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A CHARACTERIZATION OF SOME LAGUERRE-HAHN ORTHOGONAL POLYNOMIALS OF CLASS ONE

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ABSTRACT. In this present work, we give an characterization of quasi-symmetric Laguerre-Hahn orthogonal polynomial sequences of class one through the study of the differential functional equation fulfilled by its corresponding regular linear form.

1. INTRODUCTION

The study of the Laguerre-Hahn polynomial sequences is one of the interesting popular problems in the area of orthogonal polynomials. Since the system corresponding to the problem of determining all the Laguerre-Hahn linear forms of class $s \geq 1$ becomes non-linear, the problem was only solved when $s = 1$ and for the symmetric case [3]. Thus, several authors use different processes in order to obtain Laguerre-Hahn linear forms of class $s \geq 1$. For instance, we can mention the adjunction of either a Dirac mass or its derivative to Laguerre-Hahn linear forms [2], [9], [10], the product and the division of a linear form by a polynomial [8], [15]. So, some examples of Laguerre-Hahn linear forms are given in terms of classical ones. But, they are just few examples. The aim of this work is to study a family of quasi-symmetric Laguerre-Hahn orthogonal polynomial sequences $\{W_n\}_{n\geq 0}$ of class $s_w = 1$ verifying the following three-term recurrence relation:

$$(1)\qquad \begin{aligned} &W_{n+2}(x) = (x - (-1)^{n+1})W_{n+1}(x) - \gamma_{n+1}W_n(x)\ , \quad n \geq 0\ , \\ &W_1(x) = x - 1\ , \qquad W_0(x) = 1\ , \end{aligned}$$

through the study of the Pearson differential functional equation satisfied by its corresponding regular linear form w and using the framework of the quadratic decomposition.

In section 2, the preliminaries results as well as the notations in use throughout the text are given. One of the properties of the linear form w indicates that there is a relationship between w and a symmetric regular linear form ϑ. In section 3, we deal with the Laguerre-Hahn character of the symmetric linear form ϑ, which allows us to characterize the structure of the polynomial elements of the differential functional equation satisfied by the linear form w. As an exploitation of these results, we treat in detail all Laguerre-Hahn polynomial sequences of class $s_w = 1$ satisfying (1). The obtained linear forms are in connection of modified linear forms of the Laguerre-Hahn linear forms of class zero

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analogous to the classical Jacobi's one. The regularity conditions and the recurrence coefficients $\gamma_{n+1},\ n \geq 0$ are given.

2. PRELIMINARIES

Let $\mathcal{P}$ be the vector space of polynomials with complex coefficients and let $\mathcal{P}'$ be its dual. The elements of $\mathcal{P}'$ will be called either linear form or linear functional. We denote by $\langle v, f\rangle$ the action of $v \in \mathcal{P}'$ on $f \in \mathcal{P}$. For $n \geq 0$, $(v)_n = \langle v, x^n\rangle$ are the moments of v. In particular a linear form is called symmetric if all of its moments of odd order are zero [7].

We define in the space $\mathcal{P}'$ the derivative $v^{'}$ of the form v by $\langle v', f\rangle := -\langle v, f'\rangle$, the left multiplication by a polynomial hv by $\langle hv, f\rangle := \langle v, hf\rangle$, the shifted form $h_a v$, $\tau_b v$ by $\langle h_a v, f\rangle := \langle v, h_a f\rangle = \langle v, f(ax)\rangle$, $\langle \tau_b v, f\rangle := \langle v, \tau_{-b} f\rangle = \langle v, f(x+b)\rangle$, the Dirac form at origin δ_0 by $\langle \delta_0, f\rangle := f(0)$ and the inverse multiplication by a polynomial of degree one $(x-c)^{-1}v$, through

$$\big\langle (x-c)^{-1}v, f\big\rangle := \langle v, \theta_c f\rangle \text{ with } \big(\theta_c f\big)(x) := \frac{f(x)-f(c)}{x-c},\ f \in \mathcal{P}.$$

We also denote $(f(\xi))(x) = f(x)$ for the dummy variable ξ.

It is straightforward to prove that for $v \in \mathcal{P}'$ and $f \in \mathcal{P}$, we have [13]

$$x^{-1}(xv) = v - (v)_0\delta_0\ , \qquad x(x^{-1}v) = v\ , \tag{2}$$

$$f(x^{-1}v) = x^{-1}(fv) + \langle v, \theta_0 f\rangle \delta_0\ . \tag{3}$$

We also define the right-multiplication of a form v by a polynomial h with

$$(vh)(x) := \left\langle v, \frac{xh(x) - \xi h(\xi)}{x-\xi}\right\rangle\ . \tag{4}$$

Next, it is possible to define the product of two forms through

$$\langle uv, f\rangle := \langle u, vf\rangle\ ,\quad u, v \in \mathcal{P}',\ f \in \mathcal{P}.$$

For $f \in \mathcal{P}$ and $v \in \mathcal{P}'$ we have the following result [1]

$$f^2v^2 = (fv)^2 + 2xf(x)(v\theta_0 f)(x)v\ . \tag{5}$$

Let us define the operator $\sigma : \mathcal{P} \longrightarrow \mathcal{P}$ by $(\sigma f)(x) := f(x^2)$ for all $f \in \mathcal{P}$. By transposition we define σv from the following:

$$\langle \sigma v, f\rangle = \langle v, \sigma f\rangle.$$

Thus we have the well-known formulas [2], [5], [12],

$$\begin{aligned} \sigma\Big(f(x^2)v\Big) &= f(x)\sigma v\ ,\\ \sigma v^{'} &= 2\Big(\sigma(xv)\Big)^{'}\ ,\\ \sigma v^2 &= (\sigma v)^2 + x^{-1}\Big(\sigma(xv)\Big)^2\ ,\\ \sigma(x^{-1}v^2) &= 2x^{-1}(\sigma v)\sigma(xv)\ . \end{aligned} \tag{6}$$

Let us recall that a form w is said to be regular (quasi-definite) if there exists a sequence $\{W_n\}_{n\geq 0}$ of polynomials with $\deg W_n = n,\ n \geq 0$, such that

$$\langle w, W_nW_m\rangle = r_n\delta_{n,m}\quad ,\quad r_n \neq 0,\quad n \geq 0\ .$$

We can always assume that each W_n is monic, i.e. $W_n(x) = x^n +$ lower degree terms. Then the sequence $\{W_n\}_{n\geq 0}$ is said to be orthogonal with respect to w (monic orthogonal polynomial sequence (MOPS) in short). It is a very well-known fact that the sequence $\{W_n\}_{n\geq 0}$ satisfies a three-term recurrence relation (see, for instance, the monograph by Chihara [7]

$$\begin{aligned} &W_{n+2}(x) = (x-\beta_{n+1})W_{n+1}(x) - \gamma_{n+1}W_n(x)\ , \quad n\geq 0\ , \\ &W_1(x) = x-\beta_0\ , \qquad W_0(x) = 1\ . \end{aligned} \tag{7}$$

with $\big(\beta_n, \gamma_{n+1}\big) \in \mathbb{C}\times\mathbb{C}\backslash\{0\}\ ,\quad n\geq 0$. By convention we set $\gamma_0 = (w)_0$. The form w is said to be normalized if $(w)_0 = 1$. In this paper, we suppose that any form will be normalized.

Here, we will be considering a MOPS $\{W_n\}_{n\geq 0}$ with respect to the form w satisfying a three-term recurrence relation (7) with

$$\beta_n = (-1)^n\ ,\ n\geq 0\ . \tag{8}$$

In this case, we have:

Lemma 1. [6], [12] *Let $\{W_n\}_{n\geq 0}$ be a MOPS with respect to the form w. The following statements are equivalents:*

(a) $\{W_n\}_{n\geq 0}$ satisfies (7)-(8).

(b) $(w)_{2n+1} = (w)_{2n},\ n\geq 0$.

