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QUASI-PLATONIC $PSL_2(q)$-ACTIONS ON CLOSED RIEMANN SURFACES

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Abstract. This paper is the first of two papers whose combined goal is to explore the dessins d'enfant and symmetries of quasi-platonic actions of $PSL_2(q)$. A quasi-platonic action of a group G on a closed Riemann S surface is a conformal action for which S/G is a sphere and $S \to S/G$ is branched over $\{0, 1, \infty\}$. The unit interval in S/G may be lifted to a dessin d'enfant $\mathcal{D}$, an embedded bipartite graph in S. The dessin forms the edges and vertices of a tiling on S by dihedrally symmetric polygons, generalizing the idea of a platonic solid. Each automorphism ψ in the absolute Galois group determines a transform S^{ψ} by transforming the coefficients of the defining equations of S. The transform defines a possibly new quasi-platonic action and a transformed dessin $\mathcal{D}^{\psi}$.

Here, in this paper, we describe the quasi-platonic actions of $PSL_2(q)$ and the action of the absolute Galois group on $PSL_2(q)$ actions. The second paper discusses the quasi-platonic actions constructed from symmetries (reflections) and the resulting triangular tiling that refines the dessin d'enfant. In particular, the number of ovals and the separation properties of the mirrors of a symmetry are determined.

1. Introduction

Let S be a closed Riemann surface, of genus $\sigma \geq 2$; we denote the group of conformal automorphisms by $\mathrm{Aut}(S)$. We say that a group G acts conformally on S, if there is a monomorphism

$$\epsilon : G \hookrightarrow \mathrm{Aut}(S). \tag{1}$$

A *symmetry* or *reflection* of S is an anti-conformal, involutory automorphism of the surface. The symmetries of S are contained in $\mathrm{Aut}^*(S)$, the group of isometries of S, both conformal and anti-conformal. A surface with a symmetry is called a *symmetric surface* and has a defining equation with real coefficients. A conformal G-action is *symmetric* if there is a symmetry ϕ of S normalizing $\epsilon(G)$. In this case

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the action extends to a reflection group $\epsilon : G^* \hookrightarrow \mathrm{Aut}^*(S)$. A subtlety is that G may not act symmetrically even though S is symmetric. This may happen if $\epsilon(G)$ is not normal in $\mathrm{Aut}(S)$; see [16] for a discussion.

Quasi-platonic surfaces, dessins, and symmetries. Quasi-platonic G-actions extend the notion of automorphism groups of platonic solids. A G-action is called *quasi-platonic* (or *triangular* see Section 2) if the quotient S/G is a sphere $\mathbb{S}^2 = \widehat{\mathbb{C}} = P^1(\mathbb{C})$ and the quotient map $\pi_G : S \to S/G$ is branched over three points. A surface is called quasi-platonic if the (natural) action of $\mathrm{Aut}(S)$ is quasi-platonic. It turns out that if the natural action of $G \leq \mathrm{Aut}(S)$ is quasi-platonic then all intermediate groups H, $G \leq H \leq \mathrm{Aut}(S)$ have natural quasi-platonic actions. There is a great interest in quasi-platonic actions for the following reasons:

(1) They are rigid, i.e., the conformal structure of the surfaces cannot be infinitesimally deformed without losing symmetry.
(2) The surface S has a defining equation with coefficients in a number field.
(3) Assume that $\pi_G : S \to S/G$ is branched over $\{0, 1, \infty\}$. Let $I = [0, 1] \subseteq \widehat{\mathbb{C}}$ be the standard unit interval. Then $\mathcal{D} = \pi_G^{-1}(I)$ is a bipartite graph in S, called a *(regular) dessin d'enfant.* The group G acts on $\mathcal{D}$, acting simply transitively on the edges. The complement $S - \mathcal{D}$ is a disjoint union of open, congruent convex hyperbolic polygons, permuted transitively by G. Each polygon is the lift to S of $\widehat{\mathbb{C}} - I$. This geometric structure on the surface and its invariance under G, generalizes the notion of a platonic solid and the tetrahedral, cubic, octahedral, dodecahedral and icosahedral tilings and automorphism groups of the sphere.
(4) There is a rich interplay between dessins and the action of the absolute Galois group on surfaces defined over number fields. We discuss this in detail in Section 5.

Most, though not all, quasi-platonic actions are symmetric. When the action is symmetric, the dessin is refined by a triangular tiling on S, generated by reflections in the sides of triangles on S. The mirror $\mathcal{M}_\phi$ of a symmetry ϕ is the fixed point subset of set ϕ and, if non-empty, consists of a finite number of circles called *ovals,* made up of edges of the tiling. The symmetry ϕ is called separating if $S - \mathcal{M}_\phi$ consists of two components, otherwise it is called *non-separating.* This paper and its sequel [7] discuss the dessins and the mirror structure of symmetric quasi-platonic actions of $PSL_2(q)$. The current paper classifies the quasi-platonic actions including the action of the absolute Galois group. The second paper discusses in detail the mirror structure of the symmetries of the actions.

Quasi-platonic actions, large actions, and genus actions. We conclude this section with a discussion of *large actions* on surfaces and the special place that quasi-platonic actions have among large actions. We say that G is a *large* group of automorphisms (G has a large action) if the ratio $|G|/(\sigma - 1)$ is fairly large, or alternatively a fundamental region for the G-action has small hyperbolic area $\frac{2\pi}{|G|/(\sigma-1)}$. For any given group G there are surfaces S, with an arbitrarily large genus, such that $G \simeq \mathrm{Aut}(S)$ and the values $|G|/(\sigma-1)$ is arbitrarily small. However there are only a finite number of large actions once a cutoff $|G|/(\sigma-1) \geq c$ has been decided. For large actions, the restriction on the size and geometry of a fundamental region forces some structure on the surface and simplifies the geometrical and group theoretic analysis of these surfaces and their symmetries.

According to the Riemann-Hurwitz theorem, we always have

$$|G|/(\sigma - 1) \leq 84. \tag{2}$$

For groups G which are efficiently generated, such as simple groups, there will always be a surface S with $G \subset \text{Aut}(S)$ and for which $|G|/(\sigma - 1)$ is of reasonable size. For example, if G is generated by r elements, then a surface S with G-action may be constructed for which

$$\frac{2}{r-1} < |G|/(\sigma - 1) \leq 84. \tag{3}$$

If G is generated by 2 elements, then

$$2 < |G|/(\sigma - 1) \leq 84. \tag{4}$$

If G is generated by an involution and another element, then

$$4 < |G|/(\sigma - 1) \leq 84. \tag{5}$$

and, finally, if G is generated by two elements of order 2 and 3, then

$$12 < |G|/(\sigma - 1) \leq 84. \tag{6}$$

In all of the actions above S/G is a sphere $\pi_G : S \to S/G$ *is* branched over $r + 1$ points, The last three classes are all quasi-platonic $r = 3$, and the last two are of great interest to researchers on dessins d'enfant. The geometrical analyses are simplified in these four cases since there are associated tilings consisting of $(r+1)$-gons (r-generator case), triangles (2-generator case), right-angled triangles (generation by an involution and another element), and triangles with a right angle and 60° angle (generation by elements of order 2 and 3). The r generator case for $PSL_2(p)$ is discussed in [20]. For simple groups the inequality 4 always holds and inequality 5 probably always holds. In the last case the groups are finite quotients of $PSL_2(\mathbb{Z})$. For $PSL_2(q)$ it follows from the works [13, 14] that we always be able get a surface for which the inequality 6 holds. The construction of actions of groups on surfaces from group generators is well known, see [3], [4], or [22] for example. The inequalities are derived from the Riemann-Hurwitz equation.

A *(hyperbolic) genus action* is an action of G on a surface S of genus $\sigma \geq 2$ such that G acts on no surface of lower genus ≥ 2. The surface S has the smallest hyperbolic area for a G action and, hence, the largest action of G. Genus actions are broadly studied; see [5, 10, 13, 14] for instance. We shall pay special attention to the genus actions of $G = PSL_2(q)$, since for genus actions the action is quasi-platonic, and $\epsilon(G)$ is a normal subgroup of $\text{Aut}(S)$ of index 1 or 2 (see [5]).

Finally, why consider $PSL_2(q)$? They are simple groups; there are many low genus quasi-platonic actions among simple group actions; all the actions are symmetric, and the general calculations are fairly easy.

Overview of paper. Given the foregoing, we are going to focus on quasi-platonic actions in the rest of the paper and its sequel. The two papers are motivated by the prior work in [5], [6], [8], and [9], and, in particular, extend the earlier work in [8]. In the work [8] the symmetry structure of Hurwitz surfaces (surfaces for which $|G| = 84(\sigma-1)$) with $PSL_2(q)$ as automorphism group were completely determined. In that paper, all the symmetries are classified, an algorithm for computing the number of ovals is given; and it is proven that none of the symmetries on these surfaces are separating.

The remainder of this paper is organized as follows. In Section 2 we describe the construction of surfaces with quasi-platonic actions with symmetry for a given group G. In Section 3 we develop the tools to enumerate all quasi-platonic actions of $PSL_2(q)$. Our main results are Theorems 20 and 22. In Section 4 we sketch how a MAGMA [19] classification of the actions may be carried out and give complete lists for $q = 7, 8$. We then enumerate the actions for all $q < 50$ and $q = 64 = 2^6, q = 81 = 3^4$;consider a few other interesting examples; and describe four infinite families of large actions. Finally, in Section 5, we discuss the action of the absolute Galois group on the quasi-platonic actions. Our main results are Theorem 30 and 33.

