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ON THE CONNECTEDNESS OF THE BRANCH LOCUS OF THE SCHOTTKY SPACE

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Dedicated to the memory of Kay Magaard

Abstract. Schottky space $\mathcal{S}_g$ is the space that parametrizes $\mathrm{PSL}_2(\mathbb{C})$-conjugacy classes of Schottky groups of rank $g \geq 2$. The branch locus $\mathcal{B}_g$ consists of the conjugacy classes of those Schottky groups which are a finite index proper subgroup of some Kleinian group. In a previous paper we observed that $\mathcal{B}_g$ was connected for $g \geq 3$ odd and that it has at most two components for $g \geq 4$ even. In this short note, we observe that $\mathcal{B}_g$ is always connected.

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1. Introduction

A Schottky group of rank $g \geq 2$ is a purely loxodromic Kleinian group, with non-empty region of discontinuity, isomorphic to the free group of rank g. Geometrically, these groups are constructed as follows. Let C_k, C'_k, $k = 1, \cdots, g$, be $2g$ Jordan curves on the Riemann sphere $\widehat{\mathbb{C}}$ such that they are mutually disjoint and bound a $2g$-connected domain $\mathcal{D}$. Suppose that for each k there exists a fractional linear

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transformation $A_k \in \mathrm{PSL}_2(\mathbb{C})$ so that (i) $A_k(C_k)=C_k'$ and (ii) $A_k(\mathcal{D}) \cap \mathcal{D} = \emptyset$. Then the group Γ, generated by all these transformations, is a Schottky group of rank g. Every Schottky group is constructed in that way [1]. If Ω is the region of discontinuity of the Schottky group Γ, then Ω is connected and Ω/Γ is a closed Riemann surface of genus g (by the retrosection theorem, every closed Riemann surface of genus g is obtained in that way). Schottky groups are exactly those Kleinian groups providing the lowest regular planar coverings of closed Riemann surfaces. See [8, 9].

The *Schottky space of rank* $g \geq 2$, which we denote as $\mathcal{S}_g$, is the one that parametrizes $\mathrm{PSL}_2(\mathbb{C})$-conjugacy classes of Schottky groups of rank g. ($\mathcal{S}_g$ can be identified with the space of classes of conformally equivalent Kleinian structures on an oriented handlebody.) If Γ is a Schottky group, then we denote by $[\Gamma] \in \mathcal{S}_g$ its conjugacy class. The *branch locus* $\mathcal{B}_g \subset \mathcal{S}_g$ consists of the conjugacy classes of those Schottky groups which are a finite index proper subgroup of some Kleinian group.

A marked Schottky group of rank $g \geq 2$ is a tuple $(\Gamma, A_1, \ldots, A_g)$, where Γ is a Schottky group of rank g and $A_1, \ldots, A_g$ is a set of generators for it. Two marked Schottky groups of rank g, say $(\Gamma, A_1, \ldots, A_g)$ and $(\widehat{\Gamma}, \widehat{A}_1, \ldots, \widehat{A}_g)$, are said to be equivalent if there is a Möbius transformation B so that $BA_jB^{-1} = \widehat{A}_j$, for every $j = 1, \ldots, g$. The *marked Schottky space of rank* g, denoted by $\mathcal{M}S_g$, parametrizes equivalence classes of marked Schottky groups of rank g. This space can be identified with the quasiconformal deformation space of a Schottky group of rank g, so it carries a complex manifold of dimension $3(g-1)$ [2, 13]. (It can also be identified with the Teichmüller space of classes of marked Kleinian structures of an orientable handlebody of genus g.)

The group of holomorphic automorphisms of $\mathcal{M}S_g$ is isomorphic to the outer automorphism group $\mathrm{Out}(F_g)$, where F_g is the free group of rank g, and the *forgetful map* $\pi : \mathcal{M}S_g \to \mathcal{S}_g$ is a (regular) orbifold-covering whose deck group is $\mathrm{Out}(F_g)$ [4, 8, 9, 13]. In this setting, the branch locus $\mathcal{B}_g$ is the projection under π of the points in $\mathcal{M}S_g$ with non-trivial $\mathrm{Out}(F_g)$-stabilizer.

