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MATRIX MULTIVARIATE PEARSON II-RIESZ DISTRIBUTION

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Abstract. Matrix multivariate Pearson type II-Riesz distribution is defined and some of its properties are studied. In particular, the associated matrix multivariate beta distribution type I is derived. Also the singular values and eigenvalues distributions are obtained.

1. Introduction

When a new statistic theory is proposed, the statistician known well about the rigorously mathematical foundations of their discipline, however in order to reach a wider interdisciplinary public, some of the classical statistical techniques have been usually published without explaining the supporting abstract mathematical tools which governs the approach. For example, in the context of the distribution theory of random matrices, in the last 20 years, a number of more abstract and mathematical approaches have emerged for studying and generalizing the usual matrix variate distributions. In particular, this needing have appeared recently in the generalization, by using abstract algebra, of some results of real random matrices to another supporting fields, such as complex, quaternion and octonion, see [26], [27], [12], [15], among many others authors. Studying distribution theory by another algebras, beyond real, have led several generalizations of substantial understanding in the theoretical context, and we expect that it is more extensively applied when a an improvement of its unified potential can be explored in other

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contexts. Two main tendencies have been considered in literature, Jordan algebras and real normed division algebras. Some works dealing the first approach are due to [14], [24], [3], [19], [20, 21], [23], and the references therein, meanwhile, the second technique has been studied by [16], [7], [4, 5, 6], among many others.

In the same manner, different generalizations of the multivariate statistical analysis have been proposed recently. This generalized technique studies the effect of changing the usual matrix multivariate normal support by a general matrix multivariate family of distributions, such as the elliptical contoured distributions (or simply, matrix multivariate elliptical distributions), see [13] and [18]. This family of distributions involves a number of known matrix multivariate distributions such as normal, Kotz type, Bessel, Pearson type II and VII, contaminated normal and power exponential, among many others. Two important properties of these distributions must be emphasized:

i) Matrix multivariate elliptical distributions provide more flexibility in the statistical modeling by including distributions with heavier or lighter tails and/or greater or lower degree of kurtosis than matrix multivariate normal distribution;

ii) Most of the statistical tests based on matrix multivariate normal distribution are invariant under the complete family of matrix multivariate elliptical distributions.

Recently, a slight combination of these two theoretical generalizations have appeared in literature; namely, Jordan algebras has been led to the matrix multivariate Riesz distribution and its associated beta distribution. [6] proved that the above mentioned distributions can be derived from a particular matrix multivariate elliptical distribution, termed matrix multivariate Kotz-Riesz distribution. Similarly, matrix multivariate Riesz distribution is also of interest from the mathematical point of view; in fact most of their basic properties under *the structure theory of normal j-algebras* and *the theory of Vinberg algebras* in place of Jordan algebras have been studied by [22] and [2], respectively.

In this scenario, we can now propose a generalization of the matrix multivariate beta, T and Pearson type II distributions based on a matrix multivariate Kotz-Riesz distribution. As usual in the normal case, extensions of beta, T and Pearson type II distributions involves two alternatives, the matrix variate and the matrix multivariate versions[1], see [4, 5, 6], [7, 8, 9] and [10].

This article derives the matrix multivariate beta and Pearson type II distributions obtained from a matrix multivariate Kotz-Riesz distribution and some of their basic properties are studied. Section 2 gives some basic concepts and the notation of abstract algebra, Jacobians and distribution theory. The nonsingular central matrix multivariate Pearson type II-Riesz distribution and the corresponding generalized matrix multivariate beta type I distribution are studied in Section 3. Finally, the joint densities of the singular values are derived in Section 4.

[1]The term matricvariate distribution was first introduced [11], but the expression matrix-variate distribution or matrix variate distribution or matrix multivariate distribution was later used to describe any distribution of a random matrix, see [17] and [18], and the references therein. When the density function of a random matrix is written including the trace operator then the matrix multivariate designation shall be used.

2. Preliminary results

A detailed discussion of real normed division algebras can be found in [1] and [25]. For your convenience, we shall introduce some notation, although in general, we adhere to standard notation forms.

For our purposes: Let $\mathbb{F}$ be a field. An *algebra* $\mathfrak{A}$ over $\mathbb{F}$ is a pair $(\mathfrak{A}; m)$, where $\mathfrak{A}$ is a *finite-dimensional vector space* over $\mathbb{F}$ and *multiplication* $m : \mathfrak{A} \times \mathfrak{A} \to A$ is an $\mathbb{F}$-bilinear map; that is, for all $\lambda \in \mathbb{F}$, $x, y, z \in \mathfrak{A}$,

$$\begin{aligned} m(x, \lambda y + z) &= \lambda m(x; y) + m(x; z) \\ m(\lambda x + y; z) &= \lambda m(x; z) + m(y; z). \end{aligned}$$

Two algebras $(\mathfrak{A}; m)$ and $(\mathfrak{E}; n)$ over $\mathbb{F}$ are said to be *isomorphic* if there is an invertible map $\phi : \mathfrak{A} \to \mathfrak{E}$ such that for all $x, y \in \mathfrak{A}$,

$$\phi(m(x, y)) = n(\phi(x), \phi(y)).$$

By simplicity, we write $m(x; y) = xy$ for all $x, y \in \mathfrak{A}$.

Let $\mathfrak{A}$ be an algebra over $\mathbb{F}$. Then $\mathfrak{A}$ is said to be

(1) *alternative* if $x(xy) = (xx)y$ and $x(yy) = (xy)y$ for all $x, y \in \mathfrak{A}$,
(2) *associative* if $x(yz) = (xy)z$ for all $x, y, z \in \mathfrak{A}$,
(3) *commutative* if $xy = yx$ for all $x, y \in \mathfrak{A}$, and
(4) *unital* if there is a $1 \in \mathfrak{A}$ such that $x1 = x = 1x$ for all $x \in \mathfrak{A}$.

If $\mathfrak{A}$ is unital, then the identity 1 is uniquely determined.

An algebra $\mathfrak{A}$ over $\mathbb{F}$ is said to be a *division algebra* if $\mathfrak{A}$ is nonzero and $xy = 0_{\mathfrak{A}} \Rightarrow x = 0_{\mathfrak{A}}$ or $y = 0_{\mathfrak{A}}$ for all $x, y \in \mathfrak{A}$.