One of the main tools we use is Macbeath's description of generators for $PSL_2(q)$ [18]. Indeed, a number of our results are implicit in his work. The bulk of our work consists in organizing a classification. For the work on dessins and the action of the absolute Galois group, we follow some ideas in [17].

2. Symmetric quasi-platonic group actions

2.1. **Symmetric G-actions and covering groups.** We briefly discuss the general case of a symmetric group action before getting down to the specifics of quasi-platonic actions. For more on the general case see [6]. The universal cover of S is the hyperbolic plane $\mathbb{H}$ with covering map $\pi_S : \mathbb{H} \to S$. We denote the group of covering transformations of π_S by $\Pi \backsimeq \pi_1(S)$. The conformal group action of G on S has a covering action by a Fuchsian group Γ defined by an exact sequence

$$\Pi \hookrightarrow \Gamma \overset{\eta}{\twoheadrightarrow} G. \tag{7}$$

The induced isomorphism $\overline{\eta} : \Gamma/\Pi \leftrightarrow G$ defines an action $\epsilon = \overline{\eta}^{-1}$ of G on S through the natural action of Γ/Π on $S = \mathbb{H}/\Pi$.

Now consider a symmetry ϕ on S. The symmetry ϕ lifts to a reflection or glide reflection Φ on $\mathbb{H}$ which normalizes the kernel Π. The lift Φ also normalizes the covering group Γ if ϕ normalizes the G-action and we get an NEC group $\Gamma^* = \langle \Phi, \Gamma \rangle$. So assume that ϕ normalizes the G-action, define $\theta = \epsilon^{-1}\phi\epsilon \in \mathrm{Aut}(G)$, and define $G^* = \langle \theta \rangle \ltimes G$. We get extended maps

$$\epsilon : G^* \hookrightarrow \langle \phi, \epsilon(G) \rangle \leq \mathrm{Aut}^*(S), \;\; \epsilon(\theta) = \phi \tag{8}$$

$$\Pi \hookrightarrow \Gamma^* \overset{\eta}{\twoheadrightarrow} G^*, \;\; \eta(\Phi) = \theta. \tag{9}$$

Remark 1. *Using the algebraic structure of G^* we may find all the symmetries in $\epsilon(G^*)$. Every symmetry comes form an element of the form θg where $1 = (\theta g)^2 = \theta g \theta g = \theta(g)g$, or $\theta(g) = g^{-1}$.*

Of crucial importance is the tiling of $\mathbb{H}$ induced by the mirrors of symmetries in G^*. The union

$$\mathcal{M}_{G^*} = \bigcup_{\phi} \mathcal{M}_{\phi}$$

of the non-empty mirrors of all the symmetries in G^* creates a pattern of geodesic edges and ovals on S. The complement of $S - \mathcal{M}_{G^*}$ is a disjoint union of regions upon which G^* acts transitively. The decomposition of $S - \mathcal{M}_{G^*}$ into disjoint regions induces a tiling $\mathcal{T}_S$ on S where the set of faces F_S, consists of the closures of the components of $S - \mathcal{M}_{G^*}$; the set of vertices V_S consists of points of transverse intersections of ovals; and, the set of edges E_S consists of the closures of the components of $\mathcal{M}_{G^*} - V_s$. If the action is small, the faces may not be simply connected and the edges may be ovals. With large actions, typically all faces are polygons and edges are arcs, not ovals, and in the case of quasi-platonic actions the faces are triangles. We may lift the tiling on S to a tiling $\mathcal{T}$ on $\mathbb{H}$ defined by $\pi_S^{-1}(\mathcal{M}_{G^*})$. An example is given in Figure 1. We will use the interplay between the tilings $\mathcal{T}_S$ and S and $\mathcal{T}$ on $\mathbb{H}$. We note without proof the following facts about the tiling $\mathcal{T}$ on $\mathbb{H}$:

(1) every edge in $\mathcal{T}$ belongs to a line made up of edges of $\mathcal{T}$;
(2) every vertex of $\mathcal{T}$ is the unique fixed point of some element of Γ; and
(3) the group Γ^* is generated by the reflections in the sides of a single triangle, and Γ^* permutes the tiles simply transitively.

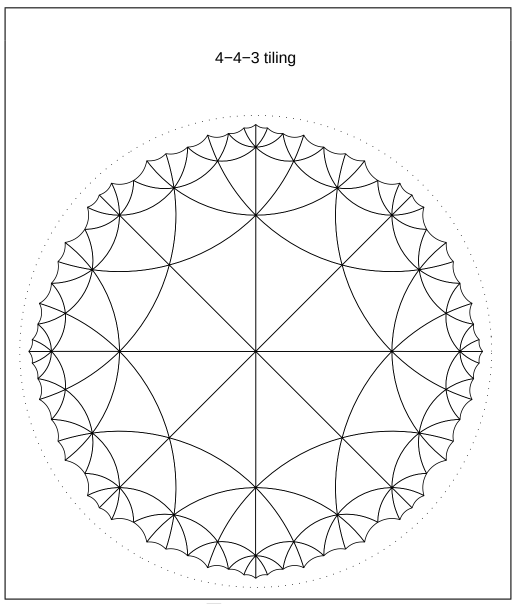

Figure 1.

2.2. **Quasi-platonic (triangular) group actions.** Throughout the remainder of this section we use the term triangular instead of quasi-platonic as it corresponds more directly to the construction. We can construct our surfaces, groups and symmetries through tilings of the hyperbolic plane by triangles. In Figure 2 we picture a (counter clockwise oriented) (l, m, n) triangle $\triangle DEF$ in the hyperbolic plane $\mathbb{H}$ (or Poincaré disc). The line segments $\overline{FD}$, $\overline{DE}$, and $\overline{EF}$ meet in the angles $\frac{\pi}{l}$, $\frac{\pi}{m}$, and $\frac{\pi}{n}$, respectively, where l, m, and n are integers ≥ 2. An (l, m, n)-triangle exists if and only if $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. The triangles in Figure 1 are $(4, 4, 3)$ triangles.

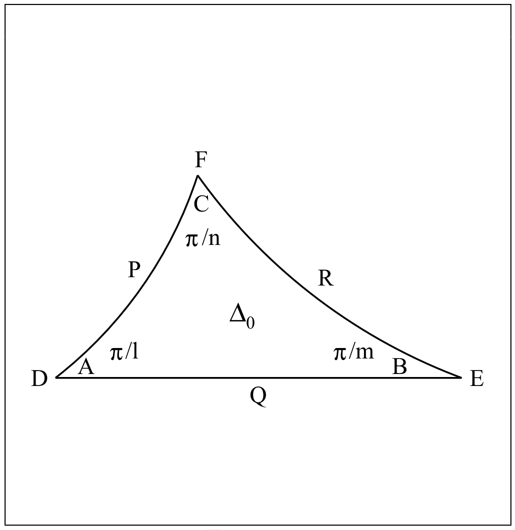

Figure 2.

Let $P, Q, R \in \text{Aut}^*(\mathbb{H})$ be the hyperbolic reflections in the lines $\overline{FD}$, $\overline{DE}$, and $\overline{EF}$, respectively, and define the rotations:

$$A = PQ,\ B = QR,\ C = RP.$$

The mappings A, B, C are counter clockwise rotations, centered at D, E, F, respectively, through the angles $\frac{2\pi}{l}$, $\frac{2\pi}{m}$, $\frac{2\pi}{n}$, respectively. It is well known that $T_{l,m,n} = \langle A, B, C \rangle \subset PSL_2(\mathbb{R})$ is a discrete group of conformal isometries of the hyperbolic plane with the following presentation

$$T_{l,m,n} = \langle A, B, C | A^l = B^m = C^n = ABC = 1 \rangle. \tag{10}$$

We call (l, m, n) the *signature* of $T_{l,m,n}$ and also call (l, m, n) the *signature* or *branching data* of the G-action on S. Now suppose that G is any group and (a, b, c) is a triple of elements generating G such that $a^l = b^m = c^n = abc = 1$. The triple (a, b, c) is a called a *generating* (l, m, n)*-triple* or *generating action triple.* If $\langle a, b, c \rangle$ is a proper subgroup of G, we just call (a, b, c) an (l, m, n)*-triple* or an *action triple.* The epimorphism of equation 7 is given by

$$\eta : T_{l,m,n} \to G,\ A \to a,\ B \to b,\ C \to c. \tag{11}$$

The kernel $\Pi = \ker(\eta)$ is torsion free and defines a closed Riemann surface $S = \mathbb{H}/\Pi$ whose genus σ satisfies the Riemann-Hurwitz equation

$$\frac{2\sigma - 2}{|G|} = 1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}. \tag{12}$$

One possibility for a symmetry on S is the involution q induced by the reflection Q in the side of $\triangle DEF$ if Q normalizes Π. The map 11 then extends to the following epimorphism

$$\eta : T^*_{l,m,n} \to G^*,\ P \to p,\ Q \to q,\ R \to r. \tag{13}$$

where

$$p = a\theta, q = \theta, r = \theta b$$

and $T^*_{l,m,n} = \langle P, Q, R\rangle$ is the group generated by the reflections in the sides of $\triangle DEF$. The automorphism θ satisfies

$$\theta(a) = a^{-1}, \theta(b) = b^{-1}. \tag{14}$$

The induced tiling on $\mathbb{H}$ is generated by reflection in the sides of the triangle. An example of the tiling on $\mathbb{H}$ for a $(4,4,3)$-action is given in Figure 1. Even if Q does not normalize Π, the tiling on $\mathbb{H}$ still projects to a tiling on the surface. For the rest of the paper we are going to assume that $\mathbb{H} \to \mathbb{H}/T_{l,m,n} = \widehat{\mathbb{C}}$ is adjusted so that $D \to 0, E \to 1, F \to \infty$ and correspondingly $S \to S/G$ maps $\overline{D} \to 0, \overline{E} \to 1, \overline{F} \to \infty$. Then the polygons of the dessin are the images of the polygons in $\mathcal{T}$ consisting of the dihedrally symmetric $2n$-gons surrounding the vertices of type F. If one of l, m equals 2 then we get regular n-gons. In Figure 1 the polygons are hexagons. The inverse image of $\mathcal{D}$ is the union of all edges of type $\overline{DE}$.