If (Γ, A_1, A_2) is a marked Schottky group of rank $g = 2$, then $E = A_1A_2 - A_2A_1$ is an elliptic transformation of order two such that $E_1 = EA_1$ and $E_2 = EA_2$ are also elliptic transformations of order two. In this case, the Kleinian group $K = \langle E, E_1, E_2 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ (called a Whittaker group) contains Γ as an index two subgroup [7]. It follows that $\mathcal{B}_2$ is connected. In the paper [6] we observed that the sublocus of $\mathcal{B}_2$ consisting of the conjugacy classes of rank two Schottky groups which are finite index proper subgroups of Kleinian groups different from the Whittaker ones, has exactly two connected components. For $g \geq 3$ odd, we proved in [6] that $\mathcal{B}_g$ is connected and, for $g \geq 4$ even, that $\mathcal{B}_g$ has at most two connected components. In this short note we complete the above results as follows:

Theorem 1. *The branch locus $\mathcal{B}_g$ is connected for every $g \geq 2$.*

As observed in the previous lines, we only need to prove the connectedness of $\mathcal{B}_g$ for the case $g \geq 4$ even.

2. Proof of Theorem 1

2.1. **Cyclic extension of Schottky groups.** First of all, we will see an interpretation of $\mathcal{M}S_g$ and $\mathcal{S}_g$ in terms of quasiconformal deformation spaces: If Γ is a Schottky group of rank $g \geq 2$, then by [2] its quasiconformal deformation space $\mathcal{Q}(\Gamma)$ turns out to be a connected complex manifold of dimension $3g-3$. As any two Schottky groups of the same rank g are quasiconformally equivalent, their respective quasiconformal deformation spaces are complex analytically equivalent. It can be seen that if Γ is a Schottky group of rank g, then $\mathcal{Q}(\Gamma)$ is isomorphic to $\mathcal{M}S_g$; that is $\mathcal{Q}(\Gamma)$ is a model of the marked Schottky space $\mathcal{M}S_g$. To obtain a model of $\mathcal{S}_g$, one has to consider the following equivalence relation on $\mathcal{Q}(\Gamma)$: two deformations ω_1 and ω_2 are equivalent if there is a Möbius transformations A so that $\omega_1\Gamma\omega_1^{-1} = A\omega_2\Gamma\omega_2^{-1}A^{-1}$. Then, the set of equivalence classes is a model for $\mathcal{S}_g$. Details can be found, for instance, in [2, 13].

Assume that there is a Kleinian group K containing Γ as a finite index normal subgroup (in particular, K is finitely generated). As each Beltrami coefficient for K is also a Beltrami differential for Γ and both K and Γ have the same limit set, there is a natural holomorphic embedding $\iota : \mathcal{Q}(K) \to \mathcal{Q}(\Gamma)$ centered at Γ. In general, if there is some $[\mu] \in \mathcal{Q}(\Gamma)$ so that the Schottky group Γ_u is contained in some Kleinian group K as a finite index normal subgroup, then it provides a holomorphic embedding $j : \mathcal{Q}(K) \to \mathcal{Q}(\Gamma)$ centered at Γ_u.

A Kleinian group K, containing a Schottky group Γ of rank $g \geq 2$ as a finite index normal subgroup so that K/Γ is a cyclic group, is called a cyclic extension Schottky group or *cyclic-Schottky group*. A geometrical picture of these Kleinian groups is provided in [5]. In the case that K/Γ is a cyclic group of rank a prime integer p, the group K is a free product, in the sense of the Klein-Maskit combination theorems, of t cyclic groups generated by loxodromic transformations, r cyclic groups generated by elliptic transformations of order p and s Abelian groups, each one generated by a loxodromic transformation and an elliptic transformation of order p both of them commuting, so that $g = 1 + p(t + r + s - 1) - r$. In particular

$$K \cong \mathbb{Z} * \overset{t}{\cdots} * \mathbb{Z} * \mathbb{Z}_p * \overset{r}{\cdots} * \mathbb{Z}_p * (\mathbb{Z} \times \mathbb{Z}_p) * \overset{s}{\cdots} * (\mathbb{Z} \times \mathbb{Z}_p). \tag{1}$$

We say that a cyclic-Schottky group K as above is of type $(g, p; t, r, s)$. In this case, the region of discontinuity Ω of K coincides with the region of discontinuity of the Schottky group Γ, and $S = \Omega/\Gamma$ is a closed Riemann surface of genus g admitting a conformal automorphism ϕ of order p with $S/\langle\phi\rangle$ of signature $(\gamma; p, \overset{2r}{\ldots}, p)$, where $\gamma = t + s$ [8, 13].