The term "division algebra", comes from the following proposition, which shows that, in such an algebra, left and right division can be unambiguously performed.

Let $\mathfrak{A}$ be an algebra over $\mathbb{F}$. Then $\mathfrak{A}$ is a division algebra if, and only if, $\mathfrak{A}$ is nonzero and for all $a, b \in \mathfrak{A}$, with $b \neq 0_{\mathfrak{A}}$, the equations $bx = a$ and $yb = a$ have unique solutions $x, y \in \mathfrak{A}$.

In the sequel we assume $\mathbb{F} = \Re$ and consider classes of division algebras over $\Re$ or "*real division algebras*" for short.

We introduce the algebras of *real numbers* $\Re$, *complex numbers* $\mathfrak{C}$, *quaternions* $\mathfrak{H}$ and *octonions* $\mathfrak{O}$. Then, if $\mathfrak{A}$ is an alternative real division algebra, then $\mathfrak{A}$ is isomorphic to $\Re$, $\mathfrak{C}$, $\mathfrak{H}$ or $\mathfrak{O}$.

Let $\mathfrak{A}$ be a real division algebra with identity 1. Then $\mathfrak{A}$ is said to be *normed* if there is an inner product $(\cdot, \cdot)$ on $\mathfrak{A}$ such that

$$(xy, xy) = (x, x)(y, y) \qquad \text{for all } x, y \in \mathfrak{A}.$$

If $\mathfrak{A}$ is a *real normed division algebra*, then $\mathfrak{A}$ is isomorphic $\Re$, $\mathfrak{C}$, $\mathfrak{H}$ or $\mathfrak{O}$.

There are exactly four normed division algebras: real numbers ($\Re$), complex numbers ($\mathfrak{C}$), quaternions ($\mathfrak{H}$) and octonions ($\mathfrak{O}$), see [1]. We take into account that should be taken into account, $\Re$, $\mathfrak{C}$, $\mathfrak{H}$ and $\mathfrak{O}$ are the only normed division algebras; furthermore, they are the only alternative division algebras.

Let $\mathfrak{A}$ be a division algebra over the real numbers. Then $\mathfrak{A}$ has dimension either 1, 2, 4 or 8. Finally, observe that

$\Re$ is a real commutative associative normed division algebra,
$\mathfrak{C}$ is a commutative associative normed division algebra,
$\mathfrak{H}$ is an associative normed division algebra,
$\mathfrak{O}$ is an alternative normed division algebra.

Let $\mathfrak{L}^{\beta}_{n,m}$ be the set of all $n \times m$ matrices of rank $m \leq n$ over $\mathfrak{A}$ with m distinct positive singular values, where $\mathfrak{A}$ denotes a *real finite-dimensional normed division algebra*. Let $\mathfrak{A}^{n\times m}$ be the set of all $n \times m$ matrices over $\mathfrak{A}$. The dimension of $\mathfrak{A}^{n\times m}$ over $\Re$ is βmn. Let $\mathbf{A} \in \mathfrak{A}^{n\times m}$, then $\mathbf{A}^* = \bar{\mathbf{A}}^T$ denotes the usual conjugate transpose.

Table 1 sets out the equivalence between the same concepts in the four normed division algebras.

TABLE 1. Notation

Real	Complex	Quaternion	Octonion	Generic notation
Semi-orthogonal	Semi-unitary	Semi-symplectic	Semi-exceptional type	$\mathcal{V}^{\beta}_{m,n}$
Orthogonal	Unitary	Symplectic	Exceptional type	$\mathfrak{U}^{\beta}(m)$
Symmetric	Hermitian	Quaternion hermitian	Octonion hermitian	$\mathfrak{S}^{\beta}_{m}$

We denote by $\mathfrak{S}^{\beta}_{m}$ the real vector space of all $\mathbf{S} \in \mathfrak{A}^{m\times m}$ such that $\mathbf{S} = \mathbf{S}^*$. In addition, let $\mathfrak{P}^{\beta}_{m}$ be the *cone of positive definite matrices* $\mathbf{S} \in \mathfrak{A}^{m\times m}$. Thus, $\mathfrak{P}^{\beta}_{m}$ consist of all matrices $\mathbf{S} = \mathbf{X}^*\mathbf{X}$, with $\mathbf{X} \in \mathfrak{L}^{\beta}_{n,m}$; then $\mathfrak{P}^{\beta}_{m}$ is an open subset of $\mathfrak{S}^{\beta}_{m}$.

Let $\mathfrak{D}^{\beta}_{m}$ consisting of all $\mathbf{D} \in \mathfrak{A}^{m\times m}$, $\mathbf{D} = \operatorname{diag}(d_1, \dots, d_m)$. Let $\mathfrak{T}^{\beta}_{U}(m)$ be the subgroup of all *upper triangular* matrices $\mathbf{T} \in \mathfrak{A}^{m\times m}$ such that $t_{ij} = 0$ for $1 < i < j \leq m$. Let $\mathbf{Z} \in \mathfrak{L}^{\beta}_{n,m}$, define the norm of $\mathbf{Z}$ as $||\mathbf{Z}|| = \sqrt{\operatorname{tr} \mathbf{Z}^*\mathbf{Z}}$.

For any matrix $\mathbf{X} \in \mathfrak{A}^{n\times m}$, $d\mathbf{X}$ denotes the *matrix of differentials* (dx_{ij}). Finally, we define the *measure* or volume element $(d\mathbf{X})$ when $\mathbf{X} \in \mathfrak{A}^{n\times m}$, $\mathfrak{S}^{\beta}_{m}$, $\mathfrak{D}^{\beta}_{m}$ or $\mathcal{V}^{\beta}_{m,n}$, see [7] and [9].

