since the the function f is the density of the standard normal distribution, it is easy to prove that the quantity $2\dot{M}_f(\rho)$ is bounded, therefore, there exists a positive constant w such that $2\dot{M}_f(\rho) \leq w$, then

$$|\frac{1}{\sqrt{n}}\ddot{\mathcal{W}}_n(\rho)| \leq w\frac{1}{n}\sum_{i=1}^{n} Y_{i-1}^2 |B(Y(i-1))|.$$

With the choice $B(Y(i-1)) = \frac{2a}{1+Y_{i-1}^2}$ with $a \neq 0$, it results that

$$|\frac{1}{\sqrt{n}}\ddot{\mathcal{W}}_n(\rho)| \leq 2w|a|\frac{1}{n}\sum_{i=1}^{n} Y_{i-1}^2.$$

By the use of the ergodicity of the model and since the model is with finite second moments, it follows that the random variable $\frac{1}{n}\sum_{i=1}^{n} Y_{i-1}^2 \xrightarrow{a.s} k$, where k is some constant, this implies that the condition $(C.3)$ is straightforward.

Proof of Proposition 3.4. The proof is similar as the proof of the Proposition (3.2).

Proof of the Theorem 2.1. Since it assumed that local asymptotic normality LAN for the log likelihood ratio is established, then we have

$$\Lambda_n = \mathcal{V}_n(\phi_0) - \frac{\tau^2(\phi_0)}{2} + o_P(1).$$

In this case the random variable $\mathcal{W}_n$ is equal to $\mathcal{V}_n$.

From the conditions (1.4) ((1.7), respectively), $(C.1)$ ($(C.2)$, respectively), it results the existence and the $\sqrt{n}$-consistency of the modified estimator $\bar{\phi}_n$ corresponding to the equation (1.5) ((1.8), respectively).
The combinaison of the condition (E_1) and the Proposition (1.2) enable us to get under H_0 the following equality

$$\mathcal{V}_n(\bar{\phi}_n) \quad = \quad \mathcal{V}_n(\phi_0) + o_P(1).$$

This last equation implies that with $o_P(1)$, the estimated central and the central sequences are equivalent, in the expression of the test (3.2), the replacing of the central sequence by the estimated central sequence has no effect.
LAN implies the contiguity of the two hypothesis (see, [5, Corrolary 4.3]), by Le Cam third lemma's (see for instance, [7, Theorem 2]), under $H_1^{(n)}$, we have

$$\mathcal{V}_n \xrightarrow{\mathcal{D}} \mathcal{N}(\tau^2, \tau^2).$$

It follows from the convergence in probability of the estimator $\bar{\phi}_n$ to ϕ_0, the continuity of the function $\tau : \cdot \longrightarrow \tau(\cdot)$ and the application of the continuous mapping theorem see, for instance ([12]) or [3], that asymptotically, the power of the test is not effected when we replace the unspecified parameter ϕ_0 by it's estimator, $\bar{\phi}_n$, hence the optimality of the test.

The power function of the test is asymptotically equal to $1 - \Phi(Z(\alpha) - \tau^2(\bar{\phi}_n))$. The proof is similar as [8, Theorem 3].

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