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ON REAL FORMS OF A BELYI ACTION OF THE ALTERNATING GROUPS

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Abstract. In virtue of the Belyi Theorem a complex algebraic curve can be defined over the algebraic numbers if and only if the corresponding Riemann surface can be uniformized by a subgroup of a Fuchsian triangle group. Such surfaces are known as Belyi surfaces. Here we study certain natural actions of the alternating groups A_n on them. We show that they are symmetric and calculate the number of connected components, called ovals, of the corresponding real forms. We show that all symmetries with ovals are conjugate and we calculate the number of purely imaginary real forms both in case of A_n considered here and S_n considered in an earlier paper [2].

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1. Introduction

There is a functorial equivalence between smooth irreducible projective complex algebraic curves and compact Riemann surfaces and in virtue of the Belyi Theorem [1] an algebraic curve can be defined over the algebraic numbers if and only if the corresponding Riemann surface can be uniformized by a subgroup of a Fuchsian triangle group. Such surfaces are known as Belyi surfaces and, by results of Köck, Lau and Singerman [7] and [8] they are symmetric if and only if the above algebraic numbers can be simultaneously chosen belonging to $\mathbb{R}$. An important class of Belyi surfaces is formed by the Riemann surfaces with so called large groups of automorphisms. More precisely, such a surface of genus g has as least $12(g-1)$ automorphisms. Necessary and sufficient algebraic conditions for these surfaces to be symmetric were found by Singerman in [10].

In [4], the third author has developed an algebraic method to calculate the number of connected components of the real forms corresponding to the symmetries given by the above theorem of Singerman. It is also worth to mention here the paper [6] which was the first tool to calculate the number of these components and its recent improvement [11] by Singerman and Watson. In [2], the method from [4] was successfully applied to study the topological type of real forms of certain symmetric Riemann surfaces determined by certain canonical actions of the symmetric groups S_n on them which correspond to certain generating pairs for S_n as described in the next section.

Here, we study the alternating groups actions within the described framework. Namely, we take certain canonical pairs of generators for the alternating groups A_n and we consider the corresponding actions of A_n on a Belyi Riemann surface. We show that such surfaces are symmetric and then we calculate the number of connected components of the corresponding real forms. We deduce that all symmetries with ovals are conjugate. The importance of A_n in this context follows from the Cayley embedding theorem which implies that an arbitrary finite group acts as a group (usually not the full group) of birational automorphisms on some algebraic curve defined over algebraic reals. Another important feature of the surfaces with the actions considered here is that they are particular examples of, so called, quasi-platonic surfaces which form the principal subject of research concerning the classical Grothendieck dessins d'enfants theory and inverse Galois problem [3].

The last section is devoted to purely imaginary real algebraic curves which are the curves which can be defined over the real but which have no $\mathbb{R}$-rational points. We show there that their number for a symmetric quasi-platonic Riemann surface X with the action of $G = \mathrm{Aut}(X)$ corresponding to a pair of generating cycles α, β for which the application $\alpha \mapsto \alpha^{-1}, \beta \mapsto \beta^{-1}$ extends to an automorphism of G does not depend on this pair if $G = \mathrm{S}_n$, while for $G = \mathrm{A}_n$ it depends on it exactly up to such extent up to which it forces $\mathrm{Aut}^{\pm}(X)$ to be S_n or $\mathrm{A}_n \times \mathrm{Z}_2$ which turn out to be the only cases that can happen for the actions considered in this paper.