If $\mathbf{X} \in \mathfrak{A}^{n\times m}$ then $(d\mathbf{X})$ (the Lebesgue measure in $\mathfrak{A}^{n\times m}$) denotes the exterior product of the βmn functionally independent variables

$$(d\mathbf{X}) = \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m} dx_{ij} \quad \text{where} \quad dx_{ij} = \bigwedge_{k=1}^{\beta} dx_{ij}^{(k)}.$$

If $\mathbf{S} \in \mathfrak{S}^{\beta}_{m}$ (or $\mathbf{S} \in \mathfrak{T}^{\beta}_{U}(m)$ with $t_{ii} > 0$, $i = 1, \dots, m$) then $(d\mathbf{S})$ (the Lebesgue measure in $\mathfrak{S}^{\beta}_{m}$ or in $\mathfrak{T}^{\beta}_{U}(m)$) denotes the exterior product of the exterior product of the $m(m-1)\beta/2 + m$ functionally independent variables,

$$(d\mathbf{S}) = \bigwedge_{i=1}^{m} ds_{ii} \bigwedge_{i>j}^{m} \bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}.$$

Observe, that for the Lebesgue measure $(d\mathbf{S})$ defined thus, it is required that $\mathbf{S} \in \mathfrak{P}^{\beta}_{m}$, that is, $\mathbf{S}$ must be a non singular Hermitian matrix (Hermitian definite positive matrix).

If $\mathbf{\Lambda} \in \mathfrak{D}^{\beta}_{m}$ then $(d\mathbf{\Lambda})$ (the Legesgue measure in $\mathfrak{D}^{\beta}_{m}$) denotes the exterior product of the βm functionally independent variables

$$(d\mathbf{\Lambda}) = \bigwedge_{i=1}^{n} \bigwedge_{k=1}^{\beta} d\lambda_{i}^{(k)}.$$

If $\mathbf{H}_1 \in \mathcal{V}_{m,n}^{\beta}$ then

$$(\mathbf{H}_1^* d\mathbf{H}_1) = \bigwedge_{i=1}^{m} \bigwedge_{j=i+1}^{n} \mathbf{h}_j^* d\mathbf{h}_i.$$

where $\mathbf{H} = (\mathbf{H}_1^*|\mathbf{H}_2^*)^* = (\mathbf{h}_1, \dots, \mathbf{h}_m|\mathbf{h}_{m+1}, \dots, \mathbf{h}_n)^* \in \mathfrak{U}^{\beta}(n)$. It can be proved that this differential form does not depend on the choice of the $\mathbf{H}_2$ matrix. When $n = 1$; $\mathcal{V}_{m,1}^{\beta}$ defines the unit sphere in $\mathfrak{A}^m$. This is, of course, an $(m-1)\beta$-dimensional surface in $\mathfrak{A}^m$. When $n = m$ and denoting $\mathbf{H}_1$ by $\mathbf{H}$, $(\mathbf{H}d\mathbf{H}^*)$ is termed the *Haar measure* on $\mathfrak{U}^{\beta}(m)$.

The surface area or volume of the Stiefel manifold $\mathcal{V}_{m,n}^{\beta}$ is

$$\text{(1)} \qquad \text{Vol}(\mathcal{V}_{m,n}^{\beta}) = \int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^{\beta}} (\mathbf{H}_1 d\mathbf{H}_1^*) = \frac{2^m \pi^{mn\beta/2}}{\Gamma_m^{\beta}[n\beta/2]},$$

where $\Gamma_m^{\beta}[a]$ denotes the multivariate *Gamma function* for the space $\mathfrak{S}_m^{\beta}$ and is defined as

$$\begin{aligned} \Gamma_m^{\beta}[a] &= \int_{\mathbf{A} \in \mathfrak{P}_m^{\beta}} \text{etr}\{-\mathbf{A}\}|\mathbf{A}|^{a-(m-1)\beta/2-1}(d\mathbf{A}) \\ &= \pi^{m(m-1)\beta/4} \prod_{i=1}^{m} \Gamma[a - (i-1)\beta/2], \end{aligned}$$

and $\text{Re}(a) > (m-1)\beta/2$. This can be obtained as a particular case of the *generalized gamma function of weight* κ for the space $\mathfrak{S}_m^{\beta}$ with $\kappa = (k_1, k_2, \dots, k_m) \in \Re^m$, taking $\kappa = (0, 0, \dots, 0) \in \Re^m$ and which for $\text{Re}(a) \geq (m-1)\beta/2 - k_m$ is defined by, see [16] and [14],

$$\begin{aligned} \text{(2)} \qquad \Gamma_m^{\beta}[a, \kappa] &= \int_{\mathbf{A} \in \mathfrak{P}_m^{\beta}} \text{etr}\{-\mathbf{A}\}|\mathbf{A}|^{a-(m-1)\beta/2-1} q_{\kappa}(\mathbf{A})(d\mathbf{A}) \\ &= \pi^{m(m-1)\beta/4} \prod_{i=1}^{m} \Gamma[a + k_i - (i-1)\beta/2] \\ \text{(3)} \qquad &= [a]_{\kappa}^{\beta} \Gamma_m^{\beta}[a], \end{aligned}$$

where $\text{etr}(\cdot) = \exp(\text{tr}(\cdot))$, $|\cdot|$ denotes the determinant, and for $\mathbf{A} \in \mathfrak{S}_m^{\beta}$

$$\text{(4)} \qquad q_{\kappa}(\mathbf{A}) = |\mathbf{A}_m|^{k_m} \prod_{i=1}^{m-1} |\mathbf{A}_i|^{k_i - k_{i+1}}$$

with $\mathbf{A}_p = (a_{rs})$, $r, s = 1, 2, \dots, p$, $p = 1, 2, \dots, m$ is termed the *highest weight vector*, see [16], [14] and [19]; And, $[a]_{\kappa}^{\beta}$ denotes the generalized Pochhammer symbol of weight κ, defined as

$$\begin{aligned} [a]_{\kappa}^{\beta} &= \prod_{i=1}^{m} (a - (i-1)\beta/2)_{k_i} \\ &= \frac{\pi^{m(m-1)\beta/4} \prod_{i=1}^{m} \Gamma[a + k_i - (i-1)\beta/2]}{\Gamma_m^{\beta}[a]} \\ &= \frac{\Gamma_m^{\beta}[a, \kappa]}{\Gamma_m^{\beta}[a]}, \end{aligned}$$

where $\text{Re}(a) > (m-1)\beta/2 - k_m$ and

$$(a)_i = a(a+1)\cdots(a+i-1),$$

is the standard Pochhammer symbol.

