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ON THE STABILITY OF BINARY FORMS AND THEIR WEIGHTED HEIGHTS

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Abstract. Let k be a number field, $\mathcal{O}_k$ its ring of integers, and $f(x,y) \in \mathcal{O}_k[x,y]$ an integral binary form of degree $d \geq 3$. Minimality of $f(x,y)$ is equivalent to residual semistability. In this paper, we give a method to explicitly determine a binary form, k-equivalent to f, which is residually semistable. for any prime $p \in \mathcal{O}_k$.

In the last part of the paper we compare the GIT height from [Zha96] with weighted height in [BGS20] and show that for strictly semistable forms their logarithmic weighted height $\mathfrak{s}_k > 0$, for $d \leq 10$. Moreover, we show that binary forms with logarithmic weighted height $\mathfrak{s}_k(\xi(f)) = 0$ exist for any degree $d \geq 3$.

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1. Introduction

Let k be a number field, $\mathcal{O}_k$ its ring of integers, and $f \in k[x,y]$ a degree $d \geq 2$ binary form. The equivalence classes of binary forms $f(x,y)$, over some algebraic closure of k, are determined by the set of generators of the ring of invariants $\mathcal{R}_d$ of degree d binary forms. It is well known that $\mathcal{R}_d$ is finitely generated. Let $\xi_0, \ldots, \xi_n$ be such generators. They are homogenous polynomial of degree $\deg \xi_i = q_i$. We denote by $\xi(f) = (\xi_0(f), \ldots, \xi_n(f))$ the values of such invariants evaluated at the form $f(x,y)$. The equivalence class of f is determined by the *weighted moduli point* $\xi(f) = [\xi_0(f) : \cdots : \xi_n(f)]$ in the weighted projective space $\mathbb{P}^n_{\mathfrak{w},k}$ with weights $\mathfrak{w} = (q_0, \ldots, q_n)$; see [BGS20] for details.

A binary form has a root of multiplicity greater than $\frac{d}{2}$ if and only if $\xi(f) = \mathbf{0}$ (cf. Lem. 1). Using Hilbert-Mumford numerical criterion (cf. Prop. 3) one proves that $f(x,y)$ is semistable (resp. stable) if and only if it has no root of multiplicity

$\geq \frac{d}{2}$ (resp. $> \frac{d}{2}$). Hence, for a prime $p \in \mathcal{O}_k$, a binary form is semistable over the residue field $\mathcal{O}_k/p\mathcal{O}_k$ if and only if $p \nmid \xi_i$, for all $i = 0, \dots, n$. The main focus of this paper is to find, for any given $f \in \mathcal{O}_k[x, y]$, a semistable model over $\mathcal{O}_k/p\mathcal{O}_k$ for all primes $p \in \mathcal{O}_k$, when such model exists.

For a fixed prime $p \in \mathcal{O}_k$, $f(x, y)$ is semistable over $\mathcal{O}_k/p\mathcal{O}_k$ if and only if $p \nmid \gcd(\xi_0(f), \dots, \xi_n(f))$. By taking a minimal model of $f(x, y)$ as described in [Sha22] we can assume that the weighted greatest common divisor $\operatorname{wgcd}(\xi(f)) = 1$; see [BGS20]. This is equivalent to assume that $f(x, y)$ can be unstable only for primes p dividing $\frac{\gcd(\xi(f))}{\operatorname{wgcd}(\xi(f))}$. For each such prime $p \in \mathcal{O}_k$ we find a matrix $A_p = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(k)$ such that $f^{A_p}(x, y) = f(ax + by, cx + dy)$ is semistable over $\mathcal{O}_k/p\mathcal{O}_k$. Taking the matrix $M = \prod A_p$ for p dividing $\frac{\gcd(\xi(f))}{\operatorname{wgcd}(\xi(f))}$ gives a binary form $f^M(x, y) \in \mathcal{O}_k[x, y]$ which is semistable over all residue fields $\mathcal{O}_k/p\mathcal{O}_k$. It turns out that this is equivalent with the reduction Type A introduced in [Sha22].

Shaska and his co-authors introduced a natural height in $\mathbb{P}^n_{\mathfrak{w},k}$ called the *weighted moduli height*; see [MS19b], [BGS20], and [SS22]. In the second part of this paper we determine stability in terms of this weighted height in the moduli space $\mathbb{P}^n_{\mathfrak{w},k}$. We give necessary and sufficient conditions in terms of weighted height and weighted greatest common divisors for a binary form to be semistable over a residue field $\mathcal{O}_k/p\mathcal{O}_k$.

If $f(x, y)$ is unstable than $\xi(f) = \mathbf{0}$. When $f(x, y)$ is strictly semistable it has a root of multiplicity $\frac{d}{2}$. Such forms do not exist when d is odd and there is only one such form (up to equivalence) for d even. We compute the weighted moduli height $\mathcal{S}_{\mathrm{k}}(\xi(f))$ for degree $d = 4, 6, 8, 10$. Moreover, we show that $\mathcal{S}_{\mathrm{k}}(\xi(f)) \geq 1$ for all semistable forms.

In [Zha96] Zhang defined the *invariant height* (also known as GIT height) and considered the question of determining the semistability (resp. stability) in terms of such height. Such question was considered in more detail for binary forms in [Rab13] and [RS15], where the authors show that such height is bounded from below for semistable points and give some lower bounds for cubic binary forms. It is unclear how the invariant height in [Zha96] relates to the weighted moduli height in [BGS20]. The comparison between the invariant height and weighted heights seems to raise many questions, which we intend to explore in [CS22].

This paper is organized as follows. In Section 2 we give a brief review of the basics in invariant theory and weighted projective spaces. We display generating invariants for the ring of invariants $\mathcal{R}_d$ for $d \geq 3$ and $d \leq 10$.

In Section 3 are covered some of the classical results of semistability of binary forms including the Hilbert-Mumford criteria. While this material is part of the classical Geometric Invariant Theory, it provides an outline here to prove the fact that a degree d binary form is semistable (resp. stable) if and only if it has a root of multiplicity $\leq \frac{d}{2}$ (resp. $< \frac{d}{2}$). Moreover, we consider binary forms over a number field k. For every prime $p \in \mathcal{O}_k$ we give a condition for a binary form to be semistable over $\mathcal{O}_k/p\mathcal{O}_k$ in terms of the coordinates of the point in the weighted moduli point and provide a method how to determine an equivalent form to the given form which is semistable over $\mathcal{O}_k/p\mathcal{O}_k$.

In Section 5 we define the weighted height and consider the weighted height of semistable points and strictly semistable points. From Lem. 5 for d odd that are no strictly semistable binary forms and for d even there is exactly one such binary form (up to equivalence). We display the corresponding points in the weighted moduli space for such binary forms and their corresponding weighted height for $d = 4, 6, 8, 10$; see Table 1.

It would also be interesting to estimate the number of stable binary forms with integer coefficients such that their weighted moduli height is less than the weighted moduli height of the strictly semistable binary form of the same even degree d. Equivalently this would estimate the number of binary forms (up to k-equivalence) with weighted height less than the strictly semistable form, such that the field of moduli is also a field of definition. Some of these problems are intended to be discussed in [CS22].

There have been many papers studying stability and heights, see [Bos96] and many others comparing different heights and finding relations between them. [dJS22], [dJ10], [dJ18]. Discussing in detail how the invariant height compares to the weighted moduli height is the focus of [CS22]. It would be interesting to discover how the weighted height relates to Neron-Tate height or Faltings height.

Notation: We fix the following notation for the remainder of this paper.

k a number field,
$\mathcal{O}_k$ the ring of integers of k,
ν an absolute value of k,
M_k the set of all absolute values of k,
M_k^0 the set of non-Archimedean absolute values of k,
M_k^∞ the set of Archimedean absolute values of k,
k_ν completion of k at ν,
n_ν local degree $[k_\nu : \mathbb{Q}_\nu]$
$p \in \mathcal{O}_k$ a prime in $\mathcal{O}_k$
$\mathcal{O}_k/p\mathcal{O}_k$ residue field at $\mathfrak{p}$
$\mathbb{P}_k^n$ projective space over a field k
$\mathbb{P}_{\mathfrak{w},k}^n$ weighted projective space with weights $\mathfrak{w} = (q_0, \dots, q_n)$

After the first version of this paper was posted we came across a more complete treatment of weighted heights in [SS22] and we decided to adapt the notation of [SS22] instead of [BGS20]. The following notation is borrowed from [SS22].

	$\mathbb{P}_k^n$	$\mathbb{P}_{\mathfrak{w},k}^n$
greatest common divisor of $\mathbf{x}$	$\gcd(\mathbf{x})$	$\mathrm{wgcd}\,(\mathbf{x})$
multiplicative height over k	H_k	$\mathcal{S}_\mathrm{k}$
logarithmic height over k	h_k	$\mathfrak{s}_\mathrm{k}$
absolute multiplicative height	H	$\mathcal{S}$
absolute logarithmic height	h	$\mathfrak{s}$

2. PRELIMINARIES

Let k be a field, $k[x, y]$ be the polynomial ring in two variables and V_d denote the $(d+1)$-dimensional subspace of $k[x, y]$ consisting of homogeneous polynomials

$$f(x, y) = a_d x^d + a_{d-1} x^{d-1} y + \cdots + a_0 y^d \tag{1}$$

of degree d. Elements in V_d are called **binary forms** of degree d. $\mathrm{GL}_2(k)$ acts as a natural group of automorphisms on $k[x,y]$. Denote by $f \to f^M$ this action. It is well known that $\mathrm{SL}_2(k)$ leaves a bilinear form (unique up to scalar multiples) on V_d invariant; see [Sha22] for details.