(c) The sequence $\{W_n\}_{n\geq 0}$ has the following quadratic decomposition

$$W_{2n}(x) = P_n(x^2)\ , \quad W_{2n+1}(x) = (x-1)R_n(x^2)\ ,\ n\geq 0\ ,$$

where the sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are respectively orthogonal with respect to $u = \sigma(w)$ and $v = \gamma_1^{-1}(x-1)\sigma(w)$. Moreover, their recurrence elements are respectively, given by (for all $n\geq 0$)

$$\begin{cases} \beta_0^P = \gamma_1 + 1\ , \\ \beta_{n+1}^P = \gamma_{2n+2} + \gamma_{2n+3} + 1\ , \\ \gamma_{n+1}^P = \gamma_{2n+1}\gamma_{2n+2}\ , \end{cases} \qquad \begin{cases} \beta_0^R = \gamma_1 + \gamma_2 + 1\ , \\ \beta_{n+1}^R = \gamma_{2n+3} + \gamma_{2n+4} + 1\ , \\ \gamma_{n+1}^R = \gamma_{2n+2}\gamma_{2n+3}\ . \end{cases} \tag{9}$$

For more details about the quadratic decomposition of MOPS, see [12]. According to the statement (b) of Lemma 1, the form $(x-1)w$ is antisymmetric, that is to say $((x-1)w)_{2n} = 0,\ n\geq 0$. Equivalently,

$$\sigma(xw) = \sigma(w)\ . \tag{10}$$

Let now λ be a non-zero complex number and ϑ be the linear form such that

$$\lambda x\vartheta = (x-1)w\ . \tag{11}$$

Equivalently, from (2)

$$\vartheta = \frac{1}{\lambda}(w - x^{-1}w) + (1-\frac{1}{\lambda})\delta_0\ . \tag{12}$$

Applying the operator σ in (11) and taking into account of (10), we get $\sigma(x\vartheta) = 0$. Hence ϑ is a symmetric form. Then, ϑ is regular if and only if $\sigma(\vartheta)$ and $x\sigma(\vartheta)$ are regular [12]. Now, multiplying (11) by x and applying the operator σ, we get $\lambda x\sigma(\vartheta) = \gamma_1 v$. That is

$\lambda x\sigma(\vartheta)$ is regular since $\lambda \neq 0$ and $\gamma_1 v = (x-1)\sigma(w)$ is regular, see [12]. So, according to [14], the form ϑ is regular if and only if $\lambda \neq \lambda_n$ where $\lambda_n\ ,\ n \geq 0$ is defined by

$$\lambda_n = -\gamma_1 \frac{R^{(1)}_{n-1}(0)}{R_n(0)}\ ,\ n \geq 0\ ,$$

with [8]

$$R^{(1)}_n(x) = \left\langle v, \frac{R_{n+1}(x) - R_{n+1}(y)}{x-y} \right\rangle\ ,\ n \geq 0\ ,\ R^{(1)}_{-1}(x) = 0\ .$$

Proposition 1. [17] *There exists a non-zero constant λ such that the form ϑ given by (11) is regular.*

Now, let us recall some features about the Laguerre-Hahn character [2], [?AM)], [4].

Definition 1. *A linear form w is said to be Laguerre-Hahn when it is regular and there exist three polynomials Φ, a monic polynomial, Ψ and B, $\deg(\Phi) = t \geq 0$, $\deg(\Psi) = p \geq 1$, $\deg(B) = r \geq 0$, such that*

$$\big(\Phi w\big)' + \Psi w + B(x^{-1}w^2) = 0\ . \tag{13}$$

The corresponding MOPS $\{W_n\}_{n\geq 0}$ is said to be Laguerre-Hahn.

Remark 1. *When $B = 0$, the form w is semiclassical.*

Proposition 2. *The Laguerre-Hahn linear form w satisfying (13) is said to be of class $s_w = \max(t-2, r-2, p-1)$ if and only if the following condition is satisfied*

$$\prod_{c\in\mathcal{Z}_\Phi} \Big(|\Phi'(c) + \Psi(c)| + |B(c)| + |\langle w, \theta_c^2\Phi + \theta_c\Psi + w\theta_0\theta_c B\rangle|\Big) \neq 0\ , \tag{14}$$

where $\mathcal{Z}_\Phi$ is the set of zeros of Φ.

The Laguerre-Hahn character of a linear form is kept by shifting. Indeed, the shifted form $\tilde{w} = (h_{a^{-1}} \circ \tau_{-b})w\ ,\ a \neq 0\ ,\ b \in \mathbb{C}$ is also Laguerre-Hahn having the same class as that w and fulfilling the equation

$$\big(\tilde{\Phi}\tilde{w}\big)' + \tilde{\Psi}\tilde{w} + \tilde{B}(x^{-1}\tilde{w}^2) = 0\ ,$$

where

$$\tilde{\Phi}(x) = a^{-t}\Phi(ax+b)\ ,\ \tilde{\Psi}(x) = a^{1-t}\Psi(ax+b)\ ,\ \tilde{B}(x) = a^{-t}B(ax+b)\ .$$

The sequence $\{\tilde{W}_n\}_{n\geq 0}$, where $\tilde{W}_n(x) = a^{-n}W_n(ax+b),\ n \geq 0$ is orthogonal with respect to $\tilde{w}$. The recurrence coefficients are given by

$$\tilde{\beta}_n = \frac{\beta_n - b}{a}\ ,\quad \tilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}\ ,\ n \geq 0\ .$$

The next result [3] characterizes the elements of the functional equation satisfied by any symmetric Laguerre-Hahn linear form.

Proposition 3. *Let w be a symmetric Laguerre-Hahn linear form of class s_w satisfying (13). The following statements hold.*

(i) When s_w is odd then Φ and B are odd and Ψ is even.

(ii) When s_w is even then Φ and B are even and Ψ is odd.

3. MAIN CHARACTERIZATION PROPERTIES

Let w be a Laguerre-Hahn linear form of class $s_w = 1$ satisfying (13) which its corresponding MOPS $\{W_n\}_{n\geq 0}$ fulfills (8).

Our aim is to characterize the structure of the polynomial elements of the functional equation (13) satisfied by the linear form w. This is possible through the study of the Laguerre-Hahn character of the form ϑ.

3.1. Class and functional equation of the form ϑ. The form ϑ define by (13) when it is regular, is also Laguerre-Hahn of class s_ϑ such that $s_\vartheta \leq s_w + 2$ and satisfying the functional equation [16]

$$(E\vartheta)^{'} + F\vartheta + G(x^{-1}\vartheta^2) = 0 \,, \tag{15}$$

with

$$\begin{aligned} &E(x) = x(x-1)\Phi(x) \,, \quad G(x) = \lambda x^2 B(x) \,, \\ &F(x) = x\Big((x-1)\Psi(x) + 2(1-\lambda)B(x) - 2\Phi(x)\Big) \,. \end{aligned} \tag{16}$$

Denoting by λ_{-1} and λ_{-2} the solutions of the equation

$$B(0)\lambda^2 + (\Phi^{'} + \Psi - 2B)(0)\lambda + B(0) - \Phi^{'}(0) - \Psi(0) - \langle w, \theta_0^2\Phi + \theta_0\Psi + w\theta_0^2 B\rangle = 0$$

if $B(0) \neq 0$ and $\lambda_{-1} = \lambda_{-2} = \frac{\langle w, \theta_0^2\Phi + \theta_0\Psi + w\theta_0^2 B\rangle}{\Phi^{'}(0) + \Psi(0)} + 1$ otherwise.

Theorem 1. *Let λ be a complex number such that $\lambda \neq \lambda_n,\ n \geq -2$. Then, the linear form ϑ is Laguerre-Hahn of class s_ϑ satisfying*

$$(\tilde{E}\vartheta)^{'} + \tilde{F}\vartheta + \tilde{G}(x^{-1}\vartheta^2) = 0 \,.$$

Moreover,

(a) If $(\Phi(1), B(1)) \neq (0,0)$, then

$$\tilde{E}(x) = E(x) \,, \quad \tilde{F}(x) = F(x) \,, \quad \tilde{G}(x) = G(x) \,,$$

and $s_\vartheta = 3$.

(b) If $(\Phi(1), B(1)) = (0,0)$ and $(\Psi(1), B^{'}(1)) \neq (0,0)$, then

$$\begin{aligned} &\tilde{E}(x) = x\Phi(x) \,, \quad \tilde{G}(x) = \lambda x^2(\theta_1 B)(x) \,, \\ &\tilde{F}(x) = x\Big(\Psi + 2(1-\lambda)(\theta_1 B) - (\theta_1\Phi)\Big)(x) \,, \end{aligned}$$

and $s_\vartheta = 2$.