2. Preliminaries and known results

We shall use the combinatorial method based on the Riemann uniformization theorem and on the algebraic theory of Fuchsian groups. Following them, a compact Riemann surface X of genus $g \geq 2$ can be represented as the orbit space $\mathcal{H}/\Gamma$ of the hyperbolic plane $\mathcal{H}$, with respect to the action of some Fuchsian surface group Γ being a discrete and cocompact subgroup of the group of isometries of $\mathcal{H}$, isomorphic to the fundamental group of the surface. Furthermore, the group of conformal automorphisms of the surface given in such a way can be represented as the factor group $G = \Delta/\Gamma$ for some other Fuchsian group Δ. Thus we can write the faithful action of a finite group G as a group of automorphisms of a Riemann surface X of genus $g \geq 2$ by a smooth epimorphism $\theta : \Delta \to G$, which means that its kernel Γ is torsion free or, equivalently, it preserves the orders of the canonical elliptic generators of Δ. However since we shall deal with surfaces with large groups of automorphisms such a group can be assumed to be a triangle group Δ with signature $(0; k, l, m)$ which means that its algebraic presentation is $\langle x_1, x_2 \,|\, x_1^k, x_2^l, (x_1x_2)^m \rangle$. Such a Fuchsian group is known to be unique up to conjugacy in the group of all isometries of $\mathcal{H}$ and so the corresponding surface is determined, up to conformal isomorphism, by a pair of generators a, b of orders k and l whose product is an element of order m. We shall refer to such surface and action as to the ones corresponding to a generating pair a, b of G. With these notations we have the following, mentioned above, result of Singerman from [10].

Theorem 2.1. *A Riemann surface X with the full large group of automorphisms G, corresponding to a generating pair (a, b), is symmetric if and only if there is an automorphism φ of G for which $\varphi(a) = a^{-1}, \varphi(b) = b^{-1}$ or $\varphi(a) = b^{-1}, \varphi(b) = a^{-1}$.*

The set of fixed points of a symmetry of a Riemann surface of genus $g \geq 2$ is homeomorphic to the set of $\mathbb{R}$-rational points of a real form of the complex algebraic curve corresponding to this surface and its symmetry. In turn, the latter consists of k disjoint Jordan curves called *ovals* for some k ranging between 0 and $g + 1$ in virtue of the classical Harnack theorem [5] with some restrictions depending on the separability of the symmetry given by Weichold in his thesis [12].

Now given an automorphism φ of G, two elements x, $y \in G$ are said to be *φ-conjugate* ($x \sim_\varphi y$) if $x = wy\varphi(w)^{-1}$ for some $w \in G$. Observe that for $\varphi = 1$ this coincides with the ordinary notion of conjugacy $\sim$. Recall also that *the isotropy group* of φ is the subgroup G_φ of G, consisting of all elements fixed by φ. With these notations we have the following result from [4] which describes the number of ovals of the conjugacy classes of symmetries from Theorem 2.1.

Theorem 2.2. *Let a and b be a generating pair of elements of a finite group G of orders $k = 2k' + 1$ and $l = 2l' + 1$ respectively so that ab has order m. Then the corresponding Riemann surface X has at most two types of symmetries: one with and one without ovals. Symmetries with ovals always exist, all of them are conjugate in G and they have N/M ovals, where N is the order of the isotropy group of φ in G and*

(1) *$M/2$ is the order of $(ab)^{m'}a^{-k'}b^{-l'}(ab)^{m'}b^{l'}a^{k'}$ if $m = 2m'$,*
(2) *M is the order of $(ab)^{-m'}b^{l'}a^{k'}$ if $m = 2m' + 1$.*

A symmetry without ovals exists if and only if $\varphi(g) = g^{-1}$ for some $g \in G$ which is not φ-conjugate to the unit of G.

Proof. For $x = ab, y = b^{-1}$, we have a generating pair for G of elements of orders m, l whose product $xy = a$ has order k. So if $m = 2m'$, by Theorem 4.1 in [4], the only symmetry up to conjugacy with fixed points has N/M ovals, where N is the order of the isotropy group of φ in G, where $\varphi(x) = x^{-1}, \varphi(y) = y^{-1}$ and $M/2$

is the order of $x^{m'}(xy)^{-k'}y^{l'}x^{m'}y^{-l'}(xy)^{k'} = (ab)^{m'}a^{-k'}b^{-l'}(ab)^{m'}b^{l'}a^{k'}$. The case (2) follows from [4] in a similar way and we omit it.

