ALBANIAN JOURNAL OF MATHEMATICS
Volume 12, Number 1, Pages 3–7
ISSN: 1930-1235; (**2018**)

RINGS WHOSE ELEMENTS ARE SUMS OF THREE OR MINUS SUMS OF TWO COMMUTING IDEMPOTENTS

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ABSTRACT. We classify up an isomorphism all rings having expressed their elements by at most three commuting idempotents. Our main result considerably extends certain important achievements established by Hirano-Tominaga [3], Ying et al. [6] and Tang et al. [5] as well as it somewhat strengthens recent results proved by the author in [1] and [2].

Mathematics Subject Classes 2010: Primary: 16U99; 16E50; Secondary: 13B99

Keywords: rings, idempotents, fields, Jacobson radical

1. Introduction and Background

Everywhere in the text of the present paper, all our rings R are assumed to be associative, containing the identity element 1, which in general differs from the zero element 0 of R, and all subrings are unital (i.e., containing the same identity as that of the former ring). Our terminology and notations are mainly in agreement with [4]. For instance, $U(R)$ denotes the set of all units in R, $Id(R)$ the set of all idempotents in R, and $Nil(R)$ the set of all nilpotents in R.

We here will be concerned with rings whose elements are representing by at most three commuting idempotents. Specifically, we start with the following new notion.

Definition 1.1. We shall say that a ring R is from the class $\mathcal{T}$ if, for each element $r \in R$ there are three commuting idempotents e_1, e_2, e_3 such that $r = e_1 + e_2 + e_3$ or $r = -e_1 - e_2$.

Under the substitution $r \to -r$, it takes the equivalent form $r = -e_1 - e_2 - e_3$ or $r = e_1 + e_2$.

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Date: Received: March 1, 2018. Accepted: July 7, 2018.

Immediate examples of such rings are $\mathbb{Z}_k$ where $k = 2, 3, 4, 5, 6$, whereas the direct product $\mathbb{Z}_5 \times \mathbb{Z}_5$ need not be so.

A brief history of the principally known results in the current subject is as follows: In [3] rings whose elements are sums of two commuting idempotents $e_1 + e_2$ were completely described. This was independently extended in both [1] and [5] to sums of three commuting idempotents $e_1 + e_2 + e_3$. Even something more, in [1] were classified those rings R whose elements are of the kind $e_1 + e_2 + e_3$ or $e_1 - e_2$. This is, however, a common expansion of the central statement from [6, Theorem 4.4], where the ring elements are written as $e_1 + e_2$ or $e_1 - e_2$. On the other hand, it is worthwhile noticing that the isomorphic structure of rings for which all elements are of the type $e_1 + e_2$ or $-e_1 - e_2$ was obtained in [2].

The goal of this article is to enlarge the aforementioned results, and especially the stated last one, by characterizing all rings from the class $\mathcal{T}$ as defined in Definition 1.1.

2. Main Results

We first begin with the following technicality.

Lemma 2.1. *Let R be a ring which belongs to the class $\mathcal{T}$. Then R can be decomposed as the direct product $R_1 \times R_2 \times R_3$, where $2^2 = 4 = 0$ in R_1, $3^2 = 9 = 0$ in R_2 and $5 = 0$ in R_3, and all of R_1, R_2, R_3 belong to the class $\mathcal{T}$.*