(c) If $(\Phi(1), B(1)) = (0,0)$ and $(\Psi(1), B^{'}(1)) = (0,0)$, then

$$\begin{aligned} &\tilde{E}(x) = x(\theta_1\Phi)(x) \,, \quad \tilde{G}(x) = \lambda x^2(\theta_1^2 B)(x) \,, \\ &\tilde{F}(x) = x\Big((\theta_1\Psi) + 2(1-\lambda)(\theta_1^2 B)\Big)(x) \,, \end{aligned}$$

and $s_\vartheta = 1$.

For the proof, we need the following lemma.

Lemma 2. *(i) For all root c of Φ, we have*

$$\begin{aligned} \langle \vartheta, \theta_c^2 E + \theta_c F + \vartheta\theta_0\theta_c G\rangle = {} & \tfrac{1}{\lambda}(c-1)^2\langle w, \theta_c^2\Phi + \theta_c\Psi + w\theta_0\theta_c B\rangle \\ & + (1-\tfrac{1}{\lambda})\Big((c-1)(\Phi^{'} + \Psi)(c) + (1-\lambda)B(c)\Big) \,, \end{aligned} \tag{17}$$

$$E^{'}(c) + F(c) = c(c-1)\big(\Phi^{'} + \Psi\big)(c) \,, \quad G(c) = \lambda c^2 B(c) \,. \tag{18}$$

(ii) For any $\lambda \neq \lambda_n,\ n \geq -2$, the class of the linear form ϑ depends only the zero $x = 1$.

Proof. (i) Let c be a root of Φ. Then we can write

$$E(x) = x(x-1)(x-c)\Phi_c(x)\ , \quad \Phi_c(x) = (\theta_c\Phi)(x)\ . \tag{19}$$

Using the definition of the operator θ_c, it is easy to prove that, for $f\ , g \in \mathcal{P}$, we have

$$\Big(\theta_c(fg)\Big)(x) = \Big(\theta_c f\Big)(x)g(x) + f(c)\Big(\theta_c g\Big)(x)\ . \tag{20}$$

So, from (12) and (19), we have

$$\langle \vartheta, \theta_c^2 E\rangle = \frac{1}{\lambda}\langle w, \theta_c\Big(\xi(\xi-1)\Phi_c\Big)(x)\rangle - \frac{1}{\lambda}\langle w, \theta_c\Big((\xi-1)\Phi_c\Big)(x)\rangle + (1-\frac{1}{\lambda})(c-1)\Phi'(c)\ .$$

Taking $f(x) = x(x-1)$ and $g(x) = \Phi_c(x)$ in (20), we obtain

$$\langle w, \theta_c\Big(\xi(\xi-1)\Phi_c\Big)(x)\rangle = \langle w, (x+c-1)\Phi_c(x)\rangle + c(c-1)\langle w, \theta_c^2\Phi\rangle\ .$$

Replacing $f(x) = x-1$ and $g(x) = \Phi_c(x)$ in (20), we deduce

$$\langle w, \theta_c\Big((\xi-1)\Phi_c\Big)(x)\rangle = \langle w, \Phi_c\rangle + (c-1)\langle w, \theta_c^2\Phi\rangle\ .$$

Therefore,

$$\langle \vartheta, \theta_c^2 E\rangle = \frac{1}{\lambda}(c-1)^2\langle w, \theta_c^2\Phi\rangle + \frac{1}{\lambda}\langle w, (x+c-2)\Phi_c(x)\rangle + (1-\frac{1}{\lambda})(c-1)\Phi'(c)\ . \tag{21}$$

Proceeding as in (21), we can easily prove that

$$\begin{aligned}\langle \vartheta, \theta_c F\rangle = {} & \tfrac{1}{\lambda}(c-1)^2\langle w, \theta_c\Psi\rangle + \tfrac{1}{\lambda}\langle w, (x+c-2)\Psi\rangle + 2(\tfrac{1}{\lambda}-1)\langle w, B\rangle \\ & +2(\tfrac{1}{\lambda}-1)(c-1)\langle w, \theta_c B\rangle - \tfrac{2}{\lambda}\langle w, (x-1)\Phi_c\rangle \\ & +(1-\tfrac{1}{\lambda})\Big((c-1)\Psi(c) + 2(1-\lambda)B(c)\Big)\ .\end{aligned} \tag{22}$$

On the other hand, from (12), we obtain

$$\begin{aligned}\langle \vartheta^2, \theta_0\theta_c G\rangle = {} & \tfrac{1}{\lambda^2}\langle w^2 + x^{-2}w^2 - 2x^{-1}w^2, \theta_0\theta_c G\rangle + \tfrac{2}{\lambda}(1-\tfrac{1}{\lambda})\langle w - x^{-1}w, \theta_0\theta_c G\rangle \\ & + \lambda(1-\tfrac{1}{\lambda})^2 B(c).\end{aligned}$$

By applying the same process as we did to obtain (21), we get after some straightforward calculations

$$\begin{aligned}\langle \vartheta^2, \theta_0\theta_c G\rangle = {} & \tfrac{1}{\lambda}(c-1)^2\langle w^2, \theta_0\theta_c B\rangle + \tfrac{1}{\lambda}\langle w^2, \theta_0\big((\xi+c-2)B\big)\rangle \\ & +2(c-1)(1-\tfrac{1}{\lambda})\langle w, \theta_c B\rangle + 2(1-\tfrac{1}{\lambda})\langle w, B\rangle + \lambda(1-\tfrac{1}{\lambda})^2 B(c)\ .\end{aligned} \tag{23}$$

Adding (21), (22) and (23), we obtain

$$\begin{aligned}\langle \vartheta, \theta_c^2 E + \theta_c F + \vartheta\theta_0\theta_c G\rangle = {} & \tfrac{1}{\lambda}(c-1)^2\langle w, \theta_c^2\Phi + \theta_c\Psi + w\theta_0\theta_c B\rangle \\ & +\tfrac{1}{\lambda}\langle(\Phi w)' + \Psi w + B(x^{-1}w^2), x+c-2\rangle \\ & +(1-\tfrac{1}{\lambda})\Big((c-1)(\Phi' + \Psi)(c) + (1-\lambda)B(c)\Big)\ .\end{aligned}$$

This yields (17), since $\langle(\Phi w)' + \Psi w + B(x^{-1}w^2), x+c-2\rangle = 0$, from (13). Next, it is easy to find (18) from (16).

(ii) Let c be a root of E such that $c \neq 1$. According to (16) we get $c\Phi(c) = 0$.

Two cases occur to discuss:

• If $c \neq 0$, then $\Phi(c) = 0$. We suppose $|E^{'}(c) + F(c)| + |G(c)| = 0$. According to (17) and (18), we obtain

$$\langle \vartheta, \theta_c^2 E + \theta_c F + \vartheta\theta_0\theta_c G\rangle = \frac{1}{\lambda}(c-1)^2\langle w, \theta_c^2\Phi + \theta_c\Psi + w\theta_0\theta_c B\rangle \neq 0,$$

since $|\Phi^{'}(c) + \Psi(c)| + |B(c)| = 0$.

• If $c = 0$, then $|E^{'}(0) + F(0)| + |G(0)| = \Phi(0)$. We suppose that $\Phi(0) = 0$.

When $|\Phi^{'}(0) + \Psi(0)| + |B(0)| = 0$, we get

$$\langle \vartheta, \theta_0^2 E + \theta_0 F + \vartheta\theta_0^2 G\rangle = \frac{1}{\lambda}\langle w, \theta_0^2\Phi + \theta_0\Psi + w\theta_0^2 B\rangle \neq 0,$$

from (17) and (18).

When $|\Phi^{'}(0) + \Psi(0)| + |B(0)| \neq 0$, the assumption $\lambda \notin \{\lambda_{-1}, \lambda_{-2}\}$ gives $\langle \vartheta, \theta_0^2 E + \theta_0 F + \vartheta\theta_0^2 G\rangle \neq 0$. Therefore, equation (15) is not simplified by $x - c$ for $c \neq 1$.