Remark 2. *There are additional possibilities for symmetries, which we explore in the sequel paper. They are irrelevant for the topic of dessins.*

Remark 3. *We get a Hurwitz surface when $\frac{2\sigma-2}{|G|}$ has the smallest possible value, if and only if $(l,m,n) = (2,3,7)$. If G is generated by the pair $\{a,b\}$ then upon setting $c = (ab)^{-1}$, we see that (a,b,c) is a generating (l,m,n)-triple for some l,m,n. Assuming that a and b have the appropriate orders, we get equations 3, 4, 5, 6. In the papers [13], [14], it is shown that all genus actions of $PSL_2(q)$ are the following types $(2,3,n), n \geq 7$, $(2,4,5)$, $(2,5,5)$, $(3,3,4)$, and $(2,5,7)$.*

Remark 4. *The triangular tiling on S determines three different dessins on S. Let $I_1 = [0,1]$, $I_2 = [1,\infty]$, $I_3 = [\infty,0]$, be considered as oriented intervals of $\widehat{\mathbb{R}} = P^1(\mathbb{R})$ and set $\mathcal{D}_i = \pi_G^{-1}(I_i)$. The polygons of $\mathcal{D}_2$ and $\mathcal{D}_3$ in S are the images of the polygons in $\mathcal{T}$ consisting of the dihedrally symmetric $2l$-gons and $2m$-gons surrounding the vertices of type D and E, respectively. The inverse images of $\mathcal{D}_2$ and $\mathcal{D}_3$ in $\mathbb{H}$ are the unions of all edges of type $\overline{EF}$, and $\overline{FD}$ respectively. The tiling $\mathcal{T}$ encodes the information of all three dessins simultaneously. The triangles on S are the closures of connected components of inverse images, by π_G, of the upper half plane (counter clockwise oriented triangles) and the lower half of plane (clockwise oriented triangles).*

Epimorphisms and equivalence. We can use the tiling on S to construct an epimorphism for the G-action. Pick a clockwise oriented triangle $\overline{\Delta}$ on S. The point $\overline{D}$ on S corresponding to D in $\triangle DEF$ is $\pi_G^{-1}(0) \cap \overline{\Delta}$. The stabilizer $G_{\overline{D}}$ of $\overline{D}$ is cyclic of order l. The rotation number map rot:$G_{\overline{D}} \to \mathbb{C}$ given by the $g \to dg$ on the tangent plane $T_{\overline{D}}(S)$ is an isomorphism of g onto the lth roots of unity. Pick a in $G_{\overline{D}}$ so that $\mathrm{rot}(a) = \mathrm{rot}(A, \overline{D}) = \exp(\frac{2\pi i}{l})$. Do the same to get b and c such that $\mathrm{rot}(b) = \exp(\frac{2\pi i}{m})$ and $\mathrm{rot}(c) = \exp(\frac{2\pi i}{n})$. Using homotopy arguments with lifts of curves, it can be shown that $abc = 1$ and that $A \to a$, $B \to b$, $C \to c$ is a uniformizing epimorphism. The selection of a different counter clockwise oriented triangle gives the triple $\mathrm{Ad}_g \cdot (a,b,c) = (gag^{-1}, gbg^{-1}, gcg^{-1})$, for some $g \in G$.

The enumeration of quasi-platonic actions is the same as the determination of $\mathrm{Aut}(G)$ equivalence classes of generating (l,m,n)-triples of G. First we define our notions of equivalence of actions.

Definition 5. *We say that two conformal actions $\epsilon_1, \epsilon_2 : G \hookrightarrow \mathrm{Aut}(S)$ are algebraically equivalent if $\epsilon_2 = \epsilon_1 \circ \omega$ for some $\omega \in \mathrm{Aut}(G)$, or equivalently if $\epsilon_1(G)$*

and $\epsilon_2(G)$ *are the same subgroup of* $\mathrm{Aut}(S)$*. Two actions* $\epsilon_1 : G \hookrightarrow \mathrm{Aut}(S_1)$ *and* $\epsilon_2 : G \hookrightarrow \mathrm{Aut}(S_2)$ *on possibly different surfaces are conformally equivalent if there is a conformal equivalence* $h : S_1 \leftrightarrow S_2$ *such that*

$$\epsilon_2(g) = h \circ \epsilon_1(\omega(g)) \circ h^{-1}, g \in G.$$

Specifically, two actions of G *on the same surface are conformally equivalent if they determine conjugate subgroups of* $\mathrm{Aut}(S)$.

Remark 6. *Conformal equivalence is a mild refinement of algebraic equivalence, but we shall not go into it deeply in this paper. See* [4] *for more detail.*

Let (a, b, c) be a generating (l, m, n)-triple of G, $\Gamma = T_{l,m,n}$,and $\omega \in \mathrm{Aut}(G)$.Then the equation 7 can be expanded to a commutative diagram

$$\begin{array}{ccccc} \Pi & \hookrightarrow & \Gamma & \overset{\eta}{\twoheadrightarrow} & G \\ \downarrow id & & \downarrow id & & \downarrow \omega \\ \Pi & \hookrightarrow & \Gamma & \overset{\omega\circ\eta}{\twoheadrightarrow} & G \end{array} \tag{15}$$

Both epimorphisms determine the same group of automorphisms of $\Gamma/\Pi \subseteq \mathrm{Aut}(S)$ acting on $S = \mathbb{H}/\Pi$. The generating triple (a', b', c') determined by $\omega \circ \eta$ is

$$(\omega(a), \omega(a), \omega(a))\,.$$

Thus, each equivalence class determined by the action $\omega\cdot(a, b, c) = (\omega(a), \omega(a), \omega(a))$, $\omega \in \mathrm{Aut}(G)$, determines a unique surface $S = \mathbb{H}/\Pi$ and unique subgroup of $\mathrm{Aut}(S)$. Correspondingly, given two epimorphisms η_1, η_2 with the same kernel as in the left half of the diagram, we have $\eta_2 = \omega\circ\, \eta_1$ for an $\omega \in \mathrm{Aut}(G)$ and so η_1 and η_2 determine the equivalent triples. Equivalent epimorphisms determine equivalent conformal actions.

There is a braid action on triples generated by these transformations: $(a, b, c) \to (b, b^{-1}ab, c)$, $(a, b, c) \to (a, c, c^{-1}bc)$, $(a, b, c) \to (a^{-1}ca, b, a)$, and their inverses. The action commutes with the $\mathrm{Aut}(G)$-action on triples and defines $\mathrm{Aut}(G)$-invariant bijections of (l, m, n) triples to (m, l, n) and (n, m, l) triples. The other permutations of indices are obtained by composition. The permutation of signatures does not produce any new actions. Consider, for instance, the permutation $(l, m, n) \to (m, l, n)$. Reflect the triangle $\triangle DEF$ in the side $\overline{EF}$ to obtain the (clockwise oriented) $\triangle ED'F$ triangle, an (m, l, n)-triangle. The rotations, in order, at the corners are $B, B^{-1}AB, C$. The very same map $\eta : \Gamma \twoheadrightarrow G$ given in equation 11 takes the triple $(B, B^{-1}AB, C)$ to $(b, b^{-1}ab, c)$ and so the same surface $S = \mathbb{H}/\Pi$, $\ker(\eta) = \Pi$ is determined. As $\Gamma = \langle B, B^{-1}AB, C\rangle$ then the same subgroup of automorphisms of S is determined and the image of G in $\mathrm{Aut}(S)$ is the same. Therefore no new actions are determined. There is a similar argument for all other permutations. Therefore, we may assume the signature has standard lexicographic form $l \le m \le n$.