□

3. Generating pairs of A_n defining our actions

We are going to consider two generating pairs for the alternating group A_n. The starting point, to consider them, goes back to Moore in 1897 who gave in [9] a complete presentation of A_n by means of defining generators and relations. In this paper we take

$$\alpha = (1,2,3) \text{ and } \beta = \begin{cases} (1,2,\dots,n) & \text{if } n \text{ is odd,} \\ (2,3,\dots,n) & \text{if } n \text{ is even.} \end{cases} \tag{1}$$

Proposition 3.1. *The actions of the group* A_n *on Belyi Riemann surfaces, corresponding to the above generating pairs are symmetric.*

Proof. It is easy to check that for

$$\gamma = \begin{cases} (1,3)(n,4)(n-1,5)(n-2,6)\dots\left(\dfrac{n+5}{2}, \dfrac{n+3}{2}\right) & \text{if } n \text{ is odd} \\ (2,3)(n,4)(n-1,5)\dots\left(\dfrac{n+6}{2}, \dfrac{n+2}{2}\right) & \text{if } n \text{ is even} \end{cases} \tag{2}$$

we have $\gamma\alpha\gamma^{-1} = \alpha^{-1}$ and $\gamma\beta\gamma^{-1} = \beta^{-1}$ and therefore the map $\alpha \mapsto \alpha^{-1}, \beta \mapsto \beta^{-1}$ induces an automorphism φ of A_n. Hence the corresponding surfaces are symmetric by Theorem 2.1.

□

4. The number of ovals

In this section we find the number of connected components, that is to say ovals, of real forms corresponding to the symmetries of Riemann surfaces with the action of A_n given by the generating pair (α, β) defined in (1). Since, by Proposition 3.1, φ is the conjugation in $G = \mathrm{A}_n$ by $\gamma \in \mathrm{S}_n$, we obtain

Lemma 4.1. *For the isotropy group* G_φ *of* φ *in* A_n *we have*

$$|G_\varphi| = \begin{cases} 2^{(n-3)/2}((n-1)/2)! & \text{if } n \text{ is odd} \\ 2^{(n-2)/2}((n-2)/2)! & \text{if } n \text{ is even} \end{cases}$$

Proof. Observe first that γ is a product of disjoint transpositions and so the isotropy group of φ in A_n coincide with the centralizer of γ in A_n. Let

$$X = \begin{cases} \{3,4,\dots,(n+3)/2\} & \text{if } n \text{ is odd} \\ \{3,4,\dots,(n+2)/2\} & \text{if } n \text{ is even} \end{cases}$$

let $Y = \{1,\dots,n\} - X$, and let for $1 \le i \le n$, $f(i)$ denotes an element for which $(i, f(i))$ is one of the transpositions composing γ. Then it is clear that $\xi \in \mathrm{A}_n$ centralizes γ if and only if

$$\xi(i) = j \Leftrightarrow \xi(f(i)) = f(j)$$

for arbitrary i, j. Now each permutation ξ of X determines an element $\xi\xi' \in \mathrm{A}_n$ centralizing γ, where ξ' is a permutation of Y defined by

$$\xi'(i) = j \Leftrightarrow \xi(f(i)) = f(j).$$

Now it is clear that to get all elements centralizing γ for n odd we need to consider the products of all of such $\xi\xi'$ with even length products of transpositions composing γ given in (2). The case of even n is a bit different since appart of transpositions composing γ we have also the transposition $(1,(n+4)/2)$ which centralizes it.

□

Theorem 4.2. *Let X be a symmetric Riemann surface corresponding to the generating pair (α,β) defined in (1). Then X has a symmetry without ovals and a symmetry with*

$$\begin{cases} 2^{(n-5)/2}\,((n-1)/2)! & \text{if } n\equiv 1 \pmod 2\\ 2^{(n-2)/2}\dfrac{((n-2)/2)!}{n-2} & \text{if } n\equiv 0 \pmod 4\\ 2^{(n-4)/2}\dfrac{((n-2)/2)!}{n-2} & \text{if } n\equiv 2 \pmod 4\end{cases}$$

ovals.