□

Proof. (of Theorem 1) We may write $E^{'}(1) + F(1) = 2(1-\lambda)B(1) - \Phi(1)$ and $G(1) = \lambda B(1)$.

(a) If $(\Phi(1), B(1)) \neq (0,0)$, then $|E^{'}(1)+F(1)|+|G(1)| \neq 0$. Thus, equation (15) cannot be simplified and so the form ϑ is of class

$$s_\vartheta = \max\big(\deg(E)-2, \deg(F)-1, \deg(G)-2\big) = \max\big(\deg(\Phi), \deg(\Psi)+1, \deg(B)\big)\ .$$

Hence, $s_\vartheta = 3$.

(b) If $(\Phi(1), B(1)) = (0,0)$, then

$$|E^{'}(1) + F(1)| + |G(1)| = 0 \text{ and } \langle \vartheta, \theta_1^2 E + \theta_1 F + \vartheta\theta_0\theta_1 G\rangle = 0,$$

according to (17) and (18). So, equation (15) can be simplified by the polynomial $x - 1$ and becomes

$$(\tilde{E}\vartheta)^{'} + \tilde{F}\vartheta + \tilde{G}(x^{-1}\vartheta^2) = 0\ , \tag{24}$$

where

$$\begin{aligned} &\tilde{E}(x) = x\Phi(x)\,, \quad \tilde{G}(x) = \lambda x^2(\theta_1 B)(x)\ , \\ &\tilde{F}(x) = x\big(\Psi + 2(1-\lambda)(\theta_1 B) - (\theta_1\Phi)\big)(x)\ . \end{aligned} \tag{25}$$

It is easy to see that (24) is not simplified, since

$$(\tilde{E}^{'}(1) + \tilde{F}(1), \tilde{G}(1)) = (\Psi(1) + 2(1-\lambda)B^{'}(1), \lambda B^{'}(1)) \neq (0,0).$$

Therefore $s_\vartheta = 2$.

(c) If $(\Phi(1), B(1)) = (0,0)$ and $(\Psi(1), B^{'}(1)) = (0,0)$, then

$$(\tilde{E}^{'}(1) + \tilde{F}(1), \tilde{G}(1)) = (\Psi(1) + 2(1-\lambda)B^{'}(1), \lambda B^{'}(1)) = (0,0).$$

A simple calculation gives $\langle \vartheta, \theta_1^2\tilde{E} + \theta_1\tilde{F} + \vartheta\theta_0\theta_1\tilde{G}\rangle = \frac{1}{\lambda}\langle w, \Psi\rangle + \frac{1}{\lambda}\langle w^2, \theta_0 B\rangle = 0$. So, (24)-(25) is simplified by the polynomial $x - 1$ and it becomes

$$(\hat{E}\vartheta)^{'} + \hat{F}\vartheta + \hat{G}(x^{-1}\vartheta^2) = 0\ , \tag{26}$$

where

$$\begin{aligned} &\hat{E}(x) = x(\theta_1\Phi)(x)\,, \quad \hat{G}(x) = \lambda x^2(\theta_1^2 B)(x)\ , \\ &\hat{F}(x) = x\big((\theta_1\Psi) + 2(1-\lambda)(\theta_1^2 B)\big)(x)\ . \end{aligned} \tag{27}$$

If 1 is a root of $\theta_1\Phi$, then $\Phi^{'}(1)+\Psi(1)=B(1)=0$. Assuming that $|\hat{E}^{'}(1)+\hat{F}(1)|+|\hat{G}(1)|=0$. A simple calculations gives

$$\langle\vartheta,\theta_1^2\hat{E}+\theta_1\hat{F}+\vartheta\theta_0\theta_1\hat{G}\rangle=\frac{1}{\lambda}\langle w,\theta_1^2\Phi+\theta_1\Psi+w\theta_0\theta_1 B\rangle\neq 0,$$

since w is a Laguerre-Hahn and it satisfies (14). Hence, equation (26)-(27) is not simplified and so $s_\vartheta=1$. □

3.2. Structure of the polynomials Φ, Ψ and B. We can decompose the polynomials Φ, Ψ, B, $\theta_1\Phi$, $\theta_1\Psi$, $\theta_1 B$ and $\theta_1^2 B$ into their odd and even parts. Set

$$\begin{aligned}&\Phi(x)=\Phi^e(x^2)+x\Phi^o(x^2)\ ,\ (\theta_1\Phi)(x)=\Phi_1^e(x^2)+x\Phi_1^o(x^2)\ ,\\&\Psi(x)=\Psi^e(x^2)+x\Psi^o(x^2)\ ,\ (\theta_1\Psi)(x)=\Psi_1^e(x^2)+x\Psi_1^o(x^2)\ ,\\&B(x)=B^e(x^2)+xB^o(x^2)\ ,\ (\theta_1 B)(x)=B_1^e(x^2)+xB_1^o(x^2)\ ,\\&(\theta_1^2 B)(x)=B_2^e(x^2)+xB_2^o(x^2)\ .\end{aligned}\tag{28}$$

Proposition 4. *Let w be a Laguerre-Hahn linear form of class $s_w=1$ satisfying (13) and $\{W_n\}_{n\geq 0}$ be its corresponding MOPS, such that $\beta_n=(-1)^n,\ n\geq 0$. The following statements hold:*

(a) If $(\Phi(1),B(1))\neq(0,0)$, then $\Phi^e(x)=\Phi^o(x)=\frac{1}{2}\big(x\Psi^o(x)-\Psi^e(x)\big)$ and $B^e=0$.

(b) If $(\Phi(1),B(1))=(0,0)$ and $(\Psi(1),B^{'}(1))\neq(0,0)$, then $\Phi^e=B_1^o=0$ and $\Psi^o(x)=\Phi_1^o(x)$.

(c) If $(\Phi(1),B(1))=(0,0)$ and $(\Psi(1),B^{'}(1))=(0,0)$, then $\Phi^e(x)+\Phi^o(x)=\Psi^e(x)+x\Psi^o(x)=0$ and $(x+1)B^e(x)+2xB^o(x)=0$.

Proof. Writing

$$\tilde{E}(x)=\tilde{E}^e(x)+x\tilde{E}^o(x)\ ,\ \tilde{F}(x)=\tilde{F}^e(x)+x\tilde{F}^o(x)\ ,\ \tilde{G}(x)=\tilde{G}^e(x)+x\tilde{G}^o(x)\ .\tag{29}$$

(a) $(\Phi(1),B(1))\neq(0,0)$. According to (28), (29) and from the expression of polynomials $\tilde{E}$, $\tilde{F}$ and $\tilde{G}$ given in Theorem 1, we get

$$\begin{aligned}&\tilde{E}^e(x)=x(\Phi^e-\Phi^o)(x)\ ,\quad \tilde{E}^o(x)=x\Phi^o(x)-\Phi^e(x)\ ,\\&\tilde{F}^e(x)=x\big(\Psi^e-\Psi^o-2\Phi^o+2(1-\lambda)B^o\big)(x)\ ,\\&\tilde{F}^o(x)=x\Psi^o(x)+\big(2(1-\lambda)B^e-\Psi^e-2\Phi^e\big)(x)\ ,\\&\tilde{G}^e(x)=\lambda xB^e(x)\ ,\quad \tilde{G}^o(x)=\lambda xB^o(x)\ .\end{aligned}$$

Then, $\tilde{E}^e=\tilde{F}^o=\tilde{G}^e=0$, from Proposition 3, since $s_\vartheta=3$. This gives (a).

(b) $(\Phi(1),B(1))=(0,0)$ and $(\Psi(1),B^{'}(1))\neq(0,0)$. Similar as above,

$$\begin{aligned}&\tilde{E}^e(x)=x\Phi^o(x)\ ,\quad \tilde{E}^o(x)=\Phi^e(x)\ ,\\&\tilde{F}^e(x)=x\big(\Psi^o+2(1-\lambda)B_1^o-\Phi_1^o\big)(x)\ ,\\&\tilde{F}^o(x)=\Psi^e(x)+2(1-\lambda)B_1^e(x)-\Phi_1^e(x)\ ,\\&\tilde{G}^e(x)=\lambda xB_1^e(x)\ ,\quad \tilde{G}^o(x)=\lambda xB_1^o(x)\ .\end{aligned}$$

If $s_\vartheta=2$, then $\tilde{E}^o=\tilde{F}^e=\tilde{G}^o=0$. This leads to result (b).