2.3. **Counting triple sets and the action of automorphism groups.** The discussion in this paragraph follows the discussion in [15]. To work with the action of $\mathrm{Aut}(G)$ on epimorphisms and conformal actions, and for later work on the Galois action on dessins, we define the following sets closely related to $\mathrm{Aut}(G)$-orbits on epimorphisms. First, an obvious one, which we call a *(generating) signature triple* *set.*

$$X_G(l,m,n) = \{(a,b,c) \in G^3 : o(a) = l, o(b) = m, o(c) = n, \\ abc = 1\}, \tag{16}$$

$$X_G^\circ(l,m,n) = \{(a,b,c) \in X_G(l,m,n) : \langle a,b,c\rangle = G\}. \tag{17}$$

For each element of $X_G(l,m,n)$, a map $\Pi \hookrightarrow \Gamma \overset{\eta}{\to} G$ with torsion free kernel Π is determined and thereby an action of $\eta(\Gamma)$ on $S = \mathbb{H}/\Pi$. Only those triples in $X_G^\circ(l,m,n)$ (generating signature triple set) determine actions of all of G. There are many cases where triples generate proper subgroups $\eta(\Gamma) \subset G$ so that $X_G^\circ(l,m,n)$ is strictly contained in $X_G(l,m,n)$. Indeed, $X_G^\circ(l,m,n)$ may even be empty. According to equation 12, the genus of the surface S is given by

$$\sigma = 1 + \frac{1}{2}\,|\eta(\Gamma)|\left(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}\right)$$

which is maximal when $\eta(\Gamma) = G$.

To understand the $\mathrm{Aut}(G)$ action on $X_G(l,m,n)$ we need to know the subgroups of G and the centralizers of subgroups. Let $H \subset G$ be a proper subgroup and (a,b,c) a triple such that $H = \langle a,b,c\rangle$. Then the size of the orbit $\mathrm{Aut}(G)\cdot(a,b,c)$ is given by

$$|\mathrm{Aut}(G)\cdot(a,b,c)| = \frac{|\mathrm{Aut}(G)|}{\left|\mathrm{Stab}_{\mathrm{Aut}(G)}((a,b,c))\right|} = \frac{|\mathrm{Aut}(G)|}{\left|\mathrm{Cent}_{\mathrm{Aut}(G)}(H)\right|}$$

After we remove all triples for all $\mathrm{Aut}(G)$-classes of proper subgroups H that have generating (l,m,n)-triples, we have only $X_G^\circ(l,m,n)$ left and so

$$|X_G^\circ(l,m,n)| = |X_G(l,m,n)| - \sum_{H=\langle a,b,c\rangle} \frac{|\mathrm{Aut}(G)|}{\mathrm{Cent}_{\mathrm{Aut}(G)}(H)} \tag{18}$$

where $H = \langle a,b,c\rangle$ denotes an $\mathrm{Aut}(G)$-class of triples generating H and its $\mathrm{Aut}(G)$-conjugates. Sometimes the right hand sum can be easily computed exactly as there may only be a small number of terms.

Closely related to the sets $X_G(l,m,n)$ are triples where the a,b,c are restricted to come from some class of elements somewhere between a conjugacy class and an automorphism class. These sets will be important when we study Galois actions on dessins in Section 5. The sets also allow more effective enumeration of actions. To this end, we define an "approximate automorphism group" to be a group of automorphisms of G satisfying $\mathrm{Inn}(G) \subseteq L \subseteq \mathrm{Aut}(G)$. The extreme cases $\mathrm{Inn}(G)$ and $\mathrm{Aut}(G)$ are denoted by K and A respectively. These notions are more appropriate when the index $|\mathrm{Aut}(G) : \mathrm{Inn}(G)| = |\mathrm{Out}(G)|$ is small, say, when the center of G is small. A specific intermediate case is $L = PGL_2(q)$ when q is a prime power. For $g \in G$ and L as above let and $g^L = \{\omega(g) : \omega \in L\}$, when $L = K$ we get conjugacy classes. For $(a,b,c) \in G^3$, we define *(generating) L-triple sets.*

$$L_G(a,b,c) = \{(x,y,z) : x \in a^L, y \in b^L, z \in c^L, xyz = 1\}, \tag{19}$$

$$L_G^\circ(a,b,c) = \{(x,y,z) : (x,y,z) \in L_G(a,b,c), \langle x,y,z\rangle = G\}\,. \tag{20}$$

If $L = \mathrm{Inn}(G)$ or $\mathrm{Aut}(G)$, we use the notation $K_G(a,b,c)$ and $K_G^\circ(a,b,c)$ or $A_G(a,b,c)$ and $A_G^\circ(a,b,c)$ respectively. Observe that $K_G(a,b,c) \subseteq L_G(a,b,c) \subseteq A_G(a,b,c) \subseteq X_G(l,m,n)$ and that both $A_G(a,b,c)$ and $X_G(l,m,n)$ are unions of

Aut(G) classes of triples. When $L = PGL_2(q)$ we call the sets *(generating) projective triple sets.* There are formulas similar to 18 for determining the number of generating triples in $K_G^\circ(a,b,c)$, $L_G^\circ(a,b,c)$, and $A_G^\circ(a,b,c)$. Moreover, if the character theory of G is tractable, then the following formula (see [2], [15]) may be used:

$$|K_G(a,b,c)| = \frac{|G|^2}{|\mathrm{Cent}(a)|\cdot|\mathrm{Cent}(b)|\cdot|\mathrm{Cent}(c)|}\sum_{\chi}\frac{\chi(a)\chi(b)\chi(c)}{\chi(1)}. \tag{21}$$

Remark 7. *The set $X_G^\circ(l,m,n)$ and their partitions into L and K classes are natural action spaces for the absolute Galois group. The sets $L_G^\circ(a,b,c)$ are useful in classifying the equivalence classes of actions. The sets $K_G^\circ(a,b,c)$ will be very useful in discussing the action of the absolute Galois group on group actions in Section 5.*

2.4. **Companion actions and a Schur cover.** For use in Section 5 we want make more precise the relation between $L_G^\circ(a,b,c)$ and $K_G^\circ(a,b,c)$ and to be able to separate the various automorphism classes of orbits in these sets. Both of these sets have orbit decompositions

$$L_G^\circ(a,b,c) = \bigcup_{(a',b',c')}(a',b',c')^L \tag{22}$$

$$K_G^\circ(a,b,c) = \bigcup_{(a'',b'',c'')}(a'',b'',c'')^G. \tag{23}$$

where the superscripts on the right hand side indicate orbits. Each orbit on the right hand side of these equations must be regular and so

$$\left|(a',b',c')^L\right| = |L|\,,\ \ \left|(a'',b'',c'')^G\right| = |\mathrm{Inn}(G)| = |K|\,. \tag{24}$$

In our investigations in Section 5, it turns out the right hand sides will have more than one orbit which leads to some indeterminacy in the action of the absolute Galois group on $PSL_2(q)$ actions. To this end, we make the following definition:

Definition 8. *Suppose that $G = \langle a,b,c\rangle$ and $\mathrm{Inn}(G) \le L \le \mathrm{Aut}(G)$ and that $(a_1',b_1',c_1')^L$, $(a_2',b_2',c_2')^L$ are two distinct orbits in the right hand side of equation 22. Then we say that the two orbits are companion L-orbits and determine companion actions with respect to L. Similar definitions apply to the decomposition in equation 23. Companion orbits may determine equivalent actions upon lifting all the way up to $A_G^\circ(a,b,c)$.*

Let $\widetilde{G}$ be a Schur cover of G. Companion classes in $K_G^\circ(a,b,c)$ are a result of projecting multiple classes $(\widetilde{a},\widetilde{b},\widetilde{c})^{\widetilde{G}}$ in $\widetilde{G}$ to different classes $(a,b,c)^G$ in $K_G^\circ(a,b,c)$. Reversing the process, we many use a Schur cover to separate companion classes. In the case at hand $\widetilde{PSL_2}(q) = SL_2(q)$. We now discuss the general case of Schur covers.

Though we are primarily interested in a Schur covering group, let $\widetilde{G} \overset{\pi}{\twoheadrightarrow} G$ be any covering group of G satisfying:

(1) there is central subgroup $Z < \widetilde{G}$ such that

$$Z \overset{\iota}{\hookrightarrow} \widetilde{G} \overset{\pi}{\twoheadrightarrow} G$$

is exact;

(2) for every proper subgroup $\widetilde{H} < \widetilde{G}$, $\pi\left(\widetilde{H}\right) < G$.

We sketch how we may use a covering group to separate companion orbits. Each conjugacy class g^G in G may have several conjugacy classes in $\widetilde{G}$ lying over it. In fact, the totality of elements lying over g^G is $\bigcup_{z \in Z} (z\widetilde{g})^{\widetilde{G}} = \bigcup_{z \in Z} z\,(\widetilde{g})^{\widetilde{G}}$ for a fixed $\widetilde{g}$ lying over g. Thus, Z acts on the conjugacy classes lying over g^G, and they all have the same size. It is easily shown that

$$\frac{\left|(\widetilde{g})^{\widetilde{G}}\right|}{|g^G|} = \frac{|Z|\,|Z_G(g)|}{\left|Z_{\widetilde{G}}(\widetilde{g})\right|}$$

Hence the number of classes lying over is g^G is $|Z|$ divided by this number giving

$$\text{number of classes lying over } g^G = \frac{\left|Z_{\widetilde{G}}(\widetilde{g})\right|}{|Z_G(g)|}.$$

In the case of $PSL_2(q)$, for odd q, this number is 2 unless g is an involution.