Consider $a_0, a_1, \dots, a_d$ as transcendentals over k (coordinate functions on V_d). Then the coordinate ring of V_d can be identified with $k[a_0, \dots, a_d]$. We define an action of $\mathrm{GL}_2(k)$ on $k[a_0, \dots, a_d]$ via

$$\begin{aligned} \mathrm{GL}_2(k) \times k[a_0, \dots, a_d] &\to k[a_0, \dots, a_d] \\ (M, F) &\to F^M := F(f^M), \quad \text{for all } \ f \in V_d. \end{aligned}$$

Thus for $F \in k[a_0, \dots, a_d]$ and $M \in \mathrm{GL}_2(k)$, define $F^M \in k[a_0, \dots, a_d]$ as

$$F^M(f) := F(f^M),$$

for all $f \in V_d$. Then $F^{MN} = (F^M)^N$. The homogeneous degree in $a_0, \dots, a_d$ is called the **degree** of F, and the homogeneous degree in x, y is called the **order** of F. An **invariant** is usually referred to an $\mathrm{SL}_2(k)$-invariant on V_d. Hilbert's theorem says that the ring of invariants $\mathcal{R}_d$ of binary forms of degree d is finitely generated. Thus, $\mathcal{R}_d$ is finitely generated, and $\mathcal{R}_d$ is a graded ring.

Let $\{\xi_0, \dots, \xi_n\}$ be a minimal generating set for $\mathcal{R}_d$. Since $\xi_i \in k[a_0, \dots, a_d]$ are homogenous polynomials we denote $\deg \xi_i = q_i$ and assume that

$$q_0 \leq q_1 \leq \cdots \leq q_n.$$

The tuple of degrees $(q_0, \dots, q_n)$ are often called **weights**.

If $f, g \in V_d$, $M \in \mathrm{GL}_2(k)$, $\lambda = (\det M)^{\frac{d}{2}}$, then $f = g^M$ if and only if

$$(\xi_0(f), \dots \xi_i(f), \dots, \xi_n(f)) = (\lambda^{q_0}\, \xi_0(g), \dots, \lambda^{q_i}\, \xi_i(g), \dots, \lambda^{q_n}\, \xi_n(g)), \tag{2}$$

see [Sha22, Prop. 1] for the proof.

Lemma 1. *If $k = \mathbb{Q}$ we can choose $\xi_0, \dots, \xi_n$ with integer coefficients and primitive polynomials in $\mathbb{Z}[a_0, \dots, a_d]$ (i.e. the greatest common divisor of coefficients of each ξ is 1).*

Proof. Without loss of generality we can assume that each $f \in \mathbb{Q}[x,y]$ has integer coefficients since binary forms are defined up to multiplication by a constant. Then for $\xi = [\xi_0, \dots, \xi_n]$ each coordinate $\xi_i(f) \in \mathbb{Q}[a_0, \dots, a_d]$. Then multiplying the tuple $\xi = [\xi_0 : \cdots : \xi_n]$ by the least common multiple of all the denominators, say λ, we get a representative $[\lambda^{q_o}\xi_0 : \cdots : \lambda^{q_i}\xi_i, \cdots, \lambda^{q_n}\xi_n]$ of ξ with integer coefficients. We can redefine each ξ_i by taking its primitive part. □

2.1. **Weighted greatest common divisors.** Let $\mathbf{x} = (x_0, \dots x_n) \in \mathbb{Z}^{n+1}$ be a tuple of integers, not all equal to zero. Their greatest common divisor, denoted by $\gcd(x_0, \dots, x_n)$, is defined as the largest integer d such that $d | x_i$, for all $i = 0, \dots, n$. Let $q_0, \dots, q_n$ be positive integers. A set of weights is called the ordered tuple $\mathfrak{w} = (q_0, \dots, q_n)$. Denote by $r = \gcd(q_0, \dots, q_n)$ the greatest common divisor of $q_0, \dots, q_n$. A *weighted integer tuple* is a tuple $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1}$ such that to each coordinate x_i is assigned the weight q_i. We multiply weighted tuples by scalars $\lambda \in \mathbb{Q}$ via

$$\lambda \star (x_0, \dots, x_n) = (\lambda^{q_0} x_0, \dots, \lambda^{q_n} x_n)$$

For an ordered tuple of integers $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1}$, whose coordinates are not all zero, the **weighted greatest common divisor with respect to the set of weights $\mathfrak{w}$** is the largest integer d such that

$$d^{q_i} \mid x_i, \quad \text{for all } i = 0, \dots, n.$$

We will call a point $\mathfrak{p} \in \mathbb{P}^n_{\mathfrak{w}}(\mathbb{Q})$ **normalized** if $\operatorname{wgcd}(\mathfrak{p}) = 1$. The **absolute weighted greatest common divisor** of an integer tuple $\mathbf{x} = (x_0, \dots, x_n)$ with respect to the set of weights $\mathfrak{w} = (q_0, \dots, q_n)$ is the largest real number d such that

$$d^{q_i} \in \mathbb{Z} \quad \textit{and} \quad d^{q_i} \mid x_i, \quad \text{for all } i = 0, \dots n.$$

2.2. **Weighted projective space.** Let $\xi_0, \dots, \xi_n$ be the generators of $\mathcal{R}_d$ with degrees $q_0, \dots, q_n$ respectively. Since all $\xi_0, \dots, \xi_i, \dots, \xi_n$ are homogenous polynomials then $\mathcal{R}_d$ is a graded ring and Proj $\mathcal{R}_d$ as a weighted projective space.

Let $\mathfrak{w} := (q_0, \dots, q_n) \in \mathbb{Z}^{n+1}$ be a fixed tuple of positive integers called **weights**. Consider the action of $k^\star = k \setminus \{0\}$ on $\mathbb{A}^{n+1}(k)$ as follows

$$\lambda \star (x_0, \dots, x_n) = (\lambda^{q_0} x_0, \dots, \lambda^{q_n} x_n)$$

for $\lambda \in k^*$. The quotient of this action is called a **weighted projective space** and denoted by $\mathbb{P}^n_{(q_0,\dots,q_n),k}$. It is the projective variety $Proj\,(k[x_0, \dots, x_n])$ associated to the graded ring $k[x_0, \dots, x_n]$ where the variable x_i has degree q_i for $i = 0, \dots, n$. We denote greatest common divisor of $q_0, \dots, q_n$ by $\gcd(q_0, \dots, q_n)$. The space $\mathbb{P}^n_{\mathfrak{w},k}$ is called **well-formed** if

$$\gcd(q_0, \dots, \hat{q}_i, \dots, q_n) = 1, \quad \text{for each } \ i = 0, \dots, n.$$

While most of the papers on weighted projective spaces are on well-formed spaces, we do not assume that here. We will denote a point $\mathfrak{p} \in \mathbb{P}^n_{\mathfrak{w},k}$ by $\mathfrak{p} = [x_0 : x_1 : \cdots : x_n]$. Summarizing the previous section we have:

Remark 1. *Let $\xi_0, \xi_1, \dots, \xi_n$ be the generators of the ring of invariants $\mathcal{R}_d$ of degree d binary forms. A k-isomorphism class of a binary form f is determined by the point*

$$\xi(f) := [\xi_0(f), \xi_1(f), \dots, \xi_n(f)] \in \mathbb{P}^n_{\mathfrak{w},k}.$$

Moreover, for any two forms f, and g we have that $f = g^M$ for some $M \in \mathrm{GL}_2(k)$ if and only if $\xi(f) = \lambda \star \xi(g)$, for $\lambda = (\det A)^{\frac{d}{2}}$.

Remark 2. *Determining generating sets of invariants for $\mathcal{R}_d$ was the focus of XIX century mathematics. It was exactly this problem that lead to Hilbert's basis theorem and Emmy Noether to the concept of finitely generated ideals and what are now called Noetherian rings.*

2.3. **Generating invariants.** Finding generators for the ring of invariants $\mathcal{R}_d$ is a classical problem in which worked many of the most important mathematicians of the XIX-century. Such invariants are generated in terms of transvections or root differences. While generating set of invariants for $\mathcal{R}_d$, for $d \leq 8$ is part of the classical invariant theory, in the last decades such sets have been determined for $d = 9, 10$. For a complete list of invariants up to $d = 10$ we refer to [Pop14].

For given binary invariants $f, g \in V_d$ the r-th transvection of f and g, denoted by $(f, g)_r$, is defined as

$$(f,g)_r := \frac{(m-r)!\,(n-r)!}{n!\,m!} \sum_{k=0}^{r} (-1)^k \binom{r}{k} \cdot \frac{\partial^r f}{\partial x^{r-k}\,\partial y^k} \cdot \frac{\partial^r g}{\partial x^k\,\partial y^{r-k}}.$$

Transvections are a convenient way of generating invariants since they are expressed in terms of the coefficients of the binary form, on contrary to the method of generating invariants through root differences which give the invariants in terms of roots of binary forms, which we will briefly describe next.

While there is no method known to determine a generating set of invariants for any $\mathcal{R}_d$, we display a minimal generating set for all $3 \leq d \leq 10$. For the rest of this section $f(x,y)$ is given as in Eq. (1) and a minimal set of invariants is always picked as in Lem. 1. Proofs of the following facts are elementary for classical invariant theory experts and we skip them here. All the polynomial expressions of invariants were generated using Maple; see [Map22].