Proof. For odd n, the generators of A_n from (1) are of $(3,n,n)$ type i.e. $\alpha,\beta,\alpha\beta$ have orders $3,n,n$ respectively. Observe that $\alpha\beta=(2,1,3,4,\dots,n-1,n)$. First we show that the surface has a symmetry without ovals. Let $\delta=(1,3)(4,n)$. Then $\varphi(\delta)=\delta=\delta^{-1}$. then for arbitrary $\eta\in\mathrm{A}_n$, we have $\eta\varphi(\eta)^{-1}=\eta\gamma\eta^{-1}\gamma^{-1}$ and so δ is not φ-conjugate to the unit since it does not belong to the commutator of S_n.

Now, in terms of (2) in Theorem 2.2, $k=3$, $l=m=n$, and so $k'=1$, $l'=m'=(n-1)/2$ and so, all symmetries with ovals are conjugate and the number of ovals is equal to N/M where N is given in the Lemma 4.1, whilst M is the order of the element $(ab)^{-(n-1)/2}b^{(n-1)/2}a$ which is equal to $(2,3)((n+3)/2,(n+5)/2)$. So $M=2$ and hence the result.

Let now $n>4$ be even. The group A_n is generated by $\alpha=(1,2,3)$ and $\beta=(2,3,\dots,n)$. We have $\alpha\beta=(1,2)(3,4,\dots,n)$ and so these generators are of type $(3,n-1,n-2)$. Therefore (1) of Theorem 2.2 has application here. In particular, all symmetries with ovals are again conjugate. In term of this theorem $k=3$, $l=n-1$, $m=n-2$, and so $k'=1$, $l'=m'=n/2-1$.

As in the case of n odd, the surface has a symmetry without ovals. For, let $\delta=(2,3)(4,n)$. Then $\varphi(\delta)=\delta^{-1}$. Now again, for arbitrary $\eta\in\mathrm{A}_n$, we have $\eta\varphi(\eta)^{-1}=\eta\gamma\eta^{-1}\gamma^{-1}$ and so δ is not φ-conjugate to the unit since it does not belong to the commutator of S_n. Take now the symmetry with ovals. Lemma 4.1 gives value of N in (1) of Theorem 2.2 and $M/2$ is the order of

$$(ab)^{n/2-1}a^{-1}b^{-(n/2-1)}(ab)^{n/2-1}b^{n/2-1}a$$

which is equal to the product

$$(1,2)^{n/2-1}(3,4,\dots,n)^{n/2-1}(1,3,2)(2,3,\dots,n)^{-(n/2-1)}\\ (1,2)^{n/2-1}(3,4,\dots,n)^{n/2-1}(2,3,\dots,n)^{n/2-1}(1,2,3).$$

When n is a multiple of 4, this permutation is

$$(2,4,5,\dots,n/2+1)(n/2+3,1,n,\dots,n/2+4),$$

and hence has order $n/2-1$, whilst in the other case, it is

$$(2,4,5,\dots,n/2+1,1,n,n-1,\dots,n/2+3)(3,n/2+2),$$

and so has order $n-2$.

Therefore M is respectively $n-2$ and $2(n-2)$, and so we have the respective results, as claimed.

□

5. Purely imaginary forms

From Theorem 2.2 we know necessary and sufficient conditions for Riemann surfaces described there to admit a symmetry without ovals which correspond to purely imaginary forms. Now we shall deal with the number of conjugacy classes of such symmetries of Riemann surfaces corresponding both to the action of $G = \mathrm{A}_n$ considered here and to the one for $G = \mathrm{S}_n$ which was considered in [2]. By [10] (see also [4] for explicit statement), $G^{\pm} = \mathrm{Aut}^{\pm}(X) = G \rtimes \mathbb{Z}_2 = \langle a, b\rangle \rtimes \langle t\rangle$, where $tgt = \varphi(g)$. With these notations we have

Theorem 5.1. *Two elements g_1, g_2 of G give rise to nonconjugate fixed point free symmetries if and only if*

(a) $\varphi(g_i) = g_i^{-1}$ *and* $g_i \not\sim_{\varphi} 1$,
(b) $g_1 \not\sim_{\varphi} g_2$ *and* $g_1 \not\sim_{\varphi} \varphi(g_2)$.