Given a generating (l, m, n)-triple (a, b, c) we have numerous generating $(\widetilde{l}, \widetilde{m}, \widetilde{n})$-triples $(\widetilde{a}, \widetilde{b}, \widetilde{c})$ covering (a, b, c). For, if $(\widetilde{a}, \widetilde{b}, \widetilde{c}) \in \widetilde{G}^3$ is any triple with $\pi(\widetilde{a}, \widetilde{b}, \widetilde{c}) = (a, b, c)$, we may manufacture a covering generating $(\widetilde{l}, \widetilde{m}, \widetilde{n})$-triple. Noting that $\pi(\widetilde{a}\widetilde{b}\widetilde{c}) = abc = 1$, then $\widetilde{a}\widetilde{b}\widetilde{c} = z \in Z$, and so $(\widetilde{a}, \widetilde{b}, \widetilde{c}z^{-1})$ satisfies $\widetilde{a}\widetilde{b}\widetilde{c}z^{-1} = 1$. Also, $\pi\left(\widetilde{a}, \widetilde{b}, \widetilde{c}z^{-1}\right) = (a, b, c)$ and $\pi\left\langle\widetilde{a}, \widetilde{b}, \widetilde{c}z^{-1}\right\rangle = \langle a, b, c\rangle = G$, so $\left\langle\widetilde{a}, \widetilde{b}, \widetilde{c}z^{-1}\right\rangle = \widetilde{G}$. Thus it makes sense to assume that $\widetilde{a}\widetilde{b}\widetilde{c} = 1$ and that $\left\langle\widetilde{a}, \widetilde{b}, \widetilde{c}\right\rangle = \widetilde{G}$. We shall call such a triple a *lift* or *covering triple* of (a, b, c). The orders $\widetilde{l}, \widetilde{m}, \widetilde{n}$ of $\widetilde{a}, \widetilde{b}, \widetilde{c}$ are generally distinct from l, m, n, but l, m, n divide $\widetilde{l}, \widetilde{m}, \widetilde{n}$ respectively and the respective quotients divide the exponent of Z. If $(\widetilde{a}_0, \widetilde{b}_0, \widetilde{c}_0)$ is lift of (a, b, c), then all other lifts are of the form

$$(\widetilde{a}_0 z_1, \widetilde{b}_0 z_2, \widetilde{c}_0 z_3)$$

where

$$1 = \widetilde{a}_0 z_1 \widetilde{b}_0 z_2 \widetilde{c}_0 z_3 = \widetilde{a}_0 \widetilde{b}_0 \widetilde{c}_0 z_1 z_2 z_3 = z_1 z_2 z_3. \tag{25}$$

The collection of lifts defined above is called the *lift orbit* of (a, b, c). Indeed, let $J \leq Z^3$ be the subgroup defined by equation 25. Then the lift orbit is the J orbit of the J action acting on covering triples. The signatures $(\widetilde{l}, \widetilde{m}, \widetilde{n})$ may be different for different lifts.

For conjugacy triple sets, we have

$$K_G^\circ(a, b, c) = \bigcup_{(\widetilde{a}, \widetilde{b}, \widetilde{c})} \pi\left(K_G^\circ(\widetilde{a}, \widetilde{b}, \widetilde{c})\right)$$

where $(\widetilde{a}, \widetilde{b}, \widetilde{c}) \in \pi^{-1}(a) \times \pi^{-1}(b) \times \pi^{-1}(c)$. With luck, each $K_G^\circ(\widetilde{a}, \widetilde{b}, \widetilde{c})$ will be a single $\widetilde{G}$ orbit and, hence, each G orbit in $K_G^\circ(a, b, c)$ will be the image of a single $K_G^\circ(\widetilde{a}, \widetilde{b}, \widetilde{c})$ triple set.

3. Classifying quasi-platonic actions of $PSL_2(q)$

Now we examine quasi-platonic actions of $PSL_2(q)$ on surfaces. To use the results of the previous section, we need to discuss the automorphisms and subgroups of $PSL_2(q)$, the covering of $PSL_2(q)$ by $SL_2(q)$, and trace triple sets of $SL_2(q)$. We set $q = p^e$ unless otherwise noted.

3.1. Properties of $PSL_2(q)$ and $SL_2(q)$.

The projective linear groups have coverings by matrix groups

$$\langle \pm 1 \rangle \hookrightarrow SL_2(q) \twoheadrightarrow PSL_2(q) \tag{26}$$

and its extension

$$\mathbb{F}_q^* \hookrightarrow GL_2(q) \twoheadrightarrow PGL_2(q). \tag{27}$$

For the prime $p = 2$, we observe that $SL_2(q) = PSL_2(q)$.

For effective computations in $PSL_2(q)$ we will need to work with elements of $SL_2(q)$ and their traces using the exact covering sequence 26. We recall some of the terminology and results of [18] and [13]. For any $U \in SL_2(q)$, U is called *parabolic, hyperbolic,* or *elliptic* if the characteristic polynomial $\lambda^2 - \mathrm{trace}(U)\lambda - 1$, has a double root, has two distinct roots over $\mathbb{F}_q$, or is irreducible over $\mathbb{F}_q$, respectively. The hyperbolic and elliptic elements are called semi-simple. The trace of an element U does not uniquely determine its $GL_2(q)$ conjugacy class, the exception being parabolic elements. Restricting our attention to the elements U of $SL_2(q) - \{1, -1\}$ we have the following proposition:

Proposition 9. *Let $U \in SL_2(q) - \{1, -1\}$. Then:*

1. *the minimal polynomial is the characteristic polynomial $\lambda^2 - \upsilon\lambda - 1$, $\upsilon = \mathrm{trace}(U)$, and the order of U is determined by the value of the trace;*
2. *two elements of $SL_2(q) - \{1, -1\}$ are $GL_2(q)$-conjugate if and only if they have the same trace υ. The elements are conjugate to $\begin{bmatrix} 0 & -1 \\ 1 & \upsilon \end{bmatrix}$.*

Automorphisms. The group $GL_2(q)$ acts on $SL_2(q)$ by conjugation and, hence, $PGL_2(q)$ acts as a group of automorphisms of both $SL_2(q)$ and $PSL_2(q)$. To find the full group of automorphisms we need to take the Galois group $\mathrm{Gal}(\mathbb{F}_q)$ of field automorphisms into account. A typical automorphism of $GL_2(q)$ has the form

$$\rho \circ Ad_U : X \to \rho\left(UXU^{-1}\right)$$

for $\rho \in \mathrm{Gal}(\mathbb{F}_q)$, $U \in GL_2(q)$, and, hence, $\mathrm{Aut}(PSL_2(q)) = \mathrm{Gal}(\mathbb{F}_q) \ltimes PGL_2(q)$. The order of the groups are as follows:

	$\lvert SL_2(q)\rvert$	$\lvert PSL_2(q)\rvert$	$\lvert GL_2(q)\rvert$	$\lvert PGL_2(q)\rvert$
$p = 2$	$(q-1)q(q+1)$	$(q-1)q(q+1)$	$(q-1)^2q(q+1)$	$(q-1)q(q+1)$
p odd	$(q-1)q(q+1)$	$\frac{(q-1)q(q+1)}{2}$	$(q-1)^2q(q+1)$	$(q-1)q(q+1)$

The Galois group $\mathrm{Gal}(\mathbb{F}_{p^e})$ is the cyclic group of order e generated by the Frobenius automorphism $x \to x^p$.

Cyclic subgroups of $PSL_2(q)$. Next we describe the cyclic and other subgroups of $PSL_2(q)$. For odd p there are three conjugacy classes of maximal cyclic subgroups of $PSL_2(q)$ of orders p, $\frac{q-1}{2}$, and $\frac{q+1}{2}$; and, for even q there are three conjugacy classes of orders $2, q-1$,and $q+1$. In Table 3.1 the third column describes the number of conjugacy classes of an element with the given trace. The fourth column describes the reducibility of the characteristic polynomial $\lambda^2 - \upsilon\lambda + 1$ with $\upsilon = \text{trace}(U)$.

Type	Order	# Classes	Reducibility over $\mathbb{F}_q$
parabolic, q odd	p	2	$\lambda^2 - \upsilon\lambda + 1$ is a square
hyperbolic, q odd	$\frac{q-1}{2}$	1	$\lambda^2 - \upsilon\lambda + 1$ distinct factors
elliptic, q odd	$\frac{q+1}{2}$	1	$\lambda^2 - \upsilon\lambda + 1$ irreducible
parabolic, q even		1	$\lambda^2 - \upsilon\lambda + 1$ is a square
hyperbolic, q even	$q-1$	1	$\lambda^2 - \upsilon\lambda + 1$ distinct factors
elliptic, q even	$q+1$	1	$\lambda^2 - \upsilon\lambda + 1$ irreducible

Table 3.1 Maximal cyclic subgroups of $PSL_2(q)$

Implicit in the table is that if s is relatively prime to the order of U, then the characteristic polynomial of U^s has the same irreducibility characteristics as that of U.

The observations about cyclic subgroups of $SL_2(q)$ and $PSL_2(q)$, recorded in the next proposition, will be useful later on. They are all easily proven by diagonalization of covering elements, possibly in an extension of $\mathbb{F}_q$.