2.3.1. *Cubics.* A generating set for $\mathcal{R}_3$ is $\xi = \{J_4\}$ for $J_4 = ((f,f)_2, (f,f)_2)_2$. Computing J_4 it yields

$$J_4 = \frac{4}{3}a_1a_3a_0a_2 - \frac{8}{27}a_1^3a_3 - \frac{8}{27}a_2^3a_0 + \frac{2}{27}a_2^2a_1^2 - 2a_0^2a_3^2$$

Since we want to take ξ_0 primitive, then we take

$$\xi_0 = \frac{1}{2}\left((f,f)_2, (f,f)_2\right)_2 = \frac{2}{3}a_1a_3a_0a_2 - \frac{4}{27}a_1^3a_3 - \frac{4}{27}a_2^3a_0 + \frac{1}{27}a_2^2a_1^2 - a_0^2a_3^2$$

2.3.2. *Quartics.* A generating set for $\mathcal{R}_4$ is $\xi = [\xi_0, \xi_1]$ with $\mathfrak{w} = (2,3)$, where

$$\xi_0 = \frac{1}{2}(f,f)_4 \quad \text{and} \quad \xi_1 = (f,(f,f)_2)_4$$

In terms of $a_0, \dots, a_4$, the invariants ξ_0 and ξ_1 are

$$\begin{aligned}
\xi_0 &= a_4a_0 - \frac{a_1a_3}{4} + \frac{a_2^2}{12} \\
\xi_1 &= a_2a_4a_0 - \frac{3}{8}a_1^2a_4 - \frac{3}{8}a_0a_3^2 + \frac{1}{8}a_2a_1a_3 - \frac{1}{36}a_2^3
\end{aligned}$$

2.3.3. *Quintics.* A generating set for $\mathcal{R}_4$ is $\xi = [\xi_0, \xi_1, \xi_2]$ with $\mathfrak{w} = (4, 8, 12)$, where

$$\xi_0 = \frac{1}{2}(c_1,c_1)_2, \quad \xi_1 = (c_4,c_1)_2, \quad \xi_2 = (c_4,c_4)_2,$$

for

$$c_1 = (f,f)_4, \ c_2 = (f,f)_2, \ c_3 = (f,c_1)_2, \ c_4 = (c_3,c_3)_2.$$

2.3.4. *Sextics.* The case of sextics was studied in detail due to their connection to genus 2 curves. Generating sets were known in detail by XIX-century mathematicians (Bolza, Clebsch, et al.) when char $k = 0$ and by Igusa for char $k > 0$. For a more modern treatment see [KSV05] or [MS17, BHK+18, MS19a], where invariants of binary sextics are used to study the moduli space of genus 2 curves and even expressed in terms of modular forms. To have a uniform treatment of invariants in this paper we will define a generating set for the case $d = 6$ slightly different from generating sets commonly used in literature.

Let $c_1 = (f,f)_4$, $c_3 = (f,c_1)_4$, $c_4 = (c_1,c_1)_2$. A generating set for $\mathcal{R}_6$ is $\xi = [\xi_0, \xi_1, \xi_2, \xi_3]$ with weights $\mathfrak{w} = (2,4,6,10)$ (we are assuming char $k \neq 2$), where

$$\xi_0 = \frac{1}{2}(f,f)_6, \ \xi_1 = \frac{1}{2}(c_1,c_1)_4, \ \xi_2 = \frac{1}{2}(c_4,c_1)_4, \ \xi_3 = (c_4,c_3^2)_4.$$

Remark 3. *The reader should be aware that usually the invariants of binary sextics are denoted by $[J_2, J_4, J_6, J_{10}]$ with J_{10} being the discriminant of the sextic, but that is not the case here.*

2.3.5. *Septics.* A generating set of $\mathcal{R}_7$ is given by $\xi = [\xi_0, \xi_1, \xi_2, \xi_3, \xi_4]$ with weights $\mathfrak{w} = (4, 8, 12, 12, 20)$. We define them as follows. Let

$$c_1 = (f, f)_6, \quad c_2 = (f, f)_4, \quad c_4 = (f, c_1)_2, \quad c_5 = (c_2, c_2)_4, \quad c_7 = (c_4, c_4)_4$$

and

$$\xi_0 = \frac{1}{2}(c_1, c_1)_2, \qquad \xi_1 = (c_7, c_1)_2, \qquad \xi_2 = \frac{1}{16}((c_5, c_5)_2, c_5)_4,$$
$$\xi_3 = \left((c_4, c_4)_2, c_1^3\right)_6, \quad \xi_4 = \frac{1}{64}\left([(c_2, c_5)_4]^2, (c_5, c_5)_2\right)_4$$

2.3.6. *Octavics.* Finding a generating set for the ring of invariants of binary octavics was one of the biggest achievements of classical invariant theory. A basis was determined by von Gall; see [Gal80]. Later the ring $\mathcal{R}_8$ was studied by Shioda in [Shi67], see Shaska in [Sha14] for details. A generating set of $\mathcal{R}_8$ is given by $\xi = [\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5]$ with weights $\mathfrak{w} = (2, 3, 4, 5, 6, 7)$. We define them as follows. Let

$$c_1 = (f, f)_6, \quad c_2 = (f, c_1)_4, \quad c_3 = (f, f)_4, \quad c_5 = (c_1, c_1)_2.$$

Then the invariants are:

$$\xi_0 = \frac{1}{2}(f, f)_8, \qquad \xi_1 = (f, c_3)_8, \qquad \xi_2 = \frac{1}{2}(c_1, c_1)_4,$$
$$\xi_3 = (c_1, c_2)_4, \qquad \xi_5 = \frac{1}{2}(c_5, c_1)_4, \qquad \xi_6 = ((c_1, c_2)_2, c_1)_4.$$

2.3.7. *Nonics.* A generating set of $\mathcal{R}_9$ is given by $\xi = [\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6]$ with weights $\mathfrak{w} = (4, 8, 10, 12, 12, 14, 16)$; see [BP10b]. Let

$$c_1 = (f, f)_8, \ c_2 = (f, f)_6, \ c_4 = (f, f)_2, \ c_5 = (f, c_1)_2, \ c_6 = (f, c_2)_6,$$
$$c_7 = (c_2, c_2)_4, \ c_9 = (c_5, c_5)_4, \ c_{21} = (f, c_2)_2, \ c_{25} = (c_4, c_4)_{10}, \ c_{27} = (c_6^3, c_6)_3$$

and

$$\xi_0 = \frac{1}{2}(c_1, c_1)_2, \quad \xi_1 = (c_2, c_6^2)_6, \quad \xi_2 = (((c_{25}, f)_6, c_{21})_5, c_2)_6, \quad \xi_3 = \frac{1}{16}((c_7, c_7)_2, c_7)_4$$
$$\xi_4 = (c_9, c_1^3)_6, \quad \xi_5 = (c_2, c_{27})_6, \quad \xi_6 = \left((c_5, c_5)_2, c_1^5\right)_{10}.$$

2.3.8. *Decimics.* A generating set of $\mathcal{R}_{10}$ is given by $\xi = [\xi_0, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8]$ with weights $\mathfrak{w} = (2, 4, 6, 6, 8, 9, 10, 14, 14)$; see [BP10a]. Let

$$c_1 = (f, f)_8, \qquad c_2 = (f, f)_6, \qquad c_5 = (f, c_1)_4, \qquad c_6 = (f, c_2)_8,$$
$$c_7 = (c_2, c_2)_6, \qquad c_8 = (c_5, c_5)_4, \qquad c_9 = (c_2, c_7)_4, \qquad c_{10} = (c_1, c_1)_2,$$
$$c_{16} = (c_5, c_5)_2, \qquad c_{19} = (c_5, c_1)_1, \qquad c_{25} = (c_7, c_7)_2$$

and

$$\xi_0 = \frac{1}{2}(f, f)_{10}, \qquad \xi_1 = \frac{1}{2}(c_1, c_1)_4, \qquad \xi_2 = (c_5, c_5)_6,$$
$$\xi_3 = (c_6, c_6)_2, \qquad \xi_4 = (c_1, c_8)_4, \qquad \xi_5 = (c_{19}, c_1^2)_8,$$
$$\xi_6 = (c_{16}, c_1^2)_8, \qquad \xi_7 = \frac{1}{4}(c_{25}, c_9)_4, \qquad \xi_8 = (c_{10}^2, c_{16})_8.$$

Remark 4. *Transvections don't always give an intuitive idea of what given invariant represents, but the benefit of using transvections to generate invariants is that such invariants are obtained in terms of the coefficients of the binary forms. On contrary, invariants generated by root differences, which we will see next, are easily associated to the action of the permutation group on the set of roots of the binary forms, however expressing such invariants in terms of the coefficients is usually computationally difficult.*

2.4. **Root differences.** Let $f \in V_d$, say

$$f(x,y) = \sum_{i=0}^{i=d} a_i x^i y^{d-i} = \prod_{i=0}^{i=d} (\beta_i x - a_i y). \tag{3}$$

Set $d_{ij} := \begin{pmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{pmatrix}$. For $M \in SL_2(k)$, we have

$$f^M = (\beta_1^{'} x - \alpha_1^{'} y) \dots (\beta_6^{'} x - \alpha_6^{'} y), \quad \text{with} \quad \begin{pmatrix} \alpha_i^{'} \\ \beta_i^{'} \end{pmatrix} = M^{-1} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}.$$

Clearly d_{ij} is invariant under this action of $SL_2(k)$ on $\mathbb{P}^1$. All invariants can be expressed in terms of root differences. For example, the discriminant of the binary form $f(x,y)$ is given by the formula

$$\Delta(f) = \prod_{i<j} d_{ij}^2$$

An excellent article on invariants including root differences is [KR84].