Additional, note that, if $\kappa = (p, \dots, p)$, then $q_\kappa(\mathbf{A}) = |\mathbf{A}|^p$. In particular if $p = 0$, then $q_\kappa(\mathbf{A}) = 1$. If $\tau = (t_1, t_2, \dots, t_m)$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$, then $q_{\kappa+\tau}(\mathbf{A}) = q_\kappa(\mathbf{A})q_\tau(\mathbf{A})$, and in particular if $\tau = (p, p, \dots, p)$, then $q_{\kappa+\tau}(\mathbf{A}) \equiv q_{\kappa+p}(\mathbf{A}) = |\mathbf{A}|^p q_\kappa(\mathbf{A})$. Finally, for $\mathbf{B} \in \mathfrak{T}_U^\beta(m)$ in such a manner that $\mathbf{C} = \mathbf{B}^*\mathbf{B} \in \mathfrak{S}_m^\beta$, $q_\kappa(\mathbf{B}^*\mathbf{A}\mathbf{B}) = q_\kappa(\mathbf{C})q_\kappa(\mathbf{A})$, and $q_\kappa(\mathbf{B}^{*-1}\mathbf{A}\mathbf{B}^{-1}) = (q_\kappa(\mathbf{C}))^{-1}q_\kappa(\mathbf{A}) = q_{-\kappa}(\mathbf{C})q_\kappa(\mathbf{A})$, see [21].

Finally, the following Jacobians involving the β parameter, reflects the generalized power of the algebraic technique; the can be seen as extensions of the full derived and unconnected results in the real, complex or quaternion cases, see [14] and [7]. These results are the base for several matrix and matrix variate generalized analysis.

Proposition 2.1. *Let $\mathbf{X}$ and $\mathbf{Y} \in \mathfrak{L}_{n,m}^\beta$ be matrices of functionally independent variables, and let $\mathbf{Y} = \mathbf{AXB} + \mathbf{C}$, where $\mathbf{A} \in \mathfrak{L}_{n,n}^\beta$, $\mathbf{B} \in \mathfrak{L}_{m,m}^\beta$ and $\mathbf{C} \in \mathfrak{L}_{n,m}^\beta$ are constant matrices. Then*

$$(d\mathbf{Y}) = |\mathbf{A}^*\mathbf{A}|^{m\beta/2}|\mathbf{B}^*\mathbf{B}|^{mn\beta/2}(d\mathbf{X}). \tag{5}$$

Proposition 2.2 (Singular Value Decomposition, SVD)**.** *Let $\mathbf{X} \in \mathcal{L}_{n,m}^\beta$ be matrix of functionally independent variables, such that $\mathbf{X} = \mathbf{W}_1\mathbf{D}\mathbf{V}^*$ with $\mathbf{W}_1 \in \mathcal{V}_{m,n}^\beta$, $\mathbf{V} \in \mathfrak{U}^\beta(m)$ and $\mathbf{D} = \text{diag}(d_1, \cdots, d_m) \in \mathfrak{D}_m^1$, $d_1 > \cdots > d_m > 0$. Then*

$$(d\mathbf{X}) = 2^{-m}\pi^\varrho \prod_{i=1}^m d_i^{\beta(n-m+1)-1} \prod_{i<j}^m (d_i^2 - d_j^2)^\beta (d\mathbf{D})(\mathbf{V}^*d\mathbf{V})(\mathbf{W}_1^*d\mathbf{W}_1), \tag{6}$$

where

$$\varrho = \begin{cases} 0, & \beta = 1; \\ -m, & \beta = 2; \\ -2m, & \beta = 4; \\ -4m, & \beta = 8. \end{cases}$$

Proposition 2.3. *Let $\mathbf{X} \in \mathfrak{L}_{n,m}^\beta$ be matrix of functionally independent variables, and write $\mathbf{X} = \mathbf{V}_1\mathbf{T}$, where $\mathbf{V}_1 \in \mathcal{V}_{m,n}^\beta$ and $\mathbf{T} \in \mathfrak{T}_U^\beta(m)$ with positive diagonal elements. Define $\mathbf{S} = \mathbf{X}^*\mathbf{X} \in \mathfrak{P}_m^\beta$. Then*

$$(d\mathbf{X}) = 2^{-m}|\mathbf{S}|^{\beta(n-m+1)/2-1}(d\mathbf{S})(\mathbf{V}_1^*d\mathbf{V}_1). \tag{7}$$

Finally, to define the matrix multivariate Pearson type II-Riesz distribution we need to recall the following two definitions of Kotz-Riesz and Riesz distributions.

From [6].

Definition 2.1. Let $\mathbf{\Sigma} \in \mathbf{\Phi}_m^\beta$, $\mathbf{\Theta} \in \mathbf{\Phi}_n^\beta$, $\boldsymbol{\mu} \in \mathfrak{L}_{n,m}^\beta$ and $\kappa = (k_1, k_2, \dots, k_m) \in \Re^m$. And let $\mathbf{Y} \in \mathfrak{L}_{n,m}^\beta$ and $\mathcal{U}(\mathbf{B}) \in \mathfrak{T}_U^\beta(n)$, such that $\mathbf{B} = \mathcal{U}(\mathbf{B})^*\mathcal{U}(\mathbf{B})$ is the Cholesky decomposition of $\mathbf{B} \in \mathfrak{S}_m^\beta$. Then it is said that $\mathbf{Y}$ has a *Kotz-Riesz distribution of* *type I* and its density function is

(8)
$$\frac{\beta^{mn\beta/2+\sum_{i=1}^{m} k_i}\Gamma_m^\beta[n\beta/2]}{\pi^{mn\beta/2}\Gamma_m^\beta[n\beta/2,\kappa]|\mathbf{\Sigma}|^{n\beta/2}|\mathbf{\Theta}|^{m\beta/2}} \times \operatorname{etr}\left\{-\beta \operatorname{tr}\left[\mathbf{\Sigma}^{-1}(\mathbf{Y}-\boldsymbol{\mu})^*\mathbf{\Theta}^{-1}(\mathbf{Y}-\boldsymbol{\mu})\right]\right\}$$
$$\times\, q_\kappa\left[\mathcal{U}(\mathbf{\Sigma})^{*-1}(\mathbf{Y}-\boldsymbol{\mu})^*\mathbf{\Theta}^{-1}(\mathbf{Y}-\boldsymbol{\mu})\mathcal{U}(\mathbf{\Sigma})^{-1}\right](d\mathbf{Y})$$

with $\operatorname{Re}(n\beta/2) > (m-1)\beta/2 - k_m$; denoting this fact as

$$\mathbf{Y} \sim \mathcal{KR}_{n\times m}^{\beta,I}(\kappa,\boldsymbol{\mu},\mathbf{\Theta},\mathbf{\Sigma}).$$