Proposition 10. *Let $U, V \in SL_2(q)$ be elements covering $u, v \in PSL_2(q)$ respectively.*

1. *U is conjugate to an element with entries in $\mathbb{F}_p[\text{trace}(U)]$ (companion matrix).*
2. *If U and V are semi-simple and the order of V divides that of U, then $V = WU^sW^{-1}$ for some $W \in GL_2(q)$, and $\text{trace}(V)$ belongs to the subfield $\mathbb{F}_p[\text{trace}(U)]$ of $\mathbb{F}_q$.*
3. *If U and V are semi-simple and project to elements of the same order in $PSL_2(q)$, then the $\mathbb{F}_p[\text{trace}(U)] = \mathbb{F}_p[\text{trace}(V)]$,*
4. *If $u \in PSL_2(q)$ is semi-simple, then*
   - (a) *if u has odd order l, then the covering elements U and $-U$ can be chosen so that they have orders l and $2l$ in $SL_2(q)$, respectively;*
   - (b) *if u has even order then both of the covering elements have order $2l$;*
   - (c) *the number of traces of elements of $SL_2(q)$ that project to a semi-simple element of order l is the Euler number $\phi(l)$;*
   - (d) *the conjugacy classes $u^{PSL_2(q)}$ and $u^{PGL_2(q)}$ are equal.*
5. *If $u \in PSL_2(q)$ is parabolic, then*
   - (a) *if u has odd order p, then the covering elements U and $-U$ can be chosen so that they have orders p and $2p$ in $SL_2(q)$, respectively;*
   - (b) *if u has even order, then both of the covering elements have order 2;*
   - (c) *the traces of elements of $SL_2(q)$ that project to a parabolic element are ± 2; and,*
   - (d) *if q is odd, the conjugacy class $u^{PGL_2(q)}$ has twice as many elements as does $u^{PSL_2(q)}$. If q is even, then $u^{PSL_2(q)}$ and $u^{PGL_2(q)}$ are equal.*

Subgroups of $PSL_2(q)$. L.E. Dixon [12] classified the subgroups of $PSL_2(q)$ into three types: affine, projective, and exceptional. The types are summarized in Table 3.2 below where we show the maximal subgroups of each type.

Type	Maximal	Order	matrix type/condition
affine - parabolic	$\mathbb{F}_q^* \ltimes \mathbb{F}_q$	$\frac{q(q-1)}{2}$	$\begin{bmatrix} x & y \\ 0 & x^{-1} \end{bmatrix}$ $x \in \mathbb{F}_q^*$, $y \in \mathbb{F}_q$
affine - hyperbolic	$\mathbb{F}_q^*$	$\frac{q-1}{2}$	$\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}$, $x \in \mathbb{F}_q^*$
affine - elliptic		$\frac{q+1}{2}$	$\begin{bmatrix} y & z \\ -\lambda z & y \end{bmatrix}$, $y, z \in \mathbb{F}_q$, $\lambda \notin \mathbb{F}_q^2$, $y^2 + \lambda z^2 = 1$
projective	$PSL_2(r)$	$\frac{r(r^2-1)}{2}$	$\mathbb{F}_r \subset \mathbb{F}_q$
projective	$PGL_2(r)$	$r(r^2-1)$	$\mathbb{F}_{r^2} \subset \mathbb{F}_q$, q odd
exceptional dihedral	D_{q-1}	$q-1$	hyperbolic cyclic normalizer
exceptional dihedral	D_{q+1}	$q+1$	elliptic cyclic normalizer
exceptional	$A_4 = PSL_2(3)$	12	
exceptional	$\Sigma_4 = PGL_2(3)$	24	
exceptional	$A_5 = PSL_2(5)$	60	

Table 3.2 Subgroups of $PSL_2(q)$

3.2. **Lifting $PSL_2(q)$ triples to $SL_2(q)$.** This section recalls the work of MacBeath in [18] on generating triples, which was used extensively in [13] and [14]. First, we translate the discussion on Schur covers in Section 2.4 to the cover $\widetilde{PSL_2(q)} = SL_2(q) \twoheadrightarrow PSL_2(q)$. Let us consider a typical (l, m, n)-action of $G = PSL_2(q)$, with generating triple (a, b, c). Let (A, B, C) be a lift of (a, b, c) to $SL_2(q)$ – recall that $ABC = I$. For simplicity of notation in the remaining sections, we choose (A, B, C) to denote the lift of (a, b, c) to $SL_2(q)$, not the lift to Γ as in previous sections, no confusion should result.

Let

$$\alpha = \text{trace}(A),\ \beta = \text{trace}(B),\ \gamma = \text{trace}(C),$$

and call (α, β, γ) a trace triple and (A, B, C) an (α, β, γ)-triple. For any other triple (A', B', C') projecting to (a, b, c) the corresponding traces satisfy $\alpha' = \pm\alpha$, $\beta' = \pm\beta$, $\gamma' = \pm\gamma$. We noted that the order of a non-identity element in $PSL_2(q)$ is determined by the trace of an element lying over it in $SL_2(q)$ and, hence, that any two $(\pm\alpha, \pm\beta, \pm\gamma)$-triples yield (l, m, n)-triples of the same type. We define the *trace triple set* $Tr(\alpha, \beta, \gamma)$ by

$$Tr(\alpha, \beta, \gamma) = \{(A, B, C) \in (SL_2(q) - \{\pm 1\})^3 : ABC = I,\ \text{trace}(A) = \alpha,\ \text{trace}(B) = \beta,\ \text{trace}(C) = \gamma\}$$

This definition is a slight variation of the definitions in [18] and [13], where A, B, C are allowed to be ± 1.

In the following Remark, we record some properties of trace triple sets, which easily follow from Proposition 10.

Remark 11. *The following properties hold for trace triple sets:*

(1) *The four triple sets* $Tr(\alpha,\beta,\gamma)$, $Tr(\alpha,-\beta,-\gamma)$, $Tr(-\alpha,\beta,-\gamma)$, *and* $Tr(-\alpha,-\beta,\gamma)$ *form a lift orbit of* $(a,b,c)^{PGL_2(q)}$ *discussed in the Section 2.4 on Schur covers.*
(2) *The four sets in a lift orbit will be distinct if at least two of the three traces are non-zero. This will occur for hyperbolic signatures since at most one of* l,m,n *will equal* 2 *and trace zero elements of* $SL_2(q)$ *project to involutions.*
(3) *If all three of* α,β,γ *are non-zero, then* $Tr(\alpha,\beta,\gamma)$ *and* $Tr(-\alpha,\beta,\gamma)$ *will both correspond to the same* (l,m,n) *but must have disjoint projections to* $L_G(a,b,c)$. *For, if* $(A,B,C)\in Tr(\alpha,\beta,\gamma)$ *and* $(A',B',C')\in Tr(-\alpha,\beta,\gamma)$ *project to the same triple* (a,b,c), *then* $A'=-A, B'=B, C'=C$ *and* $A'B'C'=-ABC=-I$, *a contradiction. Thus, the sets* $Tr(-\alpha,\beta,\gamma)$, $Tr(\alpha,-\beta,\gamma)$, $Tr(\alpha,\beta,-\gamma)$, $Tr(-\alpha,-\beta,-\gamma)$ *form a lift orbit for a companion action. Indeed, as we shall prove,* $L_G(a,b,c)$ *is the disjoint union of the two distinct of images of* $Tr(\alpha,\beta,\gamma)$ *and* $Tr(-\alpha,-\beta,-\gamma)$.

Macbeath originally proved in [18] that each $Tr(\alpha,\beta,\gamma)$ were non-empty except in a small number of special cases. A key concept introduced by Macbeath was the notion of singularity of a triple which we now describe. Select a triple $(A,B,C)\in Tr(\alpha,\beta,\gamma)$. Since A is not a scalar matrix it is conjugate to its companion matrix and, hence triple in $Tr(\alpha,\beta,\gamma)$ is conjugate to one in which

$$A=\begin{bmatrix}0 & -1\\ 1 & \alpha\end{bmatrix},\ B=\begin{bmatrix}\beta-x & y\\ z & x\end{bmatrix},\ C=(AB)^{-1}=\begin{bmatrix}\alpha x+y & x\\ x-\alpha z-\beta & -z\end{bmatrix}.$$

Now $\det(B)=1$ and $\mathrm{tr}(C)=\gamma$ so that we have

$$x\beta-x^2-zy=1,\ z=\alpha x+y-\gamma \tag{28}$$

or the following quadratic form equation:

$$x^2+\alpha xy+y^2-\beta x-\gamma y+1=0. \tag{29}$$

Note that solutions to this equation are solutions of the form $(x,y,1)$ of

$$x^2+y^2+z^2+\alpha xy-\beta xz-\gamma yz=0 \tag{30}$$

or in matrix form, for odd q,

$$X^tQX=\begin{bmatrix}x & y & z\end{bmatrix}\begin{bmatrix}1 & \frac{\alpha}{2} & \frac{-\beta}{2}\\ \frac{\alpha}{2} & 1 & \frac{-\gamma}{2}\\ \frac{-\beta}{2} & \frac{-\gamma}{2} & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=0. \tag{31}$$

The form in equation 30 factors (possibly over an extension of $\mathbb{F}_q$) if and only if the determinant of Q is zero, which is equivalent to

$$DQ(\alpha,\beta,\gamma)=\alpha^2+\beta^2+\gamma^2-\alpha\beta\gamma-4=0. \tag{32}$$

Note that this holds for $p=2$ even though the derivation would no longer hold. Macbeath calls (α,β,γ) a singular triple in this case and non-singular otherwise. By extension we say that (A,B,C) and its projection (a,b,c) are singular or non-singular. In [18] Macbeath proves the following two propositions.