The above description permits also to see that any two cyclic-Schottky groups of the same type are quasiconformally conjugated. In particular, the quasiconformal deformation space of a cyclic-Schottky groups of a fixed type (which is connected from the measurable Riemann mapping's theorem) contains all cyclic-Schottky groups of such a type.

2.2. **A cyclic decomposition of $\mathcal{B}_g$, for $g \geq 3$.** Now, let $F(g, p; t, r, s)$ be the subset of $\mathcal{B}_g$ consisting of those points $[\Gamma] \in \mathcal{S}_g$ for which there exists some $\Gamma_0 \in [\Gamma]$ and a cyclic-Schottky group K, of type $(g, p; t, r, s)$, containing Γ_0 as an index p normal subgroup.

First of all it is easy to see that $\mathcal{B}_g$ is the union of the subsets $F(g, p; t, r, s)$, where p is prime, t, r, s are non-negative integers so that $g - 1 = p(t + r + s - 1) - r$ [6]: Let W be a Kleinian group containing a Schottky group Γ as a non-trivial finite index normal subgroup and consider the natural epimorphism $\theta : W \to W/\Gamma$. Let $\phi \in W/\Gamma$ an element of prime order p. The group $K = \theta^{-1}(\langle\phi\rangle)$ is a Kleinian group containing Γ as a normal subgroup of index p. In [6] it was observed that, for $p \geq 3$,

$F(g,p;t,r,s)$ is not necessarily connected (this it might happen since K may contain different Schottky groups of rank g). However, for $p=2$, it was proved in [3] that $F(g,2;t,r,s)$ is always connected. Moreover,it can be seen that $F(g,2;t,r,s)$ is an orbifold of complex dimension$(3g-3+r)/2$. Finally, in [6] it was proved that, for $p \geq 3$, every connected component of $F(g,p;t,r,s)$ intersects some $F(g,2;t',r',s')$ (this since the orbifold $\mathcal{O} = M/\langle \phi \rangle$, where M is the handlebody uniformized by Γ and K uniformizes $\mathcal{O}$, admits an orientation-preserving self-homeomorphism τ of order two keeping Γ).

Consequentently, to prove the connectedness of $\mathcal{B}_g$ we only need to look at the possible intersections of the connected families $F(g,2;t,r,s)$. To show that two families $F(g,2;t,r,s)$, $F(g,2;t',r',s')$ intersect, we need to construct a Kleinian group K containing two cyclic-Schottky groups K_1, K_2, of type $(g,2;t,r,s)$, $(g,2;t',r',s')$ and both of them containing the same Schottky group Γ of rank g as index two subgroup.

The following intersections were obtained in [6]:

Theorem 2 ([6]). *Consider connected components $F(g,2;t,r,s)$ of $\mathcal{B}_g$. Then the following hold:*

(1) *If $g \geq 3$ is odd:*
   (a) $F(g,2;t,r,s) \cap F(g,2;(g-1)/2,2,0) \neq \emptyset$, *if t is even.*
   (b) $F(g,2;t,r,s) \cap F(g,2;(g-3)/2,4,0) \neq \emptyset$, *if t is odd.*
   (c) $F(g,2;(g-1)/2,2,0) \cap F(g,2;(g-3)/2,4,0) \neq \emptyset$.
(2) *If $g \geq 4$ is even:*
   (a) $F(g,2;t,r,s) \cap F(g,2;g/2,1,0) \neq \emptyset$, *if s and t are even.*
   (b) $F(g,2;t,r,s) \cap F(g,2;(g-2)/2,3,0) \neq \emptyset$, *if s is even and t is odd.*
   (c) $F(g,2;t,r,s) \cap F(g,2;(g-2)/2,1,1) \neq \emptyset$, *if s is odd and t is even.*
   (d) $F(g,2;t,r,s) \cap F(g,2;(g-4)/2,3,1) \neq \emptyset$, *if s and t are odd.*
   (e) $F(g,2;g/2,1,0) \cap F(g,2;(g-2)/2,3,0) \neq \emptyset$.
   (f) $F(g,2;(g-2)/2,1,1) \cap F(g,2;(g-4)/2,3,1) \neq \emptyset$.