From [19] and [4] we have

Definition 2.2. Let $\mathbf{\Xi} \in \mathbf{\Phi}_m^\beta$ and $\kappa = (k_1, k_2, \dots, k_m) \in \Re^m$. Then it is said that $\mathbf{V}$ has a *Riesz distribution of type I* if its density function is

$$\frac{\beta^{am+\sum_{i=1}^{m} k_i}}{\Gamma_m^\beta[a,\kappa]|\mathbf{\Xi}|^a q_\kappa(\mathbf{\Xi})}\operatorname{etr}\{-\beta\mathbf{\Xi}^{-1}\mathbf{V}\}|\mathbf{V}|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{V})(d\mathbf{V}), \tag{9}$$

for $\mathbf{V} \in \mathfrak{P}_m^\beta$ and $\operatorname{Re}(a) \geq (m-1)\beta/2 - k_m$; denoting this fact as $\mathbf{V} \sim \mathcal{R}_m^{\beta,I}(a,\kappa,\mathbf{\Xi})$.

3. Matrix multivariate Pearson type II-Riesz distribution

A detailed discussion of Riesz distribution may be found in [19] and [4]. In addition the Kotz-Riesz distribution is studied in detail in [6]. For convenience, we adhere to standard notation stated in [4, 6].

Theorem 3.1. *Let* $\left(S_1^{1/2}\right)^2 = S_1 \sim \mathcal{R}_1^{\beta,I}(\nu\beta/2,k,1)$*,* $k \in \Re$ *and* $\operatorname{Re}(\nu\beta/2) > -k$*; independent of* $\mathbf{Y} \sim \mathcal{KR}_{n\times m}^{\beta,I}(\tau,\mathbf{0},\mathbf{I}_n,\mathbf{I}_m)$*,* $\operatorname{Re}(n\beta/2) > (m-1)\beta/2 - t_m$*. In addition, define* $\mathbf{R} = S^{-1/2}\mathbf{Y}$ *where* $S = S_1 + ||\mathbf{Y}||^2$*. Then*

$$S \sim \mathcal{R}_1^{\beta,I}((\nu+mn)\beta/2 + \sum_{i=1}^{m} t_i, k, 1)$$

is independent of $\mathbf{R}$*. Furthermore, the density of* $\mathbf{R}$ *is*

(10)
$$\frac{\Gamma_m^\beta[n\beta/2]\Gamma_1^\beta\left[(\nu+mn)\beta/2+k+\sum_{i=1}^{m} t_i\right]}{\pi^{\beta mn/2}\Gamma_m^\beta[n\beta/2,\tau]\Gamma_1^\beta[\nu\beta/2+k]}\left(1-||\mathbf{R}||^2\right)^{\nu\beta/2+k-1} q_\tau\left(\mathbf{R}^*\mathbf{R}\right)(d\mathbf{R}),$$

where $\left(1-||\mathbf{R}||^2\right) > 0$*; which is termed the* standardized matrix multivariate Pearson type II-Riesz type distribution *and is denoted as*

$$\mathbf{R} \sim \mathcal{P}_{\mathcal{II}}\mathcal{R}_{m\times n}^{\beta,I}(\nu,k,\tau,1,\mathbf{0},\mathbf{I}_n,\mathbf{I}m).$$

Proof. From definition 2.1 and 2.2, the joint density of S_1 and $\mathbf{Y}$ is

$$\propto s_1^{\beta\nu/2+k-1}\operatorname{etr}\left\{-\beta\left(s_1+||\mathbf{Y}||^2\right)\right\}q_\tau\left(\mathbf{Y}^*\mathbf{Y}\right)(ds_1)(d\mathbf{Y})$$

where the constant of proportionality is

$$c = \frac{\beta^{\nu\beta/2+k}}{\Gamma_1^\beta[\nu\beta/2+k]} \cdot \frac{\beta^{mn\beta/2+\sum_{i=1}^{m} t_i}\,\Gamma_m^\beta[n\beta/2]}{\pi^{mn\beta/2}\Gamma_m^\beta[n\beta/2,\tau]}.$$

Making the change of variable $S = S_1 - ||\mathbf{Y}||^2$ and $\mathbf{Y} = S_1^{1/2}\mathbf{R}$, by (5)

$$(ds_1)\wedge(d\mathbf{Y}) = s^{\beta mn/2}(ds)\wedge(d\mathbf{R}).$$

Now, observing that $S = S_1 - ||\mathbf{Y}||^2 = S\left(1 - ||\mathbf{R}||^2\right)$, the joint density of S and $\mathbf{R}$ is

$$\propto \left(1 - ||\mathbf{R}||^2\right)^{\beta\nu/2+k-1} s^{\beta\nu/2+k-1} \operatorname{etr}\{-\beta s\}\, q_\tau\left(s\mathbf{R}^*\mathbf{R}\right)(ds)(d\mathbf{R}).$$

Also, note that

$$q_\tau\left(s\mathbf{R}^*\mathbf{R}\right) = q_\tau\left((s^{1/2}\mathbf{I}_m)\mathbf{R}^*\mathbf{R}(s^{1/2}\mathbf{I}_m)\right) = q_\tau\left(s\mathbf{I}_m\right) q_\tau\left(\mathbf{R}^*\mathbf{R}\right) = s^{\sum_{i=1}^m t_i} q_\tau\left(\mathbf{R}^*\mathbf{R}\right).$$

From where, the joint density of S and $\mathbf{R}$ is given by

$$\frac{\beta^{(\nu+mn)\beta/2+k+\sum_{i=1}^m t_i}}{\Gamma_1^\beta\left[(\nu+mn)\beta/2+k+\sum_{i=1}^m t_i\right]} \operatorname{etr}\{-\beta s\}\, s^{(\nu+mn)\beta/2+k+\sum_{i=1}^m t_i - 1}(ds)$$
$$\times\ \frac{\Gamma_m^\beta[n\beta/2]\Gamma_1^\beta\left[(\nu+mn)\beta/2+k+\sum_{i=1}^m t_i\right]}{\pi^{\beta mn/2}\Gamma_m^\beta[n\beta/2,\tau]\Gamma_1^\beta[\nu\beta/2+k]} \left(1 - ||\mathbf{R}||^2\right)^{\nu\beta/2+k-1} q_\tau\left(\mathbf{R}^*\mathbf{R}\right)(d\mathbf{R}),$$

which shows that

$$S \sim \mathcal{R}_1^{\beta,I}((\nu+mn)\beta/2 + \sum_{i=1}^m t_i, k, 1),$$

and is independent of $\mathbf{R}$, where $\mathbf{R}$ has the density (10).