(c) $(\Phi(1),B(1))=(0,0)$ and $(\Psi(1),B^{'}(1))=(0,0)$. In this case, we have

$$\begin{aligned}&\tilde{E}^e(x)=x\Phi_1^o(x)\ ,\quad \tilde{E}^o(x)=\Phi_1^e(x)\ ,\\&\tilde{F}^e(x)=x\big(\Psi_1^o+2(1-\lambda)B_2^o\big)(x)\ ,\quad \tilde{F}^o(x)=\Psi_1^e(x)+2(1-\lambda)B_2^e(x)\ ,\\&\tilde{G}^e(x)=\lambda xB_2^e(x)\ ,\quad \tilde{G}^o(x)=\lambda xB_2^o(x)\ .\end{aligned}$$

Since ϑ is of odd class, $\tilde{E}^e = \tilde{F}^o = \tilde{G}^e = 0$. Therefore $\Phi_1^o = B_2^e = 0$ and $\Psi_1^e = 0$. Moreover we can write $\Phi(x) = (x-1)(\theta_1\Phi)(x) = (x-1)\Phi_1^e(x^2)$, $B(x) = (x-1)^2(\theta_1^2 B)(x) = (x-1)^2 x B_2^o(x^2)$ and $\Psi(x) = (x-1)x\Psi_1^o(x^2)$. So $\Phi^e(x) = -\Phi_1^e(x)$, $\Phi^o(x) = \Phi_1^e(x)$, $B^e(x) = -2xB_2^o(x)$, $B^o(x) = (x+1)B_2^o(x)$, $\Psi^e(x) = x\Psi_1^o(x)$, $\Psi^o(x) = -\Psi_1^o(x)$. This gives the desired result. □

Theorem 2. *Let w be a Laguerre-Hahn linear form of class one satisfying (13) and $\{W_n\}_{n\geq 0}$ be its corresponding MOPS fulfilling (8). The solutions of the functional equation (13) are given by*

$$\Phi(x) = x(x^2-1)\ , \quad \Psi(x) = ax^2 + x + c\ , \quad B(x) = (x-1)(dx^2+f)\ , \tag{30}$$

with

$$\begin{aligned} &|a+2d| + |d+f| \neq 0\ , \quad |a+d+2| + |f| + |c-1| \neq 0\ , \\ &|a+c+3| + |a+d+2| \neq 0\ , \quad |a+c+1| + |d+f| \neq 0\ . \end{aligned} \tag{31}$$

For the proof, we use the following lemma.

Lemma 3. *We have the following formulas:*

$$\begin{aligned} (w)_1 &= 1\ , \\ (w)_2 = (w)_3 &= \gamma_1 + 1\ , \\ (w)_4 = (w)_5 &= (1+\gamma_1)^2 + \gamma_1\gamma_2\ , \\ (w)_6 &= \gamma_1(\gamma_2+\gamma_1+1)^2 + (\gamma_1+1)^2 + \gamma_1\gamma_2(\gamma_3+1)\ , \end{aligned} \tag{32}$$

$$\begin{aligned} (w^2)_1 &= 2\ , \\ (w^2)_2 &= 2\gamma_1 + 3\ , \\ (w^2)_3 &= 4(\gamma_1+1)\ , \\ (w^2)_4 &= 3(\gamma_1+1)^2 + 2(\gamma_1+1) + 2\gamma_1\gamma_2\ , \\ (w^2)_5 &= 6(\gamma_1+1)^2 + 4\gamma_1\gamma_2\ , \\ (w^2)_6 &= 2\gamma_1(\gamma_2+\gamma_1+1)^2 + (2\gamma_1+7)(\gamma_1+1)^2 + 2\gamma_1\gamma_2(\gamma_3+\gamma_1+3)\ . \end{aligned} \tag{33}$$

Proof. We have $(w)_{2n} = (u)_n\ ,\ n \geq 0$. Then we can easily prove that $(w)_2 = \beta_0^P$, $(w)_4 = \left(\beta_0^P\right)^2 + \gamma_1^P$ and $(w)_6 = \left(\beta_0^P\right)^3 + \left(2\beta_0^P + \beta_1^P\right)\gamma_1^P$. Hence, from (9) and the statement (b) of Lemma 1, we can deduce the desired result (32).

Finally, from (4) we have $(w^2)_n = \sum_{k=0}^{n}(w)_{n-k}(w)_k\ ,\ n \geq 0$. Thus leads to results (33) from (32). □

Proof. (of Theorem 2) When $\deg(\Phi) \leq 2$, $\deg(B) \leq 3$ and $\deg(\Psi) \leq 2$, we consider $(a,b,c,d,e,f,g) \in \mathbb{C}^7$ such that $\Psi(x) = ax^2 + bx + c$ and $B(x) = dx^3 + ex^2 + fx + g$ with $(a,d) \neq (0,0)$. From Proposition 4, we have the following:

(i) If $(\Phi(1), B(1)) \neq (0,0)$, then $\Phi^e(x) = \Phi^o(x) = \frac{1}{2}(x\Psi^o(x) - \Psi^e(x))$ and $B^e = 0$. So, from (28), $\Phi(x) = (x+1)\Phi^e(x^2)$ and $B(x) = xB^o(x^2)$. Since Φ is a monic polynomial of degree at most two, then necessarily $\Phi^e(x) = 1$. In addition, we have $x\Psi^o(x) - \Psi^e(x) = 2$. This implies that $a = b$ and $c = -2$. Then $\Phi(x) = x+1$,

$B(x) = dx^3 + fx$ and $\Psi(x) = ax^2 + ax - 2$ with $(a, d) \neq (0, 0)$. According to equation (13), we have

$$\langle (\Phi w)' + \Psi w + B(x^{-1}w^2), x^n \rangle = 0 \ , \ n = 0, 1, 2 \ . \tag{34}$$

Then, from (30) we deduce

$$\begin{aligned}
&\langle w, ax^2 + ax - 2 \rangle + \langle w^2, dx^2 + f \rangle = 0 \ , \\
&\langle w, ax^3 + ax^2 - 3x - 1 \rangle + \langle w^2, dx^3 + fx \rangle = 0 \ , \\
&\langle w, ax^4 + ax^3 - 4x^2 - 2x \rangle + \langle w^2, dx^4 + fx^2 \rangle = 0 \ .
\end{aligned}$$

Taking into account the Lemma 3, it is equivalent to

$$(\gamma_1 + 1)(a + 2d) + a + d + f - 2 = 0 \ , \tag{35}$$

$$(\gamma_1 + 1)(a + 2d) + f - 2 = 0 \ , \tag{36}$$

$$(\gamma_1 + 1)\big((\gamma_1 + 1)d + f - 2\big) + (a + 2d)\gamma_1\gamma_2 = 0 \ . \tag{37}$$

Subtracting identities (35) and (36), we obtain

$$\begin{aligned}
a + d &= 0 \ , \\
d(\gamma_1 + 1) + f - 2 &= 0 \ .
\end{aligned} \tag{38}$$

Hence, relation (37) becomes

$$d\gamma_1\gamma_2 = 0 \ .$$

Then, from the above relation, (38) and taking into account the regularity of w, we obtain $a = d = 0$, that is a contradiction with $(a, d) \neq (0, 0)$.

(ii) If $(\Phi(1), B(1)) = (0, 0)$ and $(\Psi(1), B'(1)) \neq (0, 0)$, then $\Phi^e = 0$. So, $\Phi(x) = x$, since Φ is monic polynomial and $\deg(\Phi) \leq 2$. This contradicts $\Phi(1) = 0$.