A classical nontrivial example to illustrate expressing invariants in terms of root differences is the case of binary sextics; see [KSV05] for details.

Example 1 (Binary sextics)**.** *Let $d = 6$ and $\{i,j,k,l,m,n\} = \{1,2,3,4,5,6\}$. Treating a_i as variables, we construct the following elements in $\mathcal{R}_6$.*

$$I_{10} = \prod_{i<j} d_{ij}^2, \qquad I_2 = \sum_{\substack{i<j,\\ k<l,\\ m<n}} d_{ij}^2 d_{kl}^2 d_{mn}^2,$$

$$I_4 = (4I_2^2 - B), \qquad I_6 = (8I_2^3 - 160 I_2 I_4 - C)$$

where

$$B = \sum_{\substack{i<j,\\ j<k,\\ l<m,\\ m<n}} d_{ij}^2 d_{jk}^2 d_{ki}^2 d_{lm}^2 d_{mn}^2 d_{nl}^2, \qquad C = \sum_{\substack{i<j\\ j<k\\ l<m\\ m<n\\ i<l^{'}\\ j<m^{'}\\ k<n^{'}}} d_{ij}^2 d_{jk}^2 d_{ki}^2 d_{lm}^2 d_{mn}^2 d_{nl}^2 d_{il^{'}}^2 d_{jm^{'}}^2 d_{kn^{'}}^2$$

for $l^{'}, m^{'}, n^{'} \in \{l,m,n\}$.

Remark 5. *Notice that there is a one to one correspondence between the equivalence classes of effective divisors of degree n in $\mathbb{P}^1(\bar{\mathbb{Q}})$. For any $f(x,y)$ as in Eq. (3) we can associate a divisor D as follows. Let $[a_0 : \cdots : a_n] \in \mathbb{P}^n_k$. Take the divisor of degree n, namely $D = \sum_i [x_i : y_i]$.*

Conversely for any divisor $D = \sum_i^d a_i P_i$, *where* $P_i = [x_i : y_i] \in \mathbb{P}^1_{\bar{\mathbb{Q}}}$ *and* $\sum a_i = n$, *we consider* $f(x,y) = \prod_i (xy_i - yx_i)$. *Denote* $f = \sum_{i=0}^d a_i x^i y^{d-i}$ *and consider* $[a_0 : \cdots : a_n] \in \mathbb{P}^n_k$. *These are called the* ***Chow coordinates*** *of* D.

It is the action on the roots that determines stability for binary forms. However, it would be more convenient that such action on roots is expressed as condition on invariants. While expressions of the above invariants are well known in terms of coefficients $a_0, \ldots, a_6$ from Bolza (1898) and many others since, for higher degree things get computationally more difficult.

It turns out that if the binary form has a root of high multiplicity all the invariants vanish and this is exactly the condition that determines stability as we will see next.

Lemma 2. *Let* $f \in V_d$.

(i) *If* f *a root of multiplicity* $r > \frac{d}{2}$ *then* $\xi(f) = (\xi_0, \ldots, \xi_n) = (0, \ldots, 0)$.

(ii) *If* d *is even, then all binary forms with a root of multiplicity* $\frac{d}{2}$ *have the same invariants.*

Proof. Let $m = \lfloor \frac{d}{2} \rfloor$. Hence, $d = 2m$ when d is even and $d = 2m+1$ when d is odd. Notice that every degree s invariant $J_s \in k[a_0, \ldots, a_d]$ is invariant under the permutation (a_i, a_{d-i}) for $i = 0, \cdots, m$, since such permutation corresponds to permuting x and y. If $f(x,y)$ has a root of multiplicity r then we can assume that $f(x,y) = x^r g(x,y)$ for some degree $(d-r)$ binary form $g(x,y)$, say

$$g(x,y) = b_{d-r} x^{d-r} + b_{d-r-1} x^{d-r-1} y + \cdots b_1 x y^{d-r-1} + b_0 y^{d-r}.$$

Then

$$f(x,y) = b_{d-r} x^d + b_{d-r-1} x^{d-1} y + \cdots b_1 x^{r+1} y^{d-r-1} + b_0 x^r y^{d-r}. \tag{4}$$

Hence every $\xi_i(f)$ will be written in terms of coefficients $b_0, \ldots, b_{d-r}$ or equivalently in terms of a_i, where

$$a_0 = \cdots = a_{r-1} = 0 \quad \text{and} \quad a_{r+j} = b_j, \quad \text{for } j = 0, \ldots, d-r.$$

Thus, evaluated at $f(x,y)$ as in Eq. (4) is given by a sum of degree s monomials in $a_r, \ldots, a_d$ since $a_0 = \cdots = a_{r-1} = 0$.

To prove part i) let $r = m+1$. Then all $a_0 = \cdots = a_{(m+1)-1} = 0$ which makes all $a_i = 0$, for $i = 0, \ldots, m$. Since J_s is invariant under the permutation (a_i, a_{d-i}) for $i = 0, \cdots, m$, above, then $J_s = 0$ which implies that $(\xi_0(f), \ldots, \xi_n(f)) = (0, \ldots, 0)$.

To prove ii) let $r = m$ and $d = 2m$. Then, all $a_0 = \cdots = a_{m-1} = 0$. Hence $a_m = b_0$ is the only nonzero coefficient. Since each $\xi_i(f)$ is invariant under the permutation (a_i, a_{d-i}) for $i = 0, \cdots, m$, above then each $\xi_i(f)$ is a degree q_i homogenous polynomial in b_0. Thus,

$$\xi(f) = [b_0^{q_0} \cdot \lambda_0 : b_0^{q_1} \cdot \lambda_1 : \ldots : b_0^{q_n} \cdot \lambda_n] = [\lambda_1 : \cdots : \lambda_n]$$

for some $\lambda_i \in k$. Hence, there is a unique set of invariants $\xi(f) = [\lambda_0, \lambda_1, \ldots, \lambda_n]$. This completes the proof. □

Remark 6. *The second part of the lemma says that when* d *is even there is only one binary form (up to equivalence) which has a root of multiplicity* $d/2$. *In the last section we will list all degree* d *binary forms and their invariants which have a root of multiplicity* $d/2$ *for* $4 \leq d \leq 10$.

3. Stability and the Hilbert-Mumford criterion.

Stability of binary forms has long been studied. In this section we want to give the basic definitions and terminology and give an outline of the main result which is Thm. 3. Our main references are [DM69] and [New09].

Let G be an algebraic group acting rationally on a variety $\mathcal{X}$ (that is, through a morphism

$$G \times \mathcal{X} \to \mathcal{X}$$
$$(g, x) \to g.x$$

We shall always write $G.x$ for the orbit

$$\{y \in \mathcal{X} : y = g.x \text{ for some } g \in G\}$$

of x. From now on we will assume that G is a **reductive** group.

Let $\mathcal{X} \subset \mathbb{P}^d_k$ and G act linearly on $\mathcal{X}$. Hence we can assume $G \leq \mathrm{GL}_2(k)$ acting on $\mathcal{X}$ in the natural way and $I \in k[a_0, \dots, a_d]$ a G-invariant polynomial. By $\mathcal{X}_I \subset \mathbb{P}^d(k)$ we denote the set

$$\mathcal{X}_I := \{\beta \in \mathcal{X} \,|\, I(\beta) \neq 0\}.$$

Definition 1. A point $\alpha \in \mathcal{X}$ is called **stable under the G-action** if α has a finite stabilizer G_α and there exist a G-invariant $I \in k[a_0, \dots, a_d]$ such that $\alpha \in \mathcal{X}_I$.

If we drop the condition that the stabilizer G_α is finite then $\alpha \in \mathcal{X}$ is called **semistable under the G-action**.

3.1. **Actions of $k^\star$.** Let $G = k^\star$ acting linearly on a projective variety $\mathcal{X} \subset \mathbb{P}^d(k)$. There exists a basis $\mathcal{B} := \{b_0, \dots, b_d\}$ such that this action can be diagonalized. In other words

$$t.b_i = t^{r_i} b_i,$$

for some integers r_i. Choose $\hat{\alpha}$ one of its pre-images of α under the natural projection $\pi : \mathbb{A}^{d+1}(k) \to \mathbb{P}^d_k$. Then

$$\hat{\alpha} = \sum_{i=0}^{d} \hat{\alpha}_i b_i.$$

for some $\hat{\alpha}_i \in k$. Then

$$t.\hat{\alpha} = \sum t^{r_i} \hat{a}_i b_i$$

We define

$$\mu(\alpha) := \max\{-r_i \,|\, \alpha_i \neq 0\}$$

and have the following:

Lemma 3. *For every $\alpha \in \mathcal{X}$, $\mu(\alpha)$ is the unique integer such that $\lim_{t \to 0} t^{\mu(\alpha)}(t.\alpha)$ exists and is nonzero. Moreover, $\mu(\alpha)$ is independent of $\hat{\alpha}$ or the basis $\mathcal{B}$ and*

(i) *$\mu(\alpha) > 0$ if and only if $\lim_{t \to 0}(t.\alpha)$ does not exist*
(ii) *$\mu(\alpha) = 0$ if and only if $\lim_{t \to 0}(t.\alpha)$ exist and is non-zero*

For a proof see [DM69] or [New09, pg. 7] among other places. Similarly define

$$\mu^-(\alpha) := \max\{r_i \,|\, \alpha_i \neq 0\}$$

Then we have the following:

Proposition 1. *The following hold:*

(i) *α is stable if and only if $\mu(\alpha) > 0$ and $\mu^-(\alpha) > 0$.*
(ii) *α is semistable if and only if $\mu(\alpha) \geq 0$ and $\mu^-(\alpha) \geq 0$.*

3.2. **1-parameter groups.** Consider now an arbitrary linear action of a reductive group G on a projective variety $\mathcal{X}$. A subgroup G of $\mathrm{SL}_n(k)$ is called a **1-parameter group** if there is a non-trivial homomorphism of algebraic groups $\lambda : k^* \to G$. For any 1-parameter subgroup G and a homomorphism λ, sometimes we write $\mu(\alpha, \lambda)$ for the value of $\mu(\alpha)$ for the action of k^* on $\mathcal{X}$ induced by λ.