□

The following is an immediate consequence of the previous result.

Corollary 3.1. *Let* $\mathbf{R} \sim \mathcal{P}_{\mathcal{II}}\mathcal{R}_{m\times n}^{\beta,I}(\nu, k, \tau, 1, \mathbf{0}, \mathbf{I}_n, \mathbf{I}m)$ *and define*

$$\mathbf{C} = \rho^{-1/2}\,\mathcal{U}(\boldsymbol{\Theta})^*\mathbf{R}\,\mathcal{U}(\boldsymbol{\Sigma}) + \boldsymbol{\mu}$$

where $\mathcal{U}(\mathbf{B}) \in \mathfrak{T}_U^\beta(n)$, *such that* $\mathbf{B} = \mathcal{U}(\mathbf{B})^*\,\mathcal{U}(\mathbf{B})$ *is the Cholesky decomposition of* $\mathbf{B} \in \mathfrak{S}_m^\beta$, $\boldsymbol{\Theta} \in \mathfrak{P}_n^\beta$, $\boldsymbol{\Sigma} \in \mathfrak{P}_m^\beta$, $\rho > 0$ *constant and* $\boldsymbol{\mu} \in \mathfrak{L}_{n,m}^\beta$ *is a matrix of constants. Then the density of* $\mathbf{S}$ *is*

$$\propto \left(1 - \rho \operatorname{tr} \boldsymbol{\Sigma}^{-1}(\mathbf{C}-\boldsymbol{\mu})^*\boldsymbol{\Theta}^{-1}(\mathbf{C}-\boldsymbol{\mu})\right)^{\nu\beta/2+k-1}$$
$$\times\ q_\tau\left[\mathcal{U}(\boldsymbol{\Sigma})^{*-1}(\mathbf{C}-\boldsymbol{\mu})^*\boldsymbol{\Theta}^{-1}(\mathbf{C}-\boldsymbol{\mu})\,\mathcal{U}(\boldsymbol{\Sigma})^{-1}\right](d\mathbf{S}) \tag{11}$$

where $\left(1 - \rho \operatorname{tr} \boldsymbol{\Sigma}^{-1}(\mathbf{C}-\boldsymbol{\mu})^*\boldsymbol{\Theta}^{-1}(\mathbf{C}-\boldsymbol{\mu})\right) > 0$*; with constant of proportionality*

$$\frac{\Gamma_m^\beta[n\beta/2]\Gamma_1^\beta\left[(\nu+mn)\beta/2 - k - \sum_{i=1}^m t_i\right]\rho^{mn\beta/2 - \sum_{i=1}^m t_i}}{\pi^{\beta mn/2}\Gamma_m^\beta[n\beta/2,-\tau]\Gamma_1^\beta[\nu\beta/2-k]|\boldsymbol{\Sigma}|^{\beta n/2}|\boldsymbol{\Theta}|^{\beta m/2}},$$

which is termed the matrix multivariate Pearson type II-Riesz distribution *and is denoted as* $\mathbf{C} \sim \mathcal{P}_{\mathcal{II}}\mathcal{R}_{m\times n}^{\beta,I}(\nu, k, \tau, \rho, \boldsymbol{\mu}, \boldsymbol{\Theta}, \boldsymbol{\Sigma})$.

Proof. Observe that $\mathbf{R} = \rho^{1/2}\,\mathcal{U}(\boldsymbol{\Theta})^{*-1}(\mathbf{C}-\boldsymbol{\mu})\,\mathcal{U}(\boldsymbol{\Sigma})^{-1}$ and

$$(d\mathbf{R}) = \rho^{mn\beta/2}|\boldsymbol{\Sigma}|^{-\beta n/2}|\boldsymbol{\Theta}|^{-\beta m/2}(d\mathbf{C}).$$

The desired result is obtained making this change of variable in (10).

□

Next we derive the corresponding *matrix multivariate beta type I distribution.*

Theorem 3.2. *Let*

$$\mathbf{R} \sim \mathcal{P}_{\mathcal{II}}\mathcal{R}_{n\times m}^{\beta,I}(\nu, k, \tau, \rho, \mathbf{0}, \mathbf{I}_n, \boldsymbol{\Sigma}),$$

and define $\mathbf{B} = \mathbf{R}^*\mathbf{R} \in \mathfrak{P}_m^\beta$, *with* $n \geq m$. *Then the density of* $\mathbf{B}$ *is,*

$$\propto |\mathbf{B}|^{(n-m+1)\beta/2-1}(1 - \rho \operatorname{tr} \boldsymbol{\Sigma}^{-1}\mathbf{B})^{\nu\beta/2+k-1} q_\tau(\mathbf{B})(d\mathbf{B}), \tag{12}$$

where $1 - \rho \operatorname{tr} \mathbf{\Sigma}^{-1}\mathbf{B} > 0$*; and with constant of proportionality*

$$\frac{\Gamma_1^\beta\left[(\nu+mn)\beta/2+k+\sum_{i=1}^m t_i\right]\rho^{\beta mn/2+\sum_{i=1}^m t_i}}{\Gamma_m^\beta[n\beta/2,\tau]\Gamma_1^\beta[\nu\beta/2+k]|\mathbf{\Sigma}|^{n\beta/1}q_\tau(\mathbf{\Sigma})}.$$

$\mathbf{B}$ *is said to have a* non standardized matrix multivariate beta-Riesz type I distribution.