Proof. Clearly each element of $G^{\pm} \setminus G$ has the form gt for some $g \in G$. Now for a symmetry, we have $1 = (gt)^2 = g\varphi(g)$ and gt has ovals if and only if $gt \sim_{G^{\pm}} t$ which in turn means $gt = wtw^{-1} = wtw^{-1}tt$. Consequently $g \sim_{\varphi} 1$ and so (a). Now $g_1t \sim_{G^{\pm}} g_2t$ if and only if $g_1t = w(g_2t)w^{-1} = wg_2\varphi(w)t$ or $g_1t = (wt)(g_2t)(wt)^{-1} = w\varphi(g_2)\varphi(w)^{-1}t$ for some $w \in G$ which gives (b).

□

We shall not only find the number of purely imaginary real forms of surfaces considered in this paper but we shall show, actually, that this number for symmetric quasi-platonic Riemann surfaces with the action of G depends on α and β only up to a certain extent. For effective use of this theorem for our actions we need some preparation.

It is obvious that two cycles δ_1, δ_2 of the same length, say n, are conjugated and in addition there are n conjugating elements with support contained in $\mathrm{supp}(\delta_1) \cap \mathrm{supp}(\delta_2)$. Furthermore for $\delta_2 = \delta_1^{-1}$ all conjugating elements are involutions. In this case for n odd each such involution has exactly one fixed point, while for n even half of the involutions do not have fixed points and each of the remainder ones has exactly two fixed points.

Proposition 5.2. *Let α, β be the generating pair for A_n defined in (1) and let φ be an automorphism of A_n such that*

$$\alpha^{\varphi} = \alpha^{-1} \quad \text{and} \quad \beta^{\varphi} = \beta^{-1}, \tag{3}$$

then

$$\mathrm{A}_n \rtimes \langle\varphi\rangle \cong \begin{cases} \mathrm{A}_n \times \mathbb{Z}_2 & \text{if } n \equiv 1, 2 \pmod 4 \\ \mathrm{S}_n & \text{if } n \equiv 0, 3 \pmod 4. \end{cases}$$

Proof. From the proof of Proposition 3.1 we know that φ is the conjugation by $\gamma \in \mathrm{S}_n$ defined by (2). Now it is clear that if $n \equiv 1, 2 \pmod 4$, then $\gamma \in \mathrm{A}_n$ and by standard considerations $\mathrm{A}_n \rtimes \langle\varphi\rangle \cong \mathrm{A}_n \times \mathbb{Z}_2$. If $n \equiv 0, 3 \pmod 4$, then $\gamma \in \mathrm{S}_n - \mathrm{A}_n$ and it cannot be replaced by an involution from A_n. Therefore $\mathrm{A}_n \rtimes \langle\varphi\rangle \cong \mathrm{S}_n$.

□

The case of S_n is simpler.

Proposition 5.3. *Let $\alpha = (1,2)$ and $\beta = (1,2,\dots,n)$ and let φ be an automorphism of S_n defined by (3). Then $\mathrm{S}_n \rtimes \langle\varphi\rangle \cong \mathrm{S}_n \times \mathbb{Z}_2$.*

Notice that again φ is uniquely determined by (3) and is realized by the conjugation by $\pi = (1,2)(3,n)(4,n-1)\dots(n-3,n-2)$.