Proposition 12. *Let* $(A,B,C)\in Tr(\alpha,\beta,\gamma)$, *then* $\langle A,B,C\rangle$ *is an affine subgroup of* $SL_2(q)$ *if and only if* $x^2+y^2+z^2+\alpha xy-\beta xz-\gamma yz$ *factors, i.e., equation 32 holds.*

Proposition 13. *Let notation be as above and suppose that (α, β, γ) is a non-singular triple. Then*

$$|Tr(\alpha, \beta, \gamma)| = |PGL_2(q)| . \tag{33}$$

The two results above yield the following:

Corollary 14. *Let notation be as above and suppose that (α, β, γ) is a non-singular triple. For any $(A, B, C) \in Tr(\alpha, \beta, \gamma)$ the $PGL_2(q)$ orbit of (A, B, C) equals $Tr(\alpha, \beta, \gamma)$. Consequently, every non-singular trace triple class $Tr(\alpha, \beta, \gamma)$ has at most one $PGL_2(q)$ class of generating vectors.*

Proof: The centralizer of a subgroup of $PSL_2(q)$ is non-trivial if and only if the subgroup is cyclic. But, if $\langle A, B, C\rangle$ is cyclic, then it is affine, and (α, β, γ) is a singular. This contradiction shows that $\langle A, B, C\rangle$ has a trivial centralizer and

$$\left|\langle A, B, C\rangle^{PGL_2(q)}\right| = |PGL_2(q)| = |Tr(\alpha, \beta, \gamma)|$$

so that $\langle A, B, C\rangle^{PGL_2(q)} = Tr(\alpha, \beta, \gamma)$. If (A, B, C) is a generating vector, then $\langle A, B, C\rangle^{PGL_2(q)} = Tr(\alpha, \beta, \gamma)$, and there is no room for anything else. ■

It is instructive to give an alternate version of MacBeath's proof of Proposition 13 to see directly how the singularity condition 32 is used. We give a proof for odd q only.

Proof: (Proposition 13) Let $A_0 = \begin{bmatrix} 0 & -1 \\ 1 & \alpha \end{bmatrix}$. Each $(A, B, C) \in Tr(\alpha, \beta, \gamma)$ is conjugate to a triple of the form $(A_0, B', C') \in Tr(\alpha, \beta, \gamma)$. The number of triples of the form (A_0, B', C') in $Tr(\alpha, \beta, \gamma)$ is the number of solutions to equation 29. It follows then that

$$|Tr(\alpha, \beta, \gamma)| = \frac{|GL_2(q)|}{|\mathrm{Cent}(GL_2(q), A)|} \left|\left\{(x, y) : x^2 + \alpha xy + y^2 - \beta x - \gamma y + 1 = 0\right\}\right| . \tag{34}$$

We show that this quantity is $q(q^2 - 1)$ by a case analysis in the following table, depending on the type of A. In the table, the $GL_2(q)$ conjugacy class $A^{GL_2(q)}$ has cardinality $\left|A^{GL_2(q)}\right| = \frac{|GL_2(q)|}{|\mathrm{Cent}(GL_2(q),A)|}$. The fourth column is the number of solutions to equation 29.

Type of A	$u^2 + \alpha uv + v^2$	$\left\|A^{GL_2(q)}\right\|$	# of solutions	$\|Tr(\alpha, \beta, \gamma)\|$
elliptic	irreducible	$q(q-1)$	$q+1$	$q(q^2-1)$
hyperbolic	distinct factors	$q(q+1)$	$q-1$	$q(q^2-1)$
parabolic	square	q^2-1	q	$q(q^2-1)$

First let us calculate $|\mathrm{Cent}(GL_2(q), A)|$. The centralizer $\mathrm{Cent}(GL_2(q), A)$ is contained in the $\mathbb{F}_q$ linear span of A and the identity matrix I. The set of invertible matrices in this linear span is $\mathrm{Cent}(GL_2(q), A)$, with $Z^{-1} = \frac{1}{\det(Z)} (\mathrm{trace}(Z)I - Z)$ for a typical element $Z \in \mathrm{Cent}(GL_2(q), A)$, using the Cayley Hamilton theorem. Letting

$$Z = uI + vA = \begin{bmatrix} u & -v \\ v & u + \alpha v \end{bmatrix}$$

we see that $\det(Z) = u^2 + \alpha uv + v^2$. Thus

$$|\mathrm{Cent}(GL_2(q), A)| \;=\; q^2 - \left|\left\{(u,v) : u^2 - \alpha uv + v^2 = 0\right\}\right|.$$

Next we need the number of solutions to $u^2 + \alpha uv + v^2 = 0$. If the equation $u^2 - \alpha uv + v^2$ has a non-zero solution then it is reducible and, hence, there is only one solution $(0,0)$ in the irreducible case . If $u^2 + \alpha uv + v^2$ is reducible, but not a square, then the zero set is the union of two distinct intersecting lines and, hence, has $2q-1$ points. If $\alpha = \pm 2$, then $u^2 - \alpha uv + v^2 = (u \pm v)^2$ and there are q solutions. Thus

$$|\mathrm{Cent}(GL_2(q), A)| = q^2 - 1,\ (q-1)^2,\ q(q-1)$$

in the irreducible, distinct factors and the square cases respectively. This gives us column 3 of the table. To count the number of solutions of equation 33 in column 4 we consider three cases depending on the type of A.

Elliptic case. We eliminate the linear term in the equation as follows. Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $Q = \begin{bmatrix} 1 & \frac{\alpha}{2} \\ \frac{\alpha}{2} & 1 \end{bmatrix}$, $E = \begin{bmatrix} -\beta \\ -\gamma \end{bmatrix}$ and then the matrix form of equation 29 is:

$$X^t Q X + E^t X + 1 = 0.$$

Replacing X by $Y + W$ with $Y = \begin{bmatrix} u \\ v \end{bmatrix}$ and $W = \frac{-1}{2} Q^{-1} E$ we get $Y^t Q Y = -1 + \frac{1}{4} E^t Q^{-1} E$ or

$$u^2 + \alpha uv + v^2 = \frac{\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4}{\alpha^2 - 4}. \tag{35}$$

As (α, β, γ) is a non singular triple the right hand side of the equation is non-zero. The number of solutions to $u^2 + \alpha uv + v^2 = d \in \mathbb{F}_q^*$ is independent of d. To see this, observe that the map $\det : \mathrm{Cent}(GL_2(q), A) \to \mathbb{F}_q^*, Z \to \det(Z)$ is a group homomorphism and so

$$|\mathrm{Cent}(GL_2(q), A)| = |\mathrm{Im}(\det)| \times \left|\left\{(u,v) : u^2 + \alpha uv + v^2 = 1\right\}\right|.$$

Also as $\det(zZ) = z^2 \det(Z)$ then $\mathrm{Im}(\det)$ is either $\mathbb{F}_q^*$ or the set of squares in $\mathbb{F}_q^*$ with cardinalities $q-1$ and $(q-1)/2$ respectively. It follows that the number of solutions of $u^2 + \alpha uv + v^2 = 1$, is either $q+1$ or $2(q+1)$. But the fibres of the map $pr_2 : \{(u,v) : u^2 + \alpha uv + v^2 = 1\} \to \mathbb{F}_q$, $(u,v) \to v$ have at most two elements each, which implies that there are at most $2q$ points in $\{(u,v) : u^2 + \alpha uv + v^2 = 1\}$. We then must have $\left|\left\{(u,v) : u^2 + \alpha uv + v^2 = 1\right\}\right| = q+1$, It follows that $\mathrm{Im}(\det) = \mathbb{F}_q^*$ and that $\left|\left\{(u,v) : u^2 + \alpha uv + v^2 = d\right\}\right| = q+1$, for every $d \in \mathbb{F}_q^*$.

Hyperbolic Case: We eliminate the linear term as before. The equation $u^2 + \alpha uv + v^2 = d \in \mathbb{F}_q^*$ may be rewritten $(u + r_1 v)(u + r_2 v) = d$ for $r_1, r_2 \in \mathbb{F}_q^*$, where $r_1 r_2 = 1$, $r_1 + r_2 = \alpha$. Each possible solution satisfies $u + r_1 v = e$, $u + r_2 v = d/e$ for some $e \in \mathbb{F}_q^*$ and there are $q-1$ solutions.

Parabolic case. We may assume that $\alpha = 2$. If (α, β, γ) is non-singular then $0 \neq \alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 = (\beta - \gamma)^2$. Our equation for counting is $x^2 + 2xy + y^2 + 2x - 2y + 1 = 0$. Setting $y = u - x$ in $x^2 + 2xy + y^2 - \beta x - \gamma y + 1 = 0$ we get $u^2 - \gamma u + 1 + (\gamma - \beta)x = 0$. There are q solutions to this equation, as there is a unique value of x for every value of u. ∎

3.3. **Admissible trace triples.** We call a trace triple (α, β, γ) *admissible* if it is non-singular and the associated signature (l, m, n) is hyperbolic. Specifically, we must leave out the spherical signatures $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$; and, the planar signatures $(2, 3, 6)$, $(2, 4, 4)$, $(3, 3, 3)$. Non-admissable (α, β, γ) cannot generate a hyperbolic action of $PSL_2(q)$ though admissible triples may generate a hyperbolic action of a proper subgroup. By Corollary 14 all action triples (a, b, c) corresponding to an admissible (α, β, γ) are generating action triples or generate a subgroup belonging to a single conjugacy class of proper subgroups. In the subgroup table below, we list all possible signatures for admissible trace triples that generate proper subgroups. The affine subgroups have no admissible trace triples. The dihedral groups can only have spherical signatures, and the subgroup A_4 has no hyperbolic signatures. Thus the projective subgroups $PSL_2(r)$, $PGL_2(r)$ and the exceptional subgroups Σ_4 and A_5 are the only proper groups that can be generated by an admissible trace triple.