Proof. The desired result follows from (10), by applying (7) and then (1); and observing that

$$q_\tau(\mathcal{U}(\mathbf{\Sigma})^{*-1}\mathbf{B}\,\mathcal{U}(\mathbf{\Sigma})^{-1}) = q_{-\tau}(\mathbf{\Sigma})q_\tau(\mathbf{B}).$$

□

In particular if $\mathbf{\Sigma} = \mathbf{I}_m$ in Theorem 3.2, we obtain:

Corollary 3.2. *Let*

$$\mathbf{R} \sim \mathcal{P}_{\mathcal{II}}\mathcal{R}_{n\times m}^{\beta,I}(\nu,k,\tau,1,\mathbf{0},\mathbf{I}_n,\mathbf{I}_m),$$

and define $\mathbf{B} = \mathbf{R}^*\mathbf{R} \in \mathfrak{P}_m^\beta$*, with* $n \geq m$*. Then the density of* $\mathbf{B}$ *is,*

(13)

$$\frac{\Gamma_1^\beta\left[(\nu+mn)\beta/2+k+\sum_{i=1}^m t_i\right]}{\Gamma_m^\beta[n\beta/2,\tau]\Gamma_1^\beta[\nu\beta/2+k]}|\mathbf{B}|^{(n-m+1)\beta/2-1}(1-\rho\operatorname{tr}\mathbf{B})^{\nu\beta/2+k-1}q_\tau(\mathbf{B})(d\mathbf{B}),$$

where $1 - \rho \operatorname{tr} \mathbf{B} > 0$*.* $\mathbf{B}$ *is said to have a* matrix multivariate beta-Riesz type I distribution.

Remark 3.1. Observe that alternatively to classical definitions of generalized *matricvariate* beta function (for symmetric cones), see [5], [14] and [20], defined as

$$\begin{aligned}
\mathcal{B}_m^\beta[a,\kappa;b,\tau] &= \int_{\mathbf{0}<\mathbf{S}<\mathbf{I}_m}|\mathbf{B}|^{b-(m-1)\beta/2-1}q_\tau(\mathbf{B})|\mathbf{I}_m-\mathbf{B}|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{I}_m-\mathbf{B})(d\mathbf{B})\\
&= \int_{\mathbf{F}\in\mathfrak{P}_m^\beta}|\mathbf{F}|^{b-(m-1)\beta/2-1}q_\tau(\mathbf{F})|\mathbf{I}_m+\mathbf{F}|^{-(a+b)}q_{-(\kappa+\tau)}(\mathbf{I}_m+\mathbf{F})(d\mathbf{F})\\
&= \frac{\Gamma_m^\beta[a,\kappa]\Gamma_m^\beta[b,\tau]}{\Gamma_m^\beta[a+b,\kappa+\tau]},
\end{aligned}$$

where $\kappa = (k_1, k_2, \dots, k_m) \in \Re^m$, $\tau = (t_1, t_2, \dots, t_m) \in \Re^m$, $\operatorname{Re}(a) > (m-1)\beta/2 - k_m$ and $\operatorname{Re}(b) > (m-1)\beta/2 - t_m$. From Corollary 3.2 and Díaz-García and Gutiérrez-Sánchez [10, Theorem 3.3.1], we have the following alternative definition:

Definition 3.1. The *matrix multivariate* beta function is defined an denoted as:

$$\begin{aligned}
\mathcal{B}_m^{*\,\beta}[a,k;b,\tau] &= \int_{1-\operatorname{tr}\mathbf{B}>0}|\mathbf{B}|^{b-(m-1)\beta/2-1}(1-\operatorname{tr}\mathbf{B})^{a+k-1}q_\tau(\mathbf{B})(d\mathbf{B})\\
&= \int_{\mathbf{R}\in\mathfrak{P}_m^\beta}|\mathbf{F}|^{b-(m-1)\beta/2-1}(1+\operatorname{tr}\mathbf{F})^{-(a+mb+k+\sum_{i=1}^m t_i)}q_\tau(\mathbf{F})(d\mathbf{F})\\
&= \frac{\Gamma_1^\beta[a+k]\Gamma_m^\beta[b,\tau]}{\Gamma_1^\beta\left[a+mb+k+\sum_{i=1}^m t_i\right]}.
\end{aligned}$$

Also, observe that, when $m = 1$, then $\tau = t$ and $\kappa = k$ and

$$\mathcal{B}_1^\beta[a,k;b,t] = \frac{\Gamma_1^\beta[a+k]\Gamma_1^\beta[b+t]}{\Gamma_1^\beta\left[a+b+k+t\right]} = \mathcal{B}_1^{*\,\beta}[a,k;b,t]$$

Finally observe that if in results in this section are defined $k = 0$ and $\tau = (0, \dots, 0)$, the results in [8] are obtained as particular cases.

4. Singular value densities

In this section, the joint densities of the singular values of random matrix $\mathbf{R} \sim \mathcal{P}_{\mathcal{II}}\mathcal{R}^{\beta,I}_{n\times m}(\nu, k, \tau, 1, \mathbf{0}, \mathbf{I}_n, \mathbf{I}_m)$ are derived. In addition, and as a direct consequence, the joint density of the eigenvalues of matrix multivariate beta-Riesz type I distribution is obtained for real normed division algebras.

Theorem 4.1. *Let $\delta_1, \dots, \delta_m$, $1 > \delta_1 > \cdots > \delta_m > 0$, be the singular values of the random matrix $\mathbf{R} \sim \mathcal{P}_{\mathcal{II}}\mathcal{R}^{\beta,I}_{n\times m}(\nu, k, \tau, 1, \mathbf{0}, \mathbf{I}_n, \mathbf{I}_m)$. Then its joint density is*

$$\frac{2^m \pi^{\beta m^2/2+\varrho}}{\Gamma^\beta_m[\beta m/2] {\mathcal{B}^*_m}^\beta[\nu\beta/2, k; n\beta/2, \tau]} \prod_{i=1}^m \left(\delta_i^2\right)^{(n-m+1)\beta/2-1/2} \left(1-\rho\sum_{i=1}^m \delta_i^2\right)^{\nu\beta/2+k-1}$$

$$\times \prod_{i<j}\left(\delta_i^2-\delta_j^2\right)^\beta \frac{C^\beta_\tau(\mathbf{D}^2)}{C^\beta_\tau(\mathbf{I}_m)} \left(\bigwedge_{i=1}^m d\delta_i\right) \tag{14}$$

for $1-\rho\sum_{i=1}^m \delta_i^2 > 0$. Where ϱ is defined in Lemma 2.2, $\mathbf{D} = \operatorname{diag}(\delta_1, \dots, \delta_m)$, and $C^\beta_\kappa(\cdot)$ denotes the zonal spherical functions or spherical polynomials, see [16] and Faraut and Korányi [14, Chapter XI, Section 3].