(iii) If $(\Phi(1), B(1)) = (0, 0)$ and $(\Psi(1), B'(1)) = (0, 0)$, hence

$$\Phi(x) = x - 1 \ , \quad \Psi(x) = ax(x - 1) \ , \quad B(x) = dx(x - 1)^2 \ . \tag{39}$$

Using (34) and (39), we deduce

$$\begin{aligned}
&(a + 2d)\gamma_1 = 0 \ , \\
&(a + 2d)(\gamma_2 + \gamma_1 + 1) + (d\gamma_1 - 2)\gamma_1 = 0 \ .
\end{aligned}$$

From the above equation and taking into account the regularity of w, we obtain

$$a + 2d = 0 \ , \quad d\gamma_1 - 2 = 0 \ . \tag{40}$$

Now, using (13) and (39), we get

$$a\langle w, x^6 - x^5 \rangle - 4\langle w, x^4 - x^3 \rangle + d\langle w^2, x^6 - 2x^5 + x^4 \rangle = 0 \ .$$

Hence, from the above equation and taking into account (40) and Lemma 3, we obtain $\gamma_2\gamma_1 = 0$. It is a contradiction by virtue of the orthogonality of the sequence $\{W_n\}_{n \geq 0}$.

When $\deg(\Phi) = 3$, we obtain from (28) $\deg(\Phi^e) \leq 1$ and $\deg(\Phi^o) = 1$. According to Proposition 4, we have the following:

(i) If $(\Phi(1), B(1)) \neq (0, 0)$, then $\Phi^e(x) = \Phi^o(x)$, $\Psi^e(x) = x\Psi^o(x) - 2\Phi^e(x)$ and $B^e = 0$. We get $\Phi(x) = (x + 1)\Phi^o(x^2)$, $\Psi(x) = x(x + 1)\Psi^o(x^2) - 2\Phi^e(x^2)$ and $B(x) = xB^o(x^2)$. Therefore Ψ^o is constant polynomial, $\deg(B^o) \leq 1$ and Φ^o is a monic polynomial of degree one since $\deg(\Psi) \leq 2$, $\deg(B) \leq 3$ and $\deg(\Phi) = 3$. Hence, if denoting $\Phi^o(x) = x + h$, we get

$$\Phi(x) = (x + 1)(x^2 + h) \ , \quad \Psi(x) = (b - 2)x^2 + bx - 2h \ , \quad B(x) = dx^3 + fx \ . \tag{41}$$

From (34) and (41), we obtain

$$\langle w, (b-2)x^2 + bx - 2h\rangle + \langle w^2, dx^2 + f\rangle = 0\ ,$$

$$\langle w, (b-2)x^3 + bx^2 - 2hx\rangle - \langle w, (x+1)(x^2+h)\rangle + \langle w^2, dx^3 + fx\rangle = 0\ ,$$

$$\langle w, (b-2)x^4 + bx^3 - 2hx^2\rangle - 2\langle w, (x^2+x)(x^2+h)\rangle + \langle w^2, dx^4 + fx^2\rangle = 0\ , \tag{42}$$

what implies

$$\begin{aligned}&(b+2d-2)(\gamma_1+1) + b + d + f - 2h = 0\ ,\\ &(b+2d-2)(\gamma_1+1) + f - 2h = 0\ ,\end{aligned} \tag{43}$$

$$(b+3d-4)(\gamma_1+1)^2 + (b+2d-4)\gamma_1\gamma_2 + (b+2d+2f-4h)(\gamma_1+1) + f - 2h = 0\ . \tag{44}$$

Using (43), we get

$$b + d = 0\ , \quad (d-2)(\gamma_1+1) + f - 2h = 0\ . \tag{45}$$

Now, from (13), we get $\langle w, x^3\Psi(x) - 3x^2\Phi(x)\rangle + \langle w^2, x^2B(x)\rangle = 0$. Taking into account Lemma 3 and the second identity in (45), we obtain

$$(b+3d-4)(\gamma_1+1)^2 + (b+2d-4)\gamma_1\gamma_2 + (2f-4h)(\gamma_1+1) = 0\ . \tag{46}$$

Then, from (44), (46) and taking into account the first identity in (45), we have

$$d(\gamma_1+1) + f - 2h = 0\ .$$

Thus, by virtue of (45), we get

$$\gamma_1 + 1 = 0\ , \quad f - 2h = 0\ . \tag{47}$$

Hence, from (46), we can deduce

$$(d-2)\gamma_2 = 0\ . \tag{48}$$

Now, from (42), we can write

$$2\langle w, W_2^2\rangle = \langle w, (b-2)x^4 + bx^3 - 2hx^2\rangle - 2\langle w, x(x^2+hx+h)\rangle + \langle w^2, dx^4 + fx^2\rangle\ ,$$

since $W_2(x) = x^2$. Thus, from (45), (47), (48) and taking into account Lemma 3, we have $\langle w, W_2^2\rangle = 0$, that is a contradiction of regularity of the form w.

(ii) If $(\Phi(1), B(1)) = (0,0)$ and $(\Psi(1), B'(1)) = (0,0)$, then $\Phi^e = -\Phi^o$, $\Psi^e(x) = -x\Psi^o(x)$ and $2xB^o(x) = -(x+1)B^e(x)$. Therefore

$$\Phi(x) = (x-1)(x^2+h)\ , \quad \Psi(x) = ax(x-1)\ , \quad B(x) = dx(x-1)^2\ , \tag{49}$$

with $a \neq 0$ since $1 \leq \deg(\Psi) \leq 2$. Thus, from (34) and (49), we can deduce

$$(a+2d)\gamma_1 = 0\ , \quad (a+2d-2)(\gamma_2+\gamma_1+1) + d\gamma_1 - 2h = 0\ .$$

By virtue of the above equations and taking into account the regularity of the form w, we obtain

$$a + 2d = 0\ , \quad 2(\gamma_2+\gamma_1+h+1) - d\gamma_1 = 0\ . \tag{50}$$

Now, using (13) and (49), we get

$$4\langle w, x^3(x-1)(x^2+h)\rangle - a\langle w, x^5(x-1)\rangle - d\langle w^2, x^4(x-1)^2\rangle = 0\ .$$

Hence, from the above equation and taking into account (50) and Lemma 3, we obtain $\gamma_2\gamma_3 = 0$. It is a contradiction by virtue of the regularity of w.

(iii) If $(\Phi(1), B(1)) = (0,0)$ and $(\Psi(1), B'(1)) \neq (0,0)$, then $\Phi^e = 0$, $\Psi^o(x) = \Phi_1^o(x)$ and $B_1^o = 0$. So $\Phi(x) = x(x^2-1)$, $\Psi(x) = ax^2 + x + c$ and $B(x) = (x-1)(dx^2+f)$. If $(a+2d, d+f) = (0,0)$, then $a+c+1 = 0$ and $d = -f$ since $\langle w, \Psi\rangle + \langle w^2, \theta_0 B\rangle = 0$.

Thus $\Psi(x) = (x-1)(ax+a+1)$ and $B(x) = d(x+1)(x-1)^2$ which contradiction $(\Psi(1), B^{'}(1)) \neq (0,0)$. Necessarily $(a+2d, d+f) \neq (0,0)$. Moreover the form w is of class one, we shall have the condition (14) with $\mathcal{Z}_\Phi = \{-1,0,1\}$, which leads to relation (31). □

Remark 2. *If $d = f = 0$, then we obtain the result given in* [6], [11], [17]. *Indeed the form $w = \kappa$ is semiclassical and satisfies the functional equation*

$$\big(x(x^2-1)\kappa\big)^{'} + (ax^2+x+c)\kappa = 0 \ , \tag{51}$$

with

$$a(a+c+1) \neq 0 \ , \quad |a+2| + |c-1| \neq 0 \ , \quad |a+c+3| + |a+2| \neq 0 \ . \tag{52}$$

3.3. The Computation of γ_{n+1}. We will study the form w given in Theorem 2. The form w fulfills the following equation

$$\big(x(x^2-1)w\big)^{'} + (ax^2+x+c)w + (x-1)(dx^2+f)(x^{-1}w^2) = 0 \ . \tag{53}$$

On the other hand, while applying the operator σ in (53) and using relations (6) and (10), the form $u = \sigma(w)$ is Laguerre-Hahn satisfying

$$\big(x(x-1)u\big)^{'} + \frac{1}{2}(ax+c+1)u + \frac{1}{2}(x-1)(dx+f)(x^{-1}u^2) = 0 \ . \tag{54}$$

Multiplying (54) by $(x-1)^2$, then the form $v = \gamma_1^{-1}(x-1)u$ fulfills

$$\big(x(x-1^2)v\big)^{'} + \frac{1}{2}(x-1)\big((2d+a-4)x+2f+c+1\big)v + \frac{\gamma_1}{2}(x-1)(dx+f)(x^{-1}v^2) = 0 \ , \tag{55}$$

according to (2), (3) and (5).