The following result is often used as the definition of stability of binary forms. Its proof is done usually via the Hilbert-Mumford criteria.

Theorem 2 (Hilbert-Mumford Criterion)**.** *The following hold:*

(i) *α is stable if and only if $\mu(\alpha, \lambda) > 0$ for every 1-PS λ of G.*
(ii) *α is semistable if and only if $\mu(\alpha, \lambda) \geq 0$ for every 1-PS λ of G.*

Remark 7. *For the semistable case when G is $\mathrm{SL}_n(\mathbb{C})$ there is a proof given by Hilbert using convergent power series. Mumford and Seshadri proved it for all k and all reductive G using formal power series and a theorem of Iwahori.*

3.3. **Binary forms.** Next we apply the above results to the case of binary forms. Any 1-PS subgroup of ${}_2(k)$ is conjugate to a form λ_r for some $r \geq 0$ such that

$$\lambda_r(t) = \begin{bmatrix} t^r & 0 \\ 0 & t^{-r} \end{bmatrix}$$

Hence, we have the following:

Theorem 3. *A binary form $f(x, y) \in k[x, y]$ of degree $\deg f = d$ is stable if and only if all roots of f are of multiplicity $< \frac{d}{2}$ and semistable if and only if all roots are of multiplicity $\leq \frac{d}{2}$.*

Proof. Any one parameter subgroup G of $\mathrm{SL}_2(k)$ is given by

$$\lambda(t) = \left\{ \begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix} : t \in k^\star \right\}$$

for some $r \geq 0$. Then for $f(x, y) = \sum_{i=0}^{d} a_i x^i y^{d-i}$ we have

$$\lambda(t) \cdot f(x, y) = \sum t^{r(2i-d)} a_i x^i y^{d-i}.$$

Hence

$$\mu(f, \lambda) = -\min\{2i - d : a_i \neq 0\} = \max\{2i - d : a_i \neq 0\} = 2i_0 - d,$$

where i_0 is the largest integer for which $\alpha_i \neq 0$. Hence, when $\mu(f, \lambda) \geq 0$ we have $[0, 1]$ as a root with multiplicity at most $\frac{d}{2}$ and when $\mu(f, \lambda) > 0$ then $[0, 1]$ has multiplicity strictly less than $\frac{d}{2}$. This completes the proof. □

Definition 4. If a degree $d \geq 2$ binary form $f(x, y)$ has roots of multiplicity $\frac{d}{2}$ we say that f is **strictly semistable**.

Corollary 1. *A binary form $f(x, y)$ of degree $\deg f = d$ is unstable if and only if $\xi(f) = \mathbf{0}$ in $\mathbb{P}^n_{\mathfrak{w},k}$. Moreover, if d is even there is only one strictly semistable point in the moduli space and there are no such points when d is odd.*

Proof. We can assume that $[0, 1]$ is a root of multiplicity $d/2$. Then $f(x, y)$ can be written as

$$f(x, y) = x^{\frac{d}{2}} \cdot \left(x^{d/2} + a_{\frac{d}{2}-1} x^{\frac{d}{2}-1} y + \cdots + a_1 x y^{\frac{d}{2}-1} + a_0 y^{\frac{d}{2}} \right) \tag{5}$$

From Thm. 3 there is only one point in the moduli corresponding to such binary forms. □

4. Minimal models and stability

Next we want to focus on the stability of binary forms over number fields. As above, we let k be a number field and $\mathcal{O}_k$ its ring of integers. M_k denotes the set of places of k, where M_k are the Archimedean places and M_∞ the non Archimedean places of k. For any norm $\nu \in M_k$, the completion of k at ν is denoted by k_ν.

Historically, minimal models of binary forms or hyperelliptic curves have been considered for obvious reasons. There are two main ways to consider *minimality*; minimality in terms of the coefficients, and minimality in terms of invariants. Getting sucha minimal models is normally refereed to as *reduction*. In [Sha22] the author calls them Type A and Type B reduction and points out that a similar approach was also considered in the seminal paper of Birch and Swinnerton-Dyer; see [BSD63, BSD65].

In [Bur92], Burnol proves that f is minimal if and only if its reduction is semistable under the $\mathrm{SL}_2(\bar{k})$-action. In other words, minimality is equivalent to residual semistability. It was this type of statement in terms of weighted moduli height which was one of our main motivations for this paper.

4.1. **Minimal models.** Let $f \in \mathcal{O}_k[x,y]$ and $\mathbf{x} := \xi(f) \in \mathbb{P}^n_{\mathfrak{w}}(\mathcal{O}_k)$ its corresponding weighted moduli point. The following terminology is commonly used for algebraic curves, especially hyperelliptic and superelliptic curves; see [Sha22].

A **local minimal model** for a binary form f defined over a number field k at a prime $\mathfrak{p}$ of k is an equivalent binary form g all of whose coefficients are integral at $\mathfrak{p}$, and whose moduli point $\xi(g)$ has minimal valuation at $\mathfrak{p}$ among all such equivalent binary forms. A **global minimal model** for a binary form f defined over a number field k is an equivalent binary form g which is integral and is a local minimal model at all primes $\mathfrak{p}$ of k.

We define the **weighted valuation** of the tuple $\mathbf{x} = (x_0, \dots x_n)$ at the prime $p \in \mathcal{O}_k$ as

$$\mathbf{val}_p(\mathbf{x}) := \max \left\{ j \,|\, p^j \text{ divides } x_i^{q_i} \text{ for all } i = 0, \dots n \right\}, \tag{6}$$

We say that a binary form $f(x,y)$ has a **integral minimal model** over k if it is integral (i.e. $f \in \mathcal{O}_k[x,y]$) and $\mathbf{val}_p(\xi(f))$ is minimal for every prime $p \in \mathcal{O}_k$.

Lemma 4. *A binary form $f \in V_d$ is a minimal model over $\mathcal{O}_k$ if for every prime $p \in \mathcal{O}_k$ such that $p \,|\, \mathrm{wgcd}\,(\xi(f))$ the following holds*

$$\mathbf{val}_p(\xi(f)) < \frac{d}{2} q_i, \quad \textit{for all} \quad i = 0, \dots, n. \tag{7}$$

Moreover, for every integral binary form f its minimal model exist; see [Sha22] .

Remark 8. *Notice that an integral minimal model is not necessary semistable, since its moduli point can have minimal evaluation and still can be zero.*

4.2. **Local and global stability.** Take $\mathfrak{p} = \xi(f) \in \mathbb{P}^n_{\mathfrak{w},k}$. We can assume that $\xi(f) = [\xi_0 : \cdots : \xi_n]$ has coordinates in $\mathcal{O}_k$. Further assume that $\mathfrak{p}$ is normalized (i.e. $\mathrm{wgcd}\,(\xi_0, \dots, \xi_n) = 1$). Let p be a prime in $\mathcal{O}_k$ such that $p \mid \gcd(\xi_0, \dots, \xi_n)$. Then f is unstable over the residue field $\mathcal{O}_k/p\mathcal{O}_k$. Next we show how to determine an equivalent binary form $g(x,y)$ to $f(x,y)$ which is semistable over the residue field $\mathcal{O}_k/p\mathcal{O}_k$.

Lemma 5. *Let $f \in \mathcal{O}_k[x,y]$ and $\mathfrak{p} = \xi(f) = [\xi_0, \dots, \xi_n] \in \mathbb{P}^n_{\mathfrak{w}}(\mathcal{O}_k)$.*

(i) *f is a semistable binary form over the residue field $\mathcal{O}_k/p\mathcal{O}_k$ if and only if $p \nmid \gcd(\xi_0, \dots, \xi_n)$.*

(ii) *If $p \mid \gcd(\xi_0, \dots, \xi_n)$ let*

$$\alpha_p := \min\{|x_i|_p \mid \text{ such that } x_i \neq 0 \text{ and } i = 0, \dots, n.\}.$$

Then f^M is semistable over the residue field $\mathcal{O}_k/p\mathcal{O}_k$ for $M = \begin{bmatrix} \frac{1}{p^{r_p}} & 0 \\ 0 & 1 \end{bmatrix}$, where

$$r_p = \frac{2\alpha_p}{d \cdot q_j}, \tag{8}$$

for some $j \in \{0, 1, \cdots, n\}$ such that $\xi_j \neq 0$.