Proof. This follows immediately from (10). First using (6), then applying (1) and observing that, from [16, Equation 4.8(2) and Definition 5.3] and Faraut and Korányi [14, Chapter XI, Section 3], we have that for $\mathbf{L} \in \mathfrak{P}^\beta_m$,

$$C^\beta_\tau(\mathbf{Z}) = C^\beta_\tau(\mathbf{I}_m) \int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} q_\kappa(\mathbf{HZH}^*)(d\mathbf{H}),$$

□

Finally, observe that $\delta_i = \sqrt{\operatorname{eig}_i(\mathbf{R}^*\mathbf{R})}$, where $\operatorname{eig}_i(\mathbf{A})$, $i = 1, \dots, m$, denotes the i-th eigenvalue of $\mathbf{A}$. Let $\lambda_i = \operatorname{eig}_i(\mathbf{R}^*\mathbf{R}) = \operatorname{eig}_i(\mathbf{B})$, observing that, for example, $\delta_i = \sqrt{\lambda_i}$. Then

$$\bigwedge_{i=1}^m d\delta_i = 2^{-m}\prod_{i=1}^m \lambda_i^{-1/2} \bigwedge_{i=1}^m d\lambda_i,$$

the corresponding joint densities of $\lambda_1, \dots, \lambda_m$, $1 > \lambda_1 > \cdots > \lambda_m > 0$ is obtained from (14) as

$$\frac{\pi^{\beta m^2/2+\varrho}}{\Gamma^\beta_m[\beta m/2] {\mathcal{B}^*_m}^\beta[\nu\beta/2, k; n\beta/2, \tau]} \prod_{i=1}^m \lambda_i^{(n-m+1)\beta/2-1} \left(1-\sum_{i=1}^m \lambda_i\right)^{\nu\beta/2+k-1}$$

$$\times \prod_{i<j}(\lambda_i-\lambda_j)^\beta \frac{C^\beta_\tau(\mathbf{G})}{C^\beta_\tau(\mathbf{I}_m)} \left(\bigwedge_{i=1}^m d\lambda_i\right)$$

for $1-\sum_{i=1}^m \lambda_i > 0$, where $\mathbf{G} = \operatorname{diag}(\lambda_1, \dots, \lambda_m)$.

5. Conclusions

As visual examples, different Pearson type II-Riesz densities for $m = 1$ are showed in figures 1 and 2,

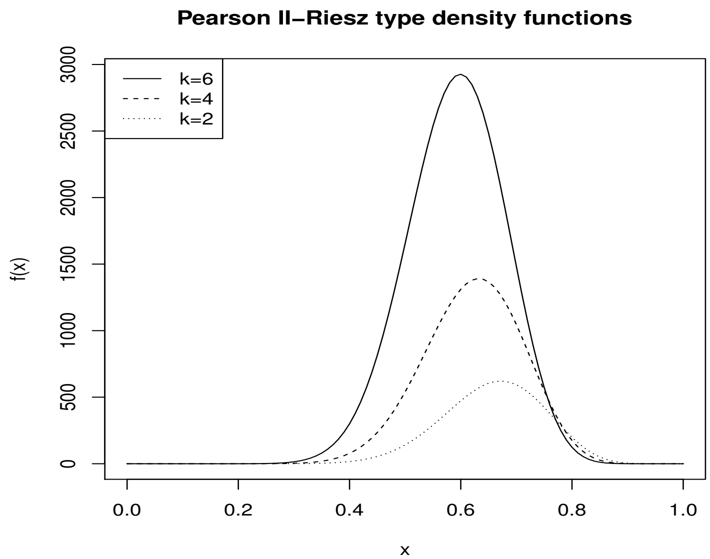

FIGURE 1. With $\nu = 15$, $n = 18$ and $t = 7$

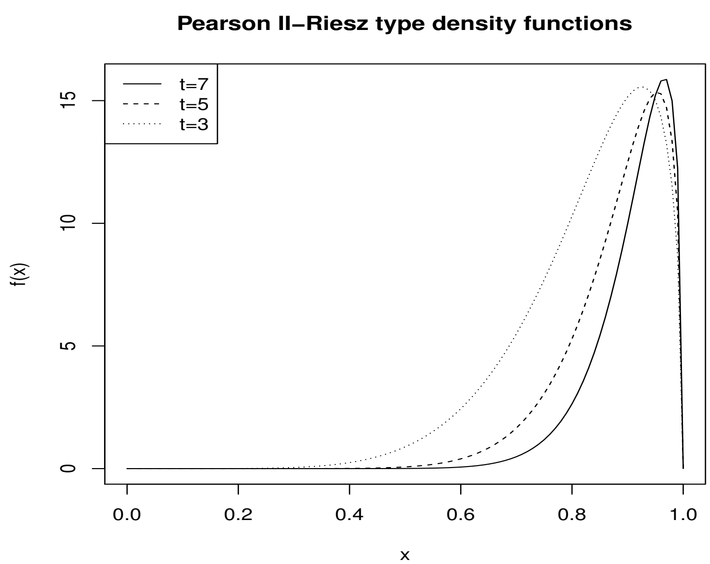

FIGURE 2. With $\nu = 3$, $n = 18$ and $k = 0$

Recall that in octonionic case, from the practical point of view, we most keep in mind the fact from [1], *there is still no proof that the octonions are useful for understanding the real world.* We can only hope that eventually this question will be settled on one way or another. In addition, as is established in [14] and [28] the result obtained in this article are valid for the *algebra of Albert*, that is when hermitian matrices ($\mathbf{S}$) or hermitian product of matrices ($\mathbf{X}^*\mathbf{X}$) are 3×3 octonionic matrices.

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