Two cases arise:

(1) If $a+d+2 \neq 0$, then we distinguish two subcases.

(i) Taking

$$\begin{cases} a = 2(1-\frac{2}{\rho})(\alpha+\beta+2\tau) - \frac{4}{\rho} \ , \\ c = (\frac{4}{\rho}-1)(\alpha+\beta+2\tau+1) + \frac{\alpha^2-\beta^2-4(\tau+1)(\tau+\beta+1)}{\alpha+\beta+2\tau+2}) \ , \\ d = 2(\frac{1}{\rho}-1)(\alpha+\beta+2\tau+1) \ , \\ f = (1-\frac{2}{\rho})(\alpha+\beta+2\tau+1) - 2\rho\frac{(\tau+1)(\tau+\beta+1)}{\alpha+\beta+2\tau+1} + \frac{4(\tau+1)(\tau+\beta+1)+\beta^2-\alpha^2}{\alpha+\beta+2\tau+2} + 1 \ . \end{cases} \tag{56}$$

Moreover, the form v is of class one fulfilling (55), (56) and

$$u = h_{-\frac{1}{2}} o \tau_{-1} \mathfrak{J}_1(\alpha,\beta,\rho,\tau,\nu_1) \ , \tag{57}$$

with

$$\nu_1 = \frac{1}{(\alpha+\beta+2\tau+2)}\Big(2\rho\frac{(\tau+1)(\tau+\beta+1)}{\alpha+\beta+2\tau+1} - \frac{\alpha^2-\beta^2}{\alpha+\beta+2\tau}\Big) - 1 \ ,$$

where $\mathfrak{J}_1(\alpha,\beta,\rho,\tau,\nu)$ is the non-singular Laguerre-Hahn form of class zero analogous to the classical Jacobi, this last form satisfies [4]

$$\big(\phi_1\mathfrak{J}_1(\alpha,\beta,\rho,\tau,\nu)\big)^{'} + \psi_1\mathfrak{J}_1(\alpha,\beta,\rho,\tau,\nu) + \varphi_1\big(x^{-1}\mathfrak{J}_1^2(\alpha,\beta,\rho,\tau,\nu)\big) = 0 \ ,$$

when

$$\begin{aligned}
&\phi_1(x) = x^2 - 1 \;, \\
&\psi_1(x) = \big((1-\tfrac{2}{\rho})(\alpha+\beta+2\tau) - \tfrac{2}{\rho}\big)x + \tfrac{2}{\rho}(\alpha+\beta+2\tau+1)\varsigma - \tfrac{\alpha^2-\beta^2}{\alpha+\beta+2\tau+2} \;, \\
&\varphi_1(x) = (\tfrac{1}{\rho}-1)(\alpha+\beta+2\tau+1)x^2 + \big\{\big((\alpha+\beta+2\tau)(1-\tfrac{2}{\rho}) - 2(\tfrac{1}{\rho}-1)\big)\varsigma \\
&\qquad + \tfrac{\alpha^2-\beta^2}{\alpha+\beta+2\tau+2}\big\}x + \tfrac{\alpha+\beta+2\tau+1}{\rho}\varsigma^2 - \tfrac{\alpha^2-\beta^2}{\alpha+\beta+2\tau+2}\varsigma + (\alpha+\beta+2\tau+3)\varpi - 1 \;,
\end{aligned}$$

with

$$\varsigma = \frac{\alpha^2-\beta^2}{(\alpha+\beta+2\tau)(\alpha+\beta+2\tau+2)} + \nu \;,$$

$$\varpi = \frac{4\rho(\tau+1)(\tau+\alpha+1)(\tau+\beta+1)(\tau+\alpha+\beta+1)}{(2\tau+\alpha+\beta+1)(2\tau+\alpha+\beta+2)^2(2\tau+\alpha+\beta+3)} \;.$$

$\mathfrak{J}_1(\alpha,\beta,\rho,\tau,\nu)$ is regular if and only if $\rho \neq 0$, $\tau \neq -(n+1)$, $\tau+\alpha \neq -(n+1)$, $\tau+\beta \neq -(n+1)$, $\tau+\alpha+\beta \neq -(n+1)$, $n \geq 0$. Moreover, the coefficients of its corresponding MOPS $\{Z_n^{(\alpha,\beta,\rho,\tau,\nu)}\}_{n\geq 0}$ are given by (for $n \geq 0$)

(58)
$$\begin{cases} \beta_0^{(\alpha,\beta,\rho,\tau,\nu)} = \varsigma \;,\; \beta_{n+1}^{(\alpha,\beta,\rho,\tau,\nu)} = \frac{\alpha^2-\beta^2}{(2n+2\tau+\alpha+\beta+2)(2n+2\tau+\alpha+\beta+4)}, \\ \gamma_1^{(\alpha,\beta,\rho,\tau,\nu)} = \varpi \;,\; \gamma_{n+2}^{(\alpha,\beta,\rho,\tau,\nu)} = \frac{4(n+\tau+2)(n+\tau+\alpha+2)(n+\tau+\beta+2)(n+\tau+\alpha+\beta+2)}{(2n+2\tau+\alpha+\beta+3)(2n+2\tau+\alpha+\beta+4)^2(2n+2\tau+\alpha+\beta+5)} \;. \end{cases}$$

Proposition 5. *Let w be a form of class one satisfying (53)-(56). The form w is regular for $\rho \neq 0$, $\tau \neq -(n+1)$, $\tau+\alpha \neq -(n+1)$, $\tau+\beta \neq -(n+1)$, $\tau+\alpha+\beta \neq -(n+1)$, $n \geq 0$. The recurrence coefficient coefficients of its MOPS $\{W_n\}_{n\geq 0}$ are given by (for $n \geq 1$)*

(59)
$$\begin{aligned} &\gamma_1 = -\rho\frac{(\tau+\beta+1)(\tau+1)}{(\alpha+\beta+2\tau+1)(\alpha+\beta+2\tau+2)} \;, \\ &\gamma_{n+1} = -\frac{1}{4}\frac{\big((n+2\tau+2+(\alpha-\frac{1}{2})(1-(-1)^n)\big)\big((n+2\tau+2\beta+2+(\alpha-\frac{1}{2})(1-(-1)^n)\big)}{(n+2\tau+\alpha+\beta+1)(n+2\tau+\alpha+\beta+2)} \;. \end{aligned}$$

Proof. Taking into account (9), we obtain

$$\frac{\gamma_{2n+3}}{\gamma_{2n+1}} = \frac{\gamma_{n+1}^R}{\gamma_{n+1}^P} \;,\; n \geq 0 \;.$$

Hence, from the following relation: [8]

$$\gamma_{n+1}^R = \frac{P_{n+2}(1)P_n(1)}{P_{n+1}^2(1)}\gamma_{n+1}^P \;,$$

it is to see that

$$\frac{\gamma_{2n+3}}{\gamma_{2n+1}} = \frac{P_{n+2}(1)P_n(1)}{P_{n+1}^2(1)} \;,\; n \geq 0 \;.$$

Then, we obtain by induction

(60)
$$\gamma_{2n+1} = -\frac{P_{n+1}(1)}{P_n(1)} \;,\; n \geq 0 \;.$$

Therefore, from (9) we can deduce

(61)
$$\gamma_{2n+2} = -\gamma_{n+1}^P\frac{P_n(1)}{P_{n+1}(1)} \;,\; n \geq 0 \;.$$

On the other hand, from (57) we have

(62)
$$P_{n+1}(1) = (-2)^{-n-1}Z_{n+1}^{(\alpha,\beta,\rho,\tau,\nu_1)}(-1) \;,\; n \geq 0 \;.$$