Proof. From Lem. 5, a binary form f is semistable if and only if there exists some ξ_j such that $\xi_j \neq 0$ in $\mathcal{O}_k/p\mathcal{O}_k$. Hence, the first claim of the theorem.

Assume $p \mid \xi_i$, for all $i = 0, \dots n$. We can further assume that $\operatorname{wgcd}(\xi(f)) = 1$ so f is minimal as in [BGS20, Prop. 6]. Pick ξ_j such that $\xi_j \neq 0$. Let $\xi_j = p^\alpha \beta$ such that $\gcd(\alpha, \beta) = 1$ and take

$$M = \begin{bmatrix} \frac{1}{p^r} & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_2(\bar{k})$$

for $r = \frac{2\alpha}{dq_j}$. Then $f^M(x, y) = f\left(\frac{x}{p^r}, y\right)$ and from Eq. (2) we have

$$\xi(f^M) = \left[\left(\frac{1}{p^r}\right)^{\frac{1}{2}dq_0} \xi_0, \dots, \beta, \dots \left(\frac{1}{p^r}\right)^{\frac{1}{2}dq_n} \xi_n \right]$$

and $p \nmid \beta$. This completes the proof. □

A point $\mathfrak{p} = \xi(f) \in \mathbb{P}^n_{\mathfrak{w},k}$ is unstable if there is a prime $p \in \mathcal{O}_k$ such that $p \mid \gcd(\xi_0, \dots, \xi_n)$. Assume there is such a $p \mid \gcd(\xi_0, \dots, \xi_n)$.

Proposition 2. *Let $f \in \mathcal{O}_k[x, y]$ be a semistable binary form and $\xi(f) = [\xi_0, \dots, \xi_n] \in \mathbb{P}^n_{\mathfrak{w}}(\mathcal{O}_k)$ its moduli point. Assume $\xi(f)$ is normalized (i.e. multiply $\xi(f)$ by $\frac{1}{\operatorname{wgcd}(\xi(f))}$). Let*

$$\lambda = \prod_{p \mid \gcd(\xi_0, \dots, \xi_n)} p^{r_p}$$

where r_p is as in Eq. (8) and take $M = \begin{bmatrix} \frac{1}{\lambda} & 0 \\ 0 & 1 \end{bmatrix}$. Then f^M is semistable over all residue fields $\mathcal{O}_k/p\mathcal{O}_k$ for all primes $\mathfrak{p} \in \mathcal{O}_k$.

Proof. If $\mathfrak{p}$ is normalized then $\operatorname{wgcd}(\xi_0(f), \dots, \xi_n(f)) = 1$. Let

$$\gcd(\xi_0(f), \dots, \xi_n(f)) = \prod_{i=1}^{s} p_i^{a_i},$$

where $p_i \in \mathcal{O}_k$ are primes. Then from the above Lemma, exists r_i such that for $M_i = \begin{bmatrix} \frac{1}{\lambda_i} & 0 \\ 0 & 1 \end{bmatrix}$ the form f^{M_i} is semistable over k_{p_i}. Let $M = \prod_{i=1}^{s} M_i$. Then f^M is semistable for every prime $p_i \mid \gcd(\xi_0(f), \dots, \xi_n(f))$, hence it is semistable over all $\mathcal{O}_k/p\mathcal{O}_k$. □

A prime $p \in \mathcal{O}_k$ is called a **bad prime** for f if $p \mid \gcd(\xi_0(f), \dots, \xi_n(f))$. In this case $f \mod p$ is unstable. However, there might still exist a twist of f, say $\tilde{f}$ such that $\tilde{f} \mod p$ is semistable.

Proposition 3. *Let f be a binary form which is semistable over k. Then for each prime $p \in \mathcal{O}_k$ there exists a twist $\tilde{f}$ of f which is semistable over $\mathcal{O}_k/p\mathcal{O}_k$.*

Proof. Let $\xi(f) = [\xi_0(f), \dots, \xi_n(f)]$ be the corresponding moduli point and p a bad prime. Then $p | \xi_j(f)$ for all $j = 1, \dots, n$. Without loss of generality we can assume that $\operatorname{wgcd}(\xi(f)) = 1$. There exists $1 \leq s \leq n$ such that

$$\mathbf{val}_p(\xi_s(f)) = \min\{\mathbf{val}_p(\xi_s(f)) \mid j = 1, \dots, n\}$$

Denote by $\lambda = \frac{1}{p^r}$, where $r = \frac{1}{q_s}\mathbf{val}_p(\xi_s(f))$. Consider $\tilde{f} = f^M$. Then,

$$\begin{aligned} \xi(\tilde{f}) &= \lambda \star [\xi_0(f) : \dots : \xi_s(f) : \dots : \xi_n(f)] \\ &= [\lambda^{q_0} \cdot \xi_0(f) : \dots : \lambda^{q_s} \cdot \xi_s(f) : \dots : \lambda^{q_n} \cdot \xi_n(f)], \end{aligned}$$

where $\xi_s(\tilde{f})$ is not divisible by p and $\tilde{f}$ is semistable over $\mathcal{O}_k/p\mathcal{O}_k$. □

A prime $p \in \mathcal{O}_k$ is called a *prime of good reduction for f* if f is semistable over $\mathcal{O}_k/p\mathcal{O}_k$.

5. Stability, weighted height, and invariant height

Let k be an algebraic number field of degree $m = [k : \mathbb{Q}]$, and $\bar{k}$ be an algebraically closed field containing k. We denote by $\mathcal{O}_k$ the ring of algebraic integers in k.

Denote by M_k the set of all places of k, i.e., the equivalent classes of absolute values on k. It is a disjoint union of M_k^0, the set of all non-archimedian places, and M_k^∞, the set of all Archimedean places of k. More precisely, if $\nu \in M_k^0$, then $\nu = \nu_\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset \mathcal{O}_k$ over a prime element p such that $\nu_\mathfrak{p}|_\mathbb{Q}$ is the p-adic absolute value. If $\nu \in M_k^\infty$, then $\nu = \nu_\infty$ and $\nu_\infty|_\mathbb{Q}$ is the usual absolute value $|\cdot|_\infty$ on $\mathbb{Q}$.

The **local degree** n_ν at $\nu \in M_k$ is defined by $n_\nu = [k_\nu : \mathbb{Q}_\nu]$, where k_ν and $\mathbb{Q}_\nu$ are the completions with respect to ν. For each $\nu \in M_k$, we let $|\cdot|_\nu$ be a representative of the equivalence class which is the n_ν-th power of the one that extends a normalized absolute value over $\mathbb{Q}$. Since k is a number field, then for every $x \in k^*$ we have the **product formula** $\prod_{\nu \in M_k} |x|_\nu = 1$.

Given a finite field extension K/k, we denote by M_K the set of places w on K such that $w\,|_k = \nu$, for some $\nu \in M_k$. Then, we have the **degree formula** as $\sum_{\substack{w \in M_K \\ w|_k = \nu}} [K_w : k_\nu] = [K : k]$.

5.1. **Heights.** For $x \in k^*$ **multiplicative** and **logarithmic height** are defined by

$$H_k(x) = \prod_{\nu \in M_k} \max\{1, |x|_\nu\} \quad \text{and} \quad h_k(x) = \log H_k(x) = \sum_{\nu \in M_k} \log |x|_\nu. \tag{9}$$

For $\tilde{x} = (x_0, \cdots, x_n) \in k^{n+1}$ and $v \in M_k$, we let

$$|\tilde{x}|_\nu = \max\{|x_i|_\nu : 0 \leq i \leq n\}.$$

One can extend such definitions to the projective space by defining the **multiplicative and logarithmic height** of $\mathbf{x} = [x_0 : \cdots : x_n] \in \mathbb{P}^n(k)$ by

$$\begin{aligned} H_k(\mathbf{x}) &= \prod_{\nu \in M_k} \max_{0 \leq i \leq n} \{|x_i|_\nu\} \\ h_k(\mathbf{x}) &= \log H_k(\mathbf{x}) = \sum_{\nu \in M_k} \max_{0 \leq i \leq n} \{\log |x_i|_\nu\}. \end{aligned} \tag{10}$$

Note that such height functions are independent of the choice of the coordinates.

Let K be a number field containing k. For $w \in M_K$, we normalize the absolute value $|\cdot|_w$ such that its restriction $|\cdot|_\nu$ on k satisfies $|\cdot|_\nu = |\cdot|_w^{[K_\nu : k_\nu]}$. Thus, for $x \in k^*$ we have

$$H_k(x) = H_K(x)^{1/[K:k]}, \text{ and } h_k(x) = \frac{1}{[K:k]} h_K(x), \tag{11}$$

and for $\mathbf{x} \in \mathbb{P}^n_k$

$$\begin{aligned} H_k(\mathbf{x}) &= H_K(\mathbf{x})^{1/[K:k]} \\ h_k(\mathbf{x}) &= \frac{1}{[K:k]} h_K(\mathbf{x}). \end{aligned} \tag{12}$$

The **field of definition** of $\mathbf{x} \in \mathbb{P}^n(\bar{k})$ is $k(\mathbf{x}) = k\left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right)$, for any i such that $x_i \neq 0$. The **absolute multiplicative** and **logarithmic global Weil heights** of $x \in \bar{k}^*$ are defined by

$$H(x) = H_K(x) \text{ and } h(x) = h_K(x),$$

and for $\mathbf{x} \in \mathbb{P}^n(\bar{k})$ by

$$H(\mathbf{x}) = H_K(\mathbf{x}) \text{ and } h(\mathbf{x}) = h_K(\mathbf{x}), \tag{13}$$

where K is a number field containing $k(\mathbf{x})$. The absolute heights are independent of the choice of K. We call $h(\mathbf{x})$ as **global Weil height** on $\mathbb{P}^n(\bar{k})$.