Type	**Name**	**Order**	**Signatures**
affine - parabolic	$\mathbb{F}_q^* \ltimes \mathbb{F}_q$	$\frac{q(q-1)}{2}$	none
affine - hyperbolic	$\mathbb{F}_q^*$	$\frac{q-1}{2}$	none
affine - elliptic		$\frac{q+1}{2}$	none
projective	$PSL_2(r)$, $r\|q$,	$r(r^2-1)/2$, $r(r^2-1)$	various
projective	$PGL_2(r)$, $r^2\|q$	$r(r^2-1)$	various
exceptional dihedral	D_{q-1}	$q-1$	none
exceptional dihedral	D_{q+1}	$q+1$	none
exceptional	$A_4 = PSL_2(3)$	12	none
exceptional	$\Sigma_4 = PGL_2(3)$	24	$(3, 4, 4)$
exceptional	$A_5 = PSL_2(5)$	60	$(2, 5, 5)$, $(3, 3, 5)$, $(3, 5, 5)$, $(5, 5, 5)$

Table 3.3 - Signatures for subgroups with admissible trace triples

Next we identify when trace triples generate projective subgroups.

Proposition 15. *Suppose that (α, β, γ) is a trace triple over $\mathbb{F}_q$ that determines determines a hyperbolic signature (l, m, n). Define e so that $\mathbb{F}_p[\alpha, \beta, \gamma] \backsimeq \mathbb{F}_{p^e}$ and $\mathbb{F}_{p^e} \subseteq F_q$. Then, for any other triple $(\alpha', \beta', \gamma')$ over any $\mathbb{F}_{p^s}$ that determines (l, m, n) we have*

$$\mathbb{F}_p[\alpha, \beta, \gamma] = \mathbb{F}_{p^e} = \mathbb{F}_p[\alpha', \beta', \gamma']$$

and e divides s. Moreover, there is at least one non-singular trace triple (α, β, γ) associated to (l, m, n).

Proof: First, consider an element a of order l. Then the order l divides p, $\frac{q-1}{2}$, or $\frac{q+1}{2}$ or in case $p = 2$, l divides one of 2, $q-1$, $q+1$. Let us deal with odd q first. If $PSL_2(q)$ has an element of order l then one of the following holds $l = p$, or $q = \pm 1 \bmod 2l$, i.e., $q^2 = 1 \bmod 2l$. If $l = p$, then $\alpha = \pm 2$ and set $e_l = 1$. Otherwise, the sequence of groups $PSL_2(p^s)$ with elements of order l is the set $\{PSL_2(p^s) : p^{2s} = 1 \bmod 2l\}$. Since p is invertible mod $2l$ the set of such integer exponents $\{s : p^{2s} = 1 \bmod 2l\}$ has the form $e_l\mathbb{Z}$ for some $e_l > 0$. So $PSL_2(q)$ can have an element of order l if and only if $\mathbb{F}_{p^{e_l}}$ is a subfield of $\mathbb{F}_q$. The subgroup $PSL_2(p^{e_l}) \subseteq PSL_2(q)$ has an element of U order l and hence trace$(U) \in \mathbb{F}_{p^{e_l}}$.

By Proposition 10 the traces of covering elements of $PSL_2(q)$ of order l lie in $\mathbb{F}_{p^{e_l}}$; in fact, they individually generate $\mathbb{F}_{p^{e_l}}$. Define e_m and e_n similarly, and let $e = \operatorname{lcm}(e_l, e_m, e_n)$. Then the components of every trace triple $(\alpha', \beta', \gamma')$ with associated signature (l, m, n) of $PSL_2(q)$ lie in $\mathbb{F}_{p^e}$; in fact, $\mathbb{F}_{p^e} = \mathbb{F}_p[\alpha', \beta', \gamma']$. The proof for $p = 2$ is entirely similar.

Now we find a non-singular trace triple. Holding β, γ fixed, a singular trace triple must satisfy

$$p_{\beta,\gamma}(\alpha) = \alpha^2 - (\beta\gamma)\alpha + (\beta^2 + \gamma^2 - 4) = 0, \tag{36}$$

a quadratic equation in α. According to Proposition 10, unless the Euler function value $\phi(l) \leq 2$ or $l = p$, there is an element A projecting to an element a of order l such that $p_{\beta,\gamma}(\text{trace}(A)) \neq 0$. But $\phi(l) > 2$ unless $l = 2, 3, 4, 6$. Thus, assuming $p \neq 2, 3$, we must choose l from $\{2, 3, 4, 6, p\}$. By a similar argument, the same applies to m and n. The squares of the traces $\tau = \alpha, \beta, \gamma$ for $r = l, m, n$, respectively, would then have the values in the following table:

r	2	3	4	6	p
τ^2	0	1	2	3	4

Assuming that all the triples (α, β, γ) are singular, then

$$2\alpha\beta\gamma = p_{\beta,-\gamma}(\alpha) - p_{\beta,\gamma}(\alpha) = 0.$$

For the moment assume that p is odd. Since at most one of l, m, n can equal 2 by hyperbolicity, we may assume that $2\beta\gamma \neq 0$ and, hence, $\alpha = 0$ and $l = 2$. The possible hyperbolic triples are then $(2, 3, p)$, $(2, 4, 6)$, $(2, 4, p)$, $(2, 6, 6)$, $(2, 6, p)$, and $(2, p, p)$. The singular triple equation now becomes $\beta^2 + \gamma^2 = 4$, which cannot hold for any of these triples. If $p = 2$ then l, m, n must be chosen from $2, 3$ and there are no hyperbolic triples. For $p = 3$ they must be chosen from $2, 3, 4$ and there are no hyperbolic triples with $l = 2$. ∎

Remark 16. *Using the above proof, we can immediately generalize the well known classification of Hurwitz surfaces with $PSL_2(p^e)$ as automorphism group (see [18]). There is a Hurwitz action of $PSL_2(p)$ if and only if $p = 7$, or $p = \pm 1 \bmod 7$. The only action of $PSL_2(p^e)$ for $e > 1$ is $PSL_2(p^3)$ where $p^3 = \pm 1 \bmod 7$, but $p \neq \pm 1 \bmod 7$.*

Corollary 17. *Let (l, m, n) be a hyperbolic triple. Then for every p there is an integer e, dependent on l, m, n and p, such that $PSL_2(p^e)$ has an (l, m, n) action on a surface. However, there is no (l, m, n) action for $PSL_2(p^{e'})$ where $e' \neq e$.*

Proof: As we saw previously, $PSL_2(p^e)$ has an action for $e = \operatorname{lcm}(e_l, e_m, e_n)$ defined in the preceding proof. For any other value of $q = p^{e'}$ divisible by p^e, all (l, m, n) triples (a, b, c) generate a proper subgroup of $PSL_2(q)$. ∎

Finally, we give a complete description of all triples corresponding to lifts of a hyperbolic triple (a, b, c) in $PSL_2(q)$. It is convenient to split the even and odd cases into two separate propositions.

Proposition 18. *Suppose that q is odd. Let (a, b, c) in $PSL_2(q)$ be an (l, m, n)-triple and let $(A, B, C) \in Tr(\alpha, \beta, \gamma)$ be a covering triple. Let $G = PSL_2(q)$ and $L = PGL_2(q)$, considered as automorphism groups of $PSL_2(q)$. Let $Tr(\alpha', \beta', \gamma')$*

be one of the disjoint sets $Tr(\pm\alpha, \pm\beta, \pm\gamma)$ (1, 2, 4, or 8 in number). Then we have the following.

1. *The set $L_G(a,b,c)$ is the image of $\bigcup_{\alpha',\beta',\gamma'} Tr(\alpha',\beta',\gamma')$ under the map $(A',B',C') \to (a',b',c')$, $(A',B',C') \in Tr(\alpha',\beta',\gamma')$.*
2. *Suppose that (a,b,c) has a hyperbolic signature. Then, at most, one of α,β,γ is zero and the projection $(A',B',C') \to (a',b',c')$ is 1-1 when restricted to $Tr(\alpha',\beta',\gamma')$.*
3. *Suppose that (A,B,C) is hyperbolic and non-singular. Then the image of $Tr(\alpha,\beta,\gamma)$ is a single $PGL_2(q)$ class of triples. If $\langle a,b,c\rangle$ is a proper subgroup of $PSL_2(q)$, then the signature must occur in Table 3.3.*
4. *Suppose that (A,B,C) is hyperbolic, non-singular and $\alpha\beta\gamma \neq 0$, so that there are eight disjoint sets among the $Tr(\pm\alpha,\pm\beta,\pm\gamma)$. Then we have these two cases:*
   (a) *Both lift orbits in $Tr(\pm\alpha,\pm\beta,\pm\gamma)$ correspond to non-singular triples, and there are two disjoint $PGL_2(q)