Thus, from (7) and (58), we obtain for $n \geq 0$

$$P_{n+1}(1) = \rho \frac{\Gamma(2\tau+\alpha+\beta+1)\Gamma(n+\tau+2)\Gamma(n+\tau+\beta+2)}{\Gamma(\tau+1)\Gamma(\tau+\beta+1)\Gamma(2n+2\tau+\alpha+\beta+3)} \,. \tag{63}$$

By virtue of (60), (61), (63) and from a simple calculation we can deduce (59). □

(ii) Putting

$$\begin{cases} a = 2(1-\frac{2}{\rho})(\alpha+\beta+2\tau) - \frac{4}{\rho} \,, \\ c = (\frac{4}{\rho}-1)(\alpha+\beta+2\tau+1) + \frac{\alpha^2-\beta^2-4(\tau+\alpha+1)(\tau+\alpha+\beta+1)}{\alpha+\beta+2\tau+2}) \,, \\ d = 2(\frac{1}{\rho}-1)(\alpha+\beta+2\tau+1) \,, \\ f = (1-\frac{2}{\rho})(\alpha+\beta+2\tau+1) - 2\rho\frac{(\tau+\alpha+1)(\tau+\alpha+\beta+1)}{\alpha+\beta+2\tau+1} \\ \qquad + \frac{4(\tau+\alpha+1)(\tau+\alpha+\beta+1)+\beta^2-\alpha^2}{\alpha+\beta+2\tau+2} + 1 \,. \end{cases} \tag{64}$$

Moreover, the form v is of class one fulfilling (55)-(64) and

$$u = h_{-\frac{1}{2}} o \tau_{-1} \mathfrak{J}_1(\alpha,\beta,\rho,\tau,\nu_2) \,, \tag{65}$$

with

$$\nu_2 = \frac{1}{\alpha+\beta+2\tau+2}\Big(2\rho\frac{(\tau+\alpha+1)(\tau+\alpha+\beta+1)}{\alpha+\beta+2\tau+1} - \frac{\alpha^2-\beta^2}{\alpha+\beta+2\tau}\Big) - 1 \,.$$

By applying the same process as we did to obtain (63) and using (65), we get after some straightforward calculation

$$P_{n+1}(1) = \rho\frac{\Gamma(2\tau+\alpha+\beta+1)\Gamma(n+\tau++\alpha+2)\Gamma(n+\tau+\alpha+\beta+2)}{\Gamma(\tau+\alpha+1)\Gamma(\tau+\alpha+\beta+1)\Gamma(2n+2\tau+\alpha+\beta+3)} \,,\; n \geq 0 \,.$$

Hence, by virtue of the previous equation, (60) and (61) we get

$$\begin{aligned} &\gamma_1 = -\rho\frac{(\tau+\alpha+\beta+1)(\tau+\alpha+1)}{(\alpha+\beta+2\tau+1)(\alpha+\beta+2\tau+2)} \,, \\ &\gamma_{n+1} = -\frac{1}{4}\frac{\Big((n+2\tau+1+(\alpha+\frac{1}{2})(1+(-1)^n)\Big)\Big((n+2\tau+2\beta+1+(\alpha+\frac{1}{2})(1+(-1)^n)\Big)}{(n+2\tau+\alpha+\beta+1)(n+2\tau+\alpha+\beta+2)} \,. \end{aligned} \tag{66}$$

consequently, The form w is regular for $\rho \neq 0$, $\tau \neq -(n+1)$, $\tau+\alpha \neq -(n+1)$, $\tau+\beta \neq -(n+1)$, $\tau+\alpha+\beta \neq -(n+1)$, $n \geq 0$.

Remark 3. *If we take $\tau = 0$ and $\rho = 1$ in (59) and (66), we obtain $w = \kappa$ where κ is the form satisfying (51)-(52).*

(2) If $a+d+2=0$, choosing

$$a = 2(\alpha-2) \,,\; c = -\alpha-\mu+1 \,,\; d = 2(1-\alpha) \,,\; f = 2\mu\frac{\alpha-1}{\alpha-2} - \alpha\nu \,. \tag{67}$$

Moreover, v is of class one fulfilling (55)-(67) and

$$u = h_{-\frac{1}{2}} o \tau_{-1} \mathfrak{J}_2(\alpha,\rho_1,\nu,\mu) \,, \tag{68}$$

with

$$\rho_1 = \frac{(\mu-\alpha)(\mu+2\nu+2-\alpha-\alpha\nu)}{(\alpha+1)(\alpha-2)} \,,$$

where is the singular Laguerre-Hahn form of class zero analogous to the classical Jacobi, this last form satisfies [4]

$$\big(\phi_2\mathfrak{J}_2(\alpha,\rho,\nu,\mu)\big)' + \psi_2\mathfrak{J}_2(\alpha,\rho,\nu,\mu) + \varphi_2\big(x^{-1}\mathfrak{J}_2^2(\alpha,\rho,\nu,\mu)\big) = 0 \,,$$

when

$$\phi_2(x) = x^2 - 1 ,$$
$$\psi_2(x) = (\alpha - 2)x + \mu ,$$
$$\varphi_2(x) = (1-\alpha)x^2 + \big(\alpha\nu - 2\mu\tfrac{\alpha-1}{\alpha-2}\big)x - \tfrac{\mu^2}{\alpha-2} + \mu\nu + \rho(\alpha+1) - 1 .$$

$\mathfrak{I}_2(\alpha,\rho,\nu,\mu)$ is regular if and only if $\rho \neq 0$, $\alpha \neq 2$, $\alpha \neq -n$, $\alpha \neq \pm\mu - 2n$, $n \geq 1$. Moreover, the coefficients of its corresponding MOPS $\{Q_n^{(\alpha,\rho,\nu,\mu)}\}_{n\geq 0}$ are given by (for $n \geq 0$)

(69)
$$\begin{cases} \beta_0^{(\alpha,\rho,\nu,\mu)} = \frac{\mu}{2-\alpha} + \nu \ , \ \beta_{n+1}^{(\alpha,\rho,\nu,\mu)} = -\frac{\alpha\mu}{(2n+\alpha)(2n+\alpha+2)}, \\ \gamma_1^{(\alpha,\rho,\nu,\mu)} = \rho \ , \ \gamma_{n+2}^{(\alpha,\rho,\nu,\mu)} = \frac{(n+1)(n+\alpha+1)(2n+\alpha-\mu+2)(2n+\alpha+\mu+2)}{(2n+\alpha+1)(2n+\alpha+2)^2(2n+\alpha+3)} \ . \end{cases}$$

Proposition 6. *Let w be a the form of class one satisfying (53)-(67). The form w is regular for $(\alpha-2)(\nu-1)+\mu \neq 0$, $\alpha \neq 2$, $\alpha \neq -n$, $\alpha \neq \pm\mu - 2n$, $n \geq -1$. The recurrence coefficients of its MOPS $\{W_n\}_{n\geq 0}$ are given by*

(70)
$$\begin{aligned} \gamma_1 &= -\frac{\mu+2+2\nu-\alpha-\alpha\nu}{2(2-\alpha)} \ , , \\ \gamma_{n+1} &= -\frac{1}{4}\frac{\Big(n+(\alpha-\frac{1}{2})(1-(-1)^n)\Big)\Big(n+\alpha+\mu+(1+2\mu)(\frac{(-1)^n-1}{2})\Big)}{(n+\alpha)(n+\alpha-1)} \ , \ n \geq 0 \ . \end{aligned}$$

Proof. From (68), we have

$$P_{n+1}(1) = (-2)^{-n-1}Q_{n+1}^{(\alpha,\rho_1,\nu,\mu)}(-1) \ , \ n \geq 0 \ .$$

Hence, by virtue (7), (68) and (69), we get for $n \geq 0$

$$P_{n+1}(1) = \frac{(\mu+2+2\nu-\alpha-\alpha\nu)\Gamma(\alpha+1)\Gamma(n+1)\Gamma(n+1+\frac{\alpha+\nu}{2})}{2(2-\alpha)\Gamma(1+\frac{\alpha+\nu}{2}))\Gamma(2n+\alpha+1)} \ .$$

Hence, by virtue of the last equation, (60) and (61) we can deduce (70). □

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