Proof. For an arbitrary element $x \in R$, we write that $x = e_1+e_2+e_3$ or $x = -e_1-e_2$ for some commuting idempotents e_1, e_2, e_3. We assert that $30^2 = 0$. In fact, if first $-3 = -e_1 - e_2$, then $-2 = (1 - e_1) - e_2$ and hence $(-2)^3 = -2$, i.e., $6 = 0$. If now $-3 = e_1 + e_2 + e_3$, one writes that $-4 = e_1 + e_2 - (1 - e_3) = e_1 + e_2 - e'_3$. Since $e_1 + e_2 - e'_3 = e_1 + e_2(1 - e'_3) - e'_3(1 - e_2) = e_1(1 - e'_3(1 - e_2)) + e_2(1 - e'_3) - e'_3(1 - e_2)(1 - e_1)$ as all of these elements in the last record are commuting idempotents such that the first and the second ones are both orthogonal with the third one, we may with no loss of generality assume by replacing the existing idempotents that $e_1e'_3 = e_2e'_3 = 0$. Therefore, $6e_1e_2 = 0$ and thus $30e_1 = 0$ by multiplying with e_1 both sides of the equality $-4 = e_1 + e_2 - e'_3$ and the result by e_2. In a way of similarity, we get that $30e_2 = 0$. Furthermore, squaring $-4 = e_1 + e_2 - e'_3$ and manipulated subsequently with the obtained above facts, we infer that $6e'_3 = 60$. Hence $-4 = e_1 + e_2 - e'_3$ multiplied by 30 leads to $180 = 0$ whence $30^2 = 0$ and so $30 \in Nil(R)$, as asserted. The Chinese Remainder Theorem now applies to write that $R \cong R_1 \times R_2 \times R_3$, where $2^2 = 4 = 0$ in R_1, $3^2 = 9 = 0$ in R_2 and, finally, $5 = 0$ in R_3, as asserted. The final part is now immediate. $\square$

We next proceed by proving the following.

Proposition 2.2. *Suppose that R is a ring of characteristic 5. Then the following three conditions are equivalent:*

(i) $x^3 = x$ or $x^4 = 1$, $\forall x \in R$.

(ii) $x^3 = x$ or $x^3 = -x$, $\forall x \in R$.

(iii) R is isomorphic to the field $\mathbb{Z}_5$.

Proof. "(i) $\Rightarrow$ (ii)". For an arbitrary but fixed $y \in R$ satisfying $y^4 = 1$ with $y^3 \neq y$, considering the element $y^2-1 \in R$, it must be that $(y^2-1)^4 = 1$ or $(y^2-1)^3 = y^2-1$.

In the first case, we receive $y^2 = -1$ and thus equivalently $y^3 = -y$, as required, while in the second one, we arrive at $y^2 = 1$ and so in an equivalent form $y^3 = y$ which is against our initial assumption.

"(ii) $\iff$ (iii)". Let P be the subring of R generated by 1, and thus note that $P \cong \mathbb{Z}_5$. We claim that $P = R$, so we assume in a way of contradiction that there exists $b \in R \setminus P$. With no loss of generality, we shall also assume that $b^3 = b$ since $b^3 = -b$ obviously implies that $(2b)^3 = 2b$ as $5 = 0$ and $b \notin P \iff 2b \notin P$.

Let us now $(1+b)^3 = -(1+b)$. Hence $b = b^3$ along with $5 = 0$ enable us that $b^2 = 1$. This allows us to conclude that $(1+2b)^3 \neq \pm(1+2b)$, however. In fact, if $(1+2b)^3 = 1+2b$, then one deduces that $2b = 3 \in P$ which is manifestly untrue. If now $(1+2b)^3 = -1-2b$, then one infers that $2b = 2 \in P$ which is obviously false. That is why, only $(1+b)^3 = 1+b$ holds. This, in turn, guarantees that $b^2 = -b$. Moreover, $b^3 = b$ is equivalent to $(-b)^3 = -b$ as well as $b^3 = -b$ to $(-b)^3 = -(-b)$ and thus, by what we have proved so far applied to $-b \notin P$, it follows that $-b = b^2 = (-b)^2 = -(-b) = b$. Consequently, $2b = 0 = 6b = b \in P$ because $5 = 0$, which is the wanted contradiction. We thus conclude that $P = R$, as expected.

Conversely, it is trivial that the elements of $\mathbb{Z}_5 = \{0, 1, 2, 3, 4 \mid 5 = 0\}$ are solutions of one of the equations $x^3 = x$ or $x^3 = -x$.