The above asserts, for $g \geq 3$ odd, that $\mathcal{B}_g$ is connected. In the case $g \geq 4$ is even, Theorem 2 permits to observe that the connectivity of $\mathcal{B}_g$ will be obtained if $F(g,2;0,g+1,0) \cap F(g,2;(g-2)/2,1,1) \neq \emptyset$.

2.3. **The connectedness of $\mathcal{B}_{\mathbf{g}}$, for $\mathbf{g} \geq \mathbf{4}$ even.** In order to obtain the connectedness of $\mathcal{B}_g$, for $g \geq 4$ even, we will construct two cyclic-Schottky groups K_1 and K_2, of respective types $(g,2;0,g+1,0)$ and $(g,2;(g-2)/2,1,1)$, each one containing the same Schottky group Γ as an index two normal subgroup. To do it, we consider the Kleinian group K constructed from the Klein-Maskit combination theorems [8,10,11] by using $(g-2)/2+4$ elliptic transformations of order two, say $E_1, \ldots, E_{(g-2)/2}, F_1, F_2, F_3, F_4$, such that $(F_2F_1)^2 = (F_3F_2)^2 = (F_4F_3)^2 = 1$, as shown in Figure 1.

The Kleinian group K has a Cantor as a limit set, and if its (connected) region of discontinuity is Ω, then the 2-orbifold Ω/K is the Riemann sphere (genus zero) with exactly $(g-2)+5$ cone points, each one of order two. Let us consider the surjective homomorphism $\theta : K \to \langle a,b \rangle \cong \mathbb{Z}_2^2$ defined by $\theta(E_1) = \cdots = \theta(E_{(g-2)/2}) = \theta(F_1) = \theta(F_4) = b$, $\theta(F_2) = a$, $\theta(F_3) = ab$.

The kernel Γ of θ is a index 4, torsion free subgroup of the Kleinian group K. The Kleinian group Γ is geometrically finite purely loxodromic Kleinian group

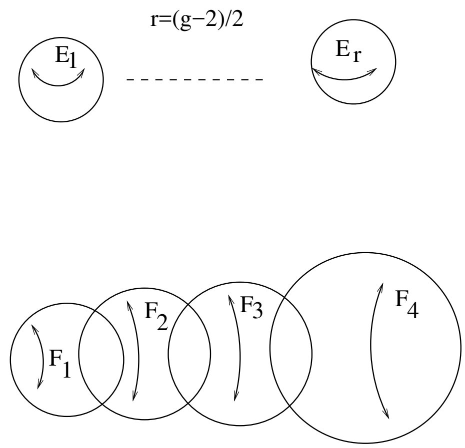

FIGURE 1. The Kleinian group K

with connected region of discontinuity. It follows from the classification of function groups [12] that Γ is necessarily a Schottky group.

Let $K_1 = \theta^{-1}(\langle a \rangle)$ and $K_2 = \theta^{-1}(\langle b \rangle)$. Both of these are index two subgroups of K and $\Gamma = K_1 \cap K_2$ has index two in each of K_1 and K_2. It can be seen that

$$K_1 = \langle F_1E_1, \ldots, F_1E_{(g-2)/2}, F_4F_3, F_2, F_3F_1 \rangle,$$
$$K_2 = \langle E_1, \ldots, E_{(g-2)/2}, F_2E_1F_2, \ldots, F_2E_{(g-2)/2}F_2, F_1, F_4, F_3F_2 \rangle.$$

The group K_1 is a cyclic-Schottky group of type $(g, 2; (g-2)/2, 1, 1)$. It induces an involution ϕ_1 in the handlebody M uniformized by Γ whose branch locus in $M/\langle \phi_1 \rangle$ consists of 1 loop and one arc of fixed points. Similarly, K_2 is a cyclic-Schottky group of type $(g, 2; 0, g+1, 0)$ inducing an involution ϕ_2 in the same handlebody M uniformized by Γ whose branch locus in $M/\langle \phi_2 \rangle$ consists of $g+1$ arcs of fixed points.

The groups K, Γ, K_1 and K_2 as above are as desired ones.

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