Now observe that for a fixed positive even integer m in the range $1 \le m \le n$, the set of all involutions $\tau \in \mathrm{A}_n$ such that $|\mathrm{supp}(\tau)| = m$ form one conjugacy class of A_n and S_n as well. The symmetric group S_n has $\lfloor n/2 \rfloor$ different conjugacy classes of involutions while the alternating group A_n has $\lfloor n/4 \rfloor$ such classes. So in particular we get

Lemma 5.4. *Let $\tilde{G} = G \rtimes \langle\varphi\rangle$.*

(*a*) *If $G = \mathrm{S}_n$ then the number of conjugacy classes of involutions from $\tilde{G} - G$ is equal to $\lfloor n/2 \rfloor$.*

(*b*) *Let $G = \mathrm{A}_n$. If $\tilde{G} \cong G \times \mathbb{Z}_2$ then the number of conjugacy classes from $\tilde{G} - G$ is equal to $\lfloor n/4 \rfloor$. If $\tilde{G} \cong \mathrm{S}_n$ then the number of conjugacy classes of involutions from $\tilde{G} - G$ is equal to $\lfloor n/2 \rfloor - \lfloor n/4 \rfloor$.*

Proof. (a) By Proposition 5.3, $\tilde{G} - G = G\tau$, where τ is an involution centralizing G. If C_γ is the conjugacy class of γ in G then $C_\gamma\tau$ is a conjugacy class of $\gamma\tau$ in $\tilde{G}$. Hence, the number of conjugacy classes of involutions of G is the same as the number of conjugacy classes of involutions of $\tilde{G}$ contained in $\tilde{G} - G$.

(b) For the case $\tilde{G} \cong G \times \mathbb{Z}_2$ the proof is the same as for (a). Let $\tau \in \tilde{G} - G$ be a fixed involution. Let $\tilde{G} = \mathrm{S}_n$. If $\tau \in \mathrm{A}_n$ is an involution, then the conjugacy class of τ in S_n is equal to the conjugacy class of this element in A_n. Hence the number of conjugacy classes of involutions of $\tilde{G} - G$ is equal to $\lfloor n/2 \rfloor - \lfloor n/4 \rfloor$. □

Suppose now that α and β are fixed arbitrary cycles generating $G \in \{\mathrm{A}_n, \mathrm{S}_n\}$, such that $\tau\alpha\tau = \alpha^{-1}$ and $\tau\beta\tau = \beta^{-1}$ for an involution τ. From the proof of Theorem 5.1, it follows that in order to calculate the number of purely imaginary forms we have to find in $\tilde{G} = G \rtimes \langle\tau\rangle$ the number of conjugacy classes of involutions of $\tilde{G}$ which are not in G and which are not conjugated to $\alpha\tau$, τ and $\beta\tau$. Observe however that for $G = \mathrm{S}_n$ with $|\mathrm{supp}(\beta)|$ odd, the elements τ and $\beta\tau$ are conjugated but τ and $\alpha\tau$ not, as $|\mathrm{supp}(\alpha)|$ is even. If both $|\mathrm{supp}(\alpha)|$ and $|\mathrm{supp}(\beta)|$ are even then $\alpha\tau$ is conjugated to $\beta\tau$ but not conjugated to τ. If $G = \mathrm{A}_n$ then the three elements τ, $\alpha\tau$, $\beta\tau$ are conjugated with each other. As a consequence we obtain our final theorem.

Theorem 5.5. *Let G be the symmetric group S_n or the alternating group A_n generated by two cycles α, β, so that the correspondence $\varphi(\alpha) = \alpha^{-1}$ and $\varphi(\beta) = \beta^{-1}$ induces an automorphism of G. Then the complex algebraic curve corresponding to α, β has*

$$\begin{array}{ll} \lfloor n/2 \rfloor - 2 & \text{if } G = \mathrm{S}_n, \\ \lfloor n/2 \rfloor - \lfloor n/4 \rfloor - 1 & \text{if } \tilde{G} \cong \mathrm{S}_n, \\ \lfloor n/4 \rfloor - 1 & \text{if } \tilde{G} \cong \mathrm{A}_n \times \mathbb{Z}_2, \end{array}$$

purely imaginary forms. □

Remark 5.6. If α and β giving the action of G are not cycles, and $\tau\alpha\tau = \alpha^{-1}$ and $\tau\beta\tau = \beta^{-1}$ for some involution τ, then in all three cases the elements τ, $\alpha\tau$, $\beta\tau$ may lie in one, two or three conjugacy classes.

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