"(iii) $\Rightarrow$ (i)". It is self-evident that all elements of $\mathbb{Z}_5 = \{0, 1, 2, 3, 4 \mid 5 = 0\}$ satisfy one of the equations $x^3 = x$ or $x^4 = 1$. □

We now have all the ingredients necessary to prove our basic result.

Theorem 2.3. *A ring R lies in the class $\mathcal{T}$ if, and only if, R is decomposable as $R_1 \times R_2 \times R_3$, where*

(1) $R_1 = \{0\}$, or R_1 is a non-zero commutative ring such that $4 = 0$ and $R_1/J(R_1)$ is a boolean ring with either $J(R_1) = \{0\}$ or nil $J(R_1) = 2Id(R_1)$;

(2) $R_2 = \{0\}$, or R_2 is a subdirect product of family of copies of the field $\mathbb{Z}_3$;

(3) $R_3 = \{0\}$ (which is mandatory when $J(R_1) \neq \{0\}$), or $R_3 \cong \mathbb{Z}_5$.

Proof. "**Necessity.**" With Lemma 2.1 at hand, one writes that $R \cong R_1 \times R_2 \times R_3$, where R_1 is either zero or R_1 is a nonzero ring in which $4 = 0$, where R_2 is either zero or R_2 is a nonzero ring in which $9 = 0$, and where R_3 is either zero or R_3 is a nonzero ring in which $5 = 0$, as well as R_1, R_2, R_3 remains in the class $\mathcal{T}$.

In order to describe the three direct factors, we distinguish three basic cases, namely:

Case 1: *Describing R_1.* We have $4 = 0$ and $2 \in Nil(R_1)$ whence $2 \in J(R_1)$, so that $R_1/J(R_1)$ is necessarily boolean being a factor-ring of characteristic 2 whose elements are sums of (at most three) commuting idempotents. What remains to prove is that $J(R_1) = 2Id(R_1)$. In showing that, the case when any element from $J(R_1)$ is written as a sum of three commuting idempotents follows analogously to [1] getting that $J(R_1) = 2Id(R_1)$. That is why, we will be now concerned with $z = -e - f$ for an arbitrary $z \in J(R_1)$, where $e, f \in Id(R_1)$ do commute. Multiplying by $1 - f$, we get that $z(1-f) = -e(1-f) \in J(R_1) \cap (-Id(R_1)) = \{0\}$ whence $e = ef$. Similarly, $ef = f$ and so $e = f$. Finally, $z = -2e \in -2Id(R_1) = 2Id(R_1)$, because $4 = 0$, as promised.

Case 2: *Describing R_2.* We have $9 = 0$ and $3 \in J(R_2)$. We assert that $J(R_2) = \{0\}$ and hence $3 = 0$ in R_2. In fact, as in the preceding case, it follows that $J(R_2) = \pm 2Id(R_2) = \mp Id(R_2) = \{0\}$, as asserted. Furthermore, it is routinely checked that $x^3 = x$ for every $x \in R_2$ and thus, the main result from [3] applies to get our stated conclusion.

Case 3: *Describing* R_3. We have $5 = 0$ and $x^5 = x$ for all elements x in R_3. Now, for each $x \in R_3$, we write that $x = e + f + h$ or $x = -e - f$ for some three commuting idempotents e, f, h. For the first record, one deduces after squaring that $2x^3 - x^2 - x = 2efh$ because $x^3 = x + ef + fh + he + efh$, so that multiplying both sides by 3 it follows that $x^3 + 2x^2 + 2x = efh$ is an idempotent. This means that $(x^3 + 2x^2 + 2x)^2 = x^3 + 2x^2 + 2x$ which, after some usual tricks, amounts to $3x^4 + 2x^3 - 2x^2 + 2x = 0$. Multiplying this by 2, we finally arrive at $x^4 - x^3 + x^2 - x = 0$. Replacing x with $x - 1$ in the given last equality, one infers that $x^4 = 1$.