5.2. **Heights on weighted projective spaces.** In this section we briefly define weighted heights (i.e. heights on weighted projective spaces), invariant heights[1] and investigate how such heights behave on strictly semistable points.

The group action $k^\star$ on $\mathbb{A}^{n+1}(k)$ induces a group action of $\mathcal{O}_k$ on $\mathbb{A}^{n+1}(k)$. By $\mathrm{Orb}(\mathfrak{p})$ we denote the $\mathcal{O}_k$-orbit in $\mathbb{A}^{n+1}(\mathcal{O}_k)$.

For any point $\mathfrak{p} = [x_0 : \cdots : x_n] \in \mathbb{P}^n_{\mathfrak{w},k}$ we can assume, without loss of generality, that $\mathfrak{p} = [x_0 : \cdots : x_n] \in \mathbb{P}^n_{\mathfrak{w},k}(\mathcal{O}_k)$. The height for weighted projective spaces will be defined in the next section.

Let $\mathfrak{w} = (q_0, \dots, q_n)$ be a set of weights and $\mathbb{P}^n_{\mathfrak{w},k}$ the weighted projective space over a number field k. Let $\mathfrak{p} \in \mathbb{P}^n_{\mathfrak{w},k}$ a point such that $\mathfrak{p} = [x_0, \dots, x_n]$. We define the **weighted multiplicative height** of $\mathfrak{p}$ as

$$\mathcal{S}_{\mathrm{k}}(\mathfrak{p}) := \prod_{v \in M_k} \max \left\{ |x_0|_v^{\frac{n_v}{q_0}}, \dots, |x_n|_v^{\frac{n_v}{q_n}} \right\}. \tag{14}$$

The **logarithmic height** of the point $\mathfrak{p}$ is defined as follows

$$\mathfrak{s}_{\mathrm{k}}(\mathfrak{p}) := \log \mathcal{S}_{\mathrm{k}}(\mathfrak{p}) = \sum_{v \in M_k} \max_{0 \le j \le n} \left\{ \frac{n_v}{q_j} \cdot \log |x_j|_v \right\}. \tag{15}$$

$\mathcal{S}_{\mathrm{k}}(\mathfrak{p})$ is well defined and $\mathcal{S}_{\mathrm{k}}(\mathfrak{p}) \geq 1$ for any $\mathfrak{p} \in \mathbb{P}^n_{\mathfrak{w},k}$, see [BGS20] or [SS22]. The **absolute (multiplicative) weighted height** of $\mathfrak{p} \in \mathbb{P}^n_{\mathfrak{w},k}$ is the function

$$\begin{aligned} \mathcal{S} : \mathbb{P}^n_{\mathfrak{w},\bar{\mathbb{Q}}} &\to [1, \infty) \\ \mathcal{S}(\mathfrak{p}) &= \mathcal{S}_{\mathrm{k}}(\mathfrak{p})^{1/[k:\mathbb{Q}]}, \end{aligned}$$

[1] Sometimes called GIT heights

where $\mathfrak{p} \in \mathbb{P}^n_{\mathfrak{w},k}$, for any k which contains $\mathbb{Q}(\overline{\mathrm{wgcd}}\,(\mathfrak{p}))$. The **absolute (logarithmic) weighted height** on $\mathbb{P}^n_{\mathfrak{w},\bar{\mathbb{Q}}}$ is the function

$$\begin{aligned}\mathfrak{s} : \mathbb{P}^n_{\mathfrak{w},\bar{\mathbb{Q}}} &\to [0,\infty)\\ \mathfrak{s}(\mathfrak{p}) = \log\, \mathcal{S}_{\mathrm{k}}(\mathfrak{p}) &= \frac{1}{[k:\mathbb{Q}]}\, \mathcal{S}_k(\mathfrak{p}).\end{aligned}$$

where again $\mathfrak{p} \in \mathbb{P}^n_{\mathfrak{w},k}$, for any k which contains $\mathbb{Q}(\overline{\mathrm{wgcd}}\,(\mathfrak{p}))$. For more details on the theory of weighted heights see [BGS20] and [SS22].

5.3. **Invariant height.** Let $v \in M_k$ be a place and $\xi(f)$ the set of invariants of a degree d binary form $f \in V_d$. We define the norm

$$|\ \xi(f)\ | := \max_{0\leq i\leq n} \left\{ \|\xi_i\|_v^{\frac{1}{q_i}} \right\},$$

and

$$\|\xi\|_v^t\,(f) := \frac{|\ \xi(f)\ |^t}{\max_i\{|\ f_i\ |_v^t\}}$$

see [Rab13] or [RS15] for details.

Let D be the divisor determined by the roots of $f(x,y)$ via the Chow coordinates as in Remark 5. The invariant height of the divisor D is defined

$$h(D) := \left(\prod_{v\in M_k} \inf_{M\in \mathrm{SL}_2(\mathbb{C})} \left(\|\xi\|_v^t\,(f^M) \right)^{-\frac{1}{t}} \right)^{\frac{1}{[k:\mathbb{Q}]}} \tag{16}$$

The **logarithmic invariant height** of D is defined as

$$\hat{h}(D) = \frac{1}{[k:\mathbb{Q}]} \sum_{v\in M_k} \inf_{M\in \mathrm{SL}_2(\mathbb{C})} \left(\frac{-\log \|\xi\|_v^t\,(f^M)}{t} \right) \tag{17}$$

as defined in [Zha96] or [Rab13].

Remark 9. *In* [Rab13, Thm.4.4.2] *it is claimed that if f is a degree* $\deg f = d$ *semi-stable binary form then* $\hat{h}(f) \geq 0$. *Moreover, if* $\Delta_f \neq 0$ *then*

$$\hat{h}(f) \geq \frac{d}{2} \log \frac{4}{3}$$

If f is a stable binary form then $\hat{h}(f) > 0$.

In our comparison between the weighted height and invariant height for binary forms $f(x,y) = x^d - y^d$ we will use repeatedly the following theorem.

Theorem 5 (Them. 4.3.3 [Rab13])**.** *Let $d \geq 3$. Then if $d = p^r$ for some prime p then*

$$\hat{h}(x^d - y^d) = \frac{d}{2}\log 2 - \frac{p}{2(p-1)}\log p,$$

otherwise $\hat{h}(x^d - y^d) = \frac{d}{2}\log 2$.

5.4. **Weighted height of semistable binary forms.** On contrary to the invariant height in [Zha96] the weighted moduli height is very easily computed once the set of invariants is known. While the generators $\xi_0, \dots, \xi_n$ of the invariant ring $\mathcal{R}_d$ are chosen as primitive polynomials (see Lem. 1), then weighted height $\mathcal{S}_{\mathrm{k}}$ is smaller than the invariant height as we will see next. First we determine the heights of strictly semistable and semistable points in $\mathbb{P}^n_{\mathfrak{w},k}$.

Theorem 6. *Let $d \geq 3$, k be a number field, $f \in V_d$ an integral form defined over k, and $\mathfrak{p} = \xi(f) \in \mathbb{P}^n_{\mathfrak{w},k}$ the moduli point in the corresponding weighted projective space. If f is semistable, then $\mathfrak{s}(\xi(f)) \geq 0$. Moreover, for every $d \geq 4$, there exist an integral binary form $g \in V_d$, defined over k, such that $\mathfrak{s}\left(\xi(g)\right) = 0$.*

Proof. Let f be semistable and $\mathfrak{p} = \xi(f)$ the corresponding moduli point $\mathfrak{p} \in \mathbb{P}^n_{\mathfrak{w},k}$. Then at least one of the coordinates of $\mathfrak{p}$ is nonzero. Without loss of generality we can assume that $\mathfrak{p} \in \mathbb{P}^n_{\mathfrak{w},k}(\mathcal{O}_k)$. Then, $\mathcal{S}(\mathfrak{p}) \geq 1$. Hence, $\mathfrak{s}(\xi(f)) \geq 0$.

To prove the second statement we take $\mathfrak{p}$ with all coordinates 0 or ± 1, but not all coordinates zero. All such points have weighted moduli height $\mathcal{S}(\mathfrak{p}) = 1$ and $\mathfrak{s}(\mathfrak{p}) = 0$. Since a generic form has no automorphisms, from a theorem of Shimura such binary forms are defined over k. ☐

Remark 10. *It is interesting to know some statistical evidence on the number of moduli points of height $\mathfrak{s}(f) = 0$ which are defined over their field of moduli. For example, for $d = 6$ such evidence was provided in* [BHK+18]*; see last table.*

A natural question is to understand what happens to the lower bound of the weighted height as d increases. There is no known estimate for $\frac{\mathcal{S}(\xi(f))}{d}$ as$d \to \infty$. We compute such heights for some small values of d for strictly semistable binary forms.

Lemma 6. *If f is strictly semistable then $d = \deg f$ is even and its absolute weighted moduli height $\mathcal{S}(\xi(f)$ and absolute logarithmic weighted height $\mathfrak{s}(\xi(f))$, for $d = 4, 6, 8, 10$ are determined in Table 1.*

Proof. Let f be a binary form which is strictly semistable. Then f can be written as in Eq. (5). Without loss of generality we can further assume that $a_0 = 1$. Then, computing invariants for $d = 4$ we get

$$\xi(f) = [\xi_0 : \xi_1] = \left[\frac{1}{12} : -\frac{1}{36}\right] = [3 : -6].$$

Its weighted height is

$$\mathcal{S}\left([3:-6]\right) = \max\left\{\sqrt{3}, 6^{\frac{1}{3}}\right\} = 6^{\frac{1}{3}} \approx 1.817$$

Let f be a sextic with a root of multiplicity 3. Invariants of f with weights $\mathfrak{w} = (2, 4, 6, 10)$ are given by

$$\begin{aligned}[\xi_0 : \xi_1 : \xi_2 : \xi_3] &= \left[-\frac{1}{(2)^3\,(5)}, \frac{1}{(2)^2\,(3)\,(5)^4}, -\frac{1}{(2)^3\,(3)^2\,(5)^6}, -\frac{1}{(2)^4\,(3)^2\,(5)^{10}}\right] \\ &= \left[-3\cdot 5 : 2^4\cdot 3 : 2^6\cdot 3 : 2^{10}\cdot 3^3\cdot 5\right]\end{aligned}$$

Its weighted height is

$$\mathcal{S}(\xi(f)) \approx 3.872$$

An octavic $f(x,y)$ is strictly semistable if and only if the basic invariants with weights $\mathfrak{w} = (2,3,4,5,6,7)$ take the form

$$
\begin{aligned}
\xi(f) &= [\xi_0 : \xi_1 : \xi_2 : \xi_3 : \xi_4 : \xi_5] \\
&= \left[\frac{1}{2^2 \cdot 5 \cdot 7} : \frac{1}{2^2 \cdot 5^2 \cdot 7^3} : \frac{1}{2^4 \cdot 3 \cdot 7^4} : \frac{1}{2^3 \cdot 5 \cdot 7^5} : \frac{1}{2^6 \cdot 3^2 \cdot 7^6} : \frac{1}{2^3 \cdot 3 \cdot 5 \cdot 7^7} \right] \\
&= \left[3^2 \cdot 5 \cdot 7 : 2 \cdot 3^5 \cdot 5 : 3^3 \cdot 5^4 : 2^2 \cdot 3^5 \cdot 5^4 : 3^4 \cdot 5^6 : 2^2 \cdot 3^6 \cdot 5^6 \right]
\end{aligned}
$$

Its weighted height is

$$\mathcal{S}(\xi(f)) = 3\sqrt{5 \cdot 7} \approx 17.748$$

For the decimic binary form

$$f(x,y) = x^5 \left(x^5 + x^4 y a_4 + x^3 y^2 a_3 + x^2 y^3 a_2 + x\, y^4 a_1 + y^5 \right)$$

the strictly semistable point is given below:

$$
\begin{aligned}
\xi(f) &= [\xi_0 : \xi_1 : \xi_2 : \xi_3 : \xi_4 : \xi_5 : \xi_6 : \xi_7 : \xi_8] \\
&= \left[-\frac{1}{2^3 3^2 7}, \frac{1}{2^4 3^7 7^2}, -\frac{5}{2^4 3^{10} 7^4}, -\frac{1}{2^7 3^8 7^4}, \frac{1}{2^4 3^{14} 7^5}, 0, \right. \\
&\qquad \left. -\frac{1}{2^6 3^{16} 7^7}, -\frac{1}{2^{21} 3^{15} 5 \cdot 7^{14}}, -\frac{1}{2^{10} 3^{24} 7^9} \right] \\
&= \left[-2 \cdot 3^2 \cdot 5 \cdot 7 : 2^4 \cdot 3 \cdot 5^2 \cdot 7^2 : -2^8 \cdot 3^2 \cdot 5^4 \cdot 7^2 : -2^5 \cdot 3^4 \cdot 5^4 \cdot 7^2 : \right. \\
&\qquad \left. 2^{12} \cdot 3^2 \cdot 5^4 \cdot 7^3 : 0 : -2^{14} \cdot 3^4 \cdot 5^5 \cdot 7^3 : -2^7 \cdot 3^{13} \cdot 5^6 : -2^{18} \cdot 3^4 \cdot 5^7 \cdot 7^5 \right]
\end{aligned}
$$

Its weighted height is

$$\mathcal{S}_{\mathrm{k}}(\xi(f)) = 3\sqrt{70} \approx 25.099$$

This completes the proof. ☐

TABLE 1. Strictly semistable points and their weighted heights

d	$\xi(f)$	$\mathcal{S}\,\xi(f)$	$\mathfrak{s}\,\xi(f)$	$\hat{h}(f)$
4	$[3 : -6]$	1.817	0.259	0.249
6	$\left[-3 \cdot 5 : 2^4 \cdot 3 : 2^6 \cdot 3 : 2^{10} \cdot 3^3 \cdot 5\right]$	3.872	0.588	0.375
8	$\left[3^2 \cdot 5 \cdot 7 : 2 \cdot 3^5 \cdot 5 : 3^3 \cdot 5^4 : 2^2 \cdot 3^5 \cdot 5^4 : 3^4 \cdot 5^6 : 2^2 \cdot 3^6 \cdot 5^6\right]$	17.748	1.249	0.499
10	$\left[-2 \cdot 3^2 \cdot 5 \cdot 7 : 2^4 \cdot 3 \cdot 5^2 \cdot 7^2 : -2^8 \cdot 3^2 \cdot 5^4 \cdot 7^2 : -2^5 \cdot 3^4 \cdot 5^4 \cdot 7^2 : 2^{12} \cdot 3^2 \cdot 5^4 \cdot 7^3 : -2^{14} \cdot 3^4 \cdot 5^5 \cdot 7^3 : -2^7 \cdot 3^{13} \cdot 5^6 : -2^{18} \cdot 3^4 \cdot 5^7 \cdot 7^5\right]$	25.099	1.399	0.625

In the fourth column of Table 1 we have presented the logarithmic weighted height. It seems that such logarithmic height increases steadily as d increases. It would be interesting to determine how fast the logarithmic height increases and how does it compare to the invariant height (which is also a logarithmic height) as defined in [Zha96].

One obvious observation from Thm. 6 and Table 1 seems that the weighted height $\mathfrak{s}(\xi(f))$ seems to be growing fast as d increases. This seems to be different from the behavior of the invariant height and the results in [Rab13]. In order to compare in more detail the weighted height with the invariant height, we consider the family of binary forms

$$f(x,y) = x^d - y^d,$$

for which we have a lower bound for the height, see Thm. 5.

Table 2. Weighted heights for $f(x^d - y^d)$

d	$\mathfrak{w}$	$\xi(f) = [\xi_0(f) : \cdots : \xi_n(f)]$	$\mathcal{S}(\xi(f))$	$\mathfrak{s}(f)$	$\hat{h}(f)$
3	(4)	[-1]	1	0	0.09
4	$(2,3)$	[-1, 0]	1	0	0.301
5	$(4,8,12)$	[-1, 0, 0]	1	0	0.315
6	$(2,4,6,10)$	$\left[-1, \frac{1}{3}, \frac{1}{9}, 0\right]$	$\sqrt{3}$	0.239	0.903
7	$(4,8,12,12,20)$	$\left[-1, 0, -\frac{1}{140625}, 0, -\frac{1}{87890625}\right]$	$75^{\frac{1}{4}}$	0.469	0.56
8	$(2,3,4,5,6,7)$	$\left[-1, 0, \frac{1}{3}, 0, \frac{1}{9}, 0\right]$	$\sqrt{3}$	0.239	0.903
9	(4,8,10,12,12,14,16)	$\left[-1, 0, 0, -\frac{1}{140625}, 0, 0, 0\right]$	$75^{\frac{1}{4}}$	0.469	0.996
10	(2,4,6,6,8,9, 10, 14, 14)	$\left[-1 : \frac{1}{3} : 0 : 0 : 0 : 0 : 0 : -\frac{1}{12353145} : 0\right]$	$\sqrt{105}$	1.011	1.505

Lemma 7. *Let $f(x,y) = x^d - y^d$ and $\mathfrak{s}(f)$ its absolute logarithmic weighted height obtained by the choice of invariants $\xi = [\xi_0 : \cdots : \xi_n]$, as in Section 2.3. Then, for $3 \leq d \leq 10$, $\mathfrak{s}(f)$ is computed in Table 2*

Proof. Computing the weighted height of cases $d = 3, \ldots, 5$ is quite easy. We illustrate with $d = 6$. The weights are $\mathfrak{w} = (2,4,6,10)$ and

$$\xi\left(x^6 - y^6\right) = \mathfrak{p} = \left[-1 : \frac{1}{3} : \frac{1}{9} : 0\right]$$

Take $\lambda = \sqrt{3}$ and we have

$$\lambda \star \mathfrak{p} = [-3 : 3 : 3 : 0]$$

Then

$$\mathcal{S}(\mathfrak{p}) = \max\{3^{\frac{1}{2}}, 3^{\frac{1}{4}}, 3^{\frac{1}{6}}, 0\} = \sqrt{3}$$

Hence, we have $\mathfrak{s}(x^6 - y^6) = 0.239$. Computation for the rest of the cases goes similarly. All results are presented in Table 2. In the 4-th and 5-th column of the table are displayed logarithmic weighted height and logarithmic invariant height. □

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