As for the second record, one derives after squaring that $x^3 - 2x^2 - 3x = x^3 - 2x^2 + 2x = 0$ because $x^3 = x - ef$ and so $(x - x^3)^2 = x - x^3$. Replacing x by $x - 1$ in the given equation, one infers that $x^3 = x$.

Now, since for any $x \in R_3$ it must be that $x^3 = x$ or $x^4 = 1$, we henceforth can successfully apply Proposition 2.2 to conclude that R_3 has to be isomorphic to the five element field $\mathbb{Z}_5$, as stated.

So, finally, the full description of R over, as formulated.

"**Sufficiency.**" A direct consultation with [3] enables us that every element of R_2 is a sum of two idempotents. Since it is pretty easy that each element in $\mathbb{Z}_5$ is a sum of three idempotents (e.g., 0, 1, 2 and 3) or minus a sum of two idempotents (e.g., 0 and 4), what remains to prove is that any element from R_1 is a sum of three idempotents. It is, really, well known that if $2 = 0$ in R_1 it must have a sum of two idempotents or even just a single idempotent. To that purpose, taking an arbitrary $r \in R_1$, we may write that $r + J(R_1)$ is an idempotent and thus $r - r^2 \in J(R_1) = 2Id(R_1)$. But $J(R_1)$ is nil with $J^2 = \{0\}$ (as $4 = 0$) and hence there exists an idempotent $g \in R_1$ with $r - g \in 2Id(R_1)$. This containment allows us to write that $r = g + 2h = g + h + h$ for some $h \in Id(R_1)$, as required. □

Remark 2.4. Considering the rings $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_3 \times \mathbb{Z}_5$ or $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$, one says that they still are in the class $\mathcal{T}$, whereas as commented above the rings $\mathbb{Z}_4 \times \mathbb{Z}_5$ and $\mathbb{Z}_5 \times \mathbb{Z}_5$ are not. However, for any element x lying in the last direct product, x or $-x$ is a sum of three idempotents. That is why, it will be of interest to consider those rings having the mentioned property – see the problem posed below.

As for the direct product $\mathbb{Z}_4 \times \mathbb{Z}_5$, consider the element $(1, 4)$ which is not presentable neither as a sum of three idempotents nor as a minus sum of two idempotents. Nevertheless, $(1, 4) = (1, 0) - (0, 1)$. Reciprocally, the element $(2, 4) = -(1, 0) - (1, 1)$, but a routine check shows that $(2, 4)$ is not neither the sum of three idempotents nor the difference of two idempotents, as expected. Resuming, $(1, 4)$ lies in $\mathcal{K} \setminus \mathcal{T}$ as opposite to $(2, 4)$ which lies in $\mathcal{T} \setminus \mathcal{K}$.

Meanwhile, surprisingly, against this element-wise discrepancy, the present class $\mathcal{T}$ from Definition 1.1 coincides with the class $\mathcal{K}$ from [1]. Likewise, the class $\mathcal{C}$ from [2] is contained in the class $\mathcal{K}$ from [1]. As a matter of fact, if $x = e_1 + e_2$ or $x = -e_1 - e_2$ for any element x, then one can write that $x - 1 = e_1 + e_2$ or $x - 1 = -e_1 - e_2$ and thus one gets that $x = e_1 + e_2 + 1$ or $x = (1 - e_1) - e_2$, as required.

So, we end our work with the following well-motivated problem:

Problem 2.5. Describe the isomorphic structure of those rings whose elements are sums or minus sums of three commuting idempotents, that is, for any element a of a ring R it is fulfilled that $a = e_1 + e_2 + e_3$ or $a = -e_1 - e_2 - e_3$ for some three commuting idempotents e_1, e_2, e_3 of R. In other words, $\forall a \in R$: a or $-a$ is a sum of three commuting idempotents.

Here the ring $\mathbb{Z}_7$ arose quite naturally, which however not occurred in the statements above, so that some new techniques should be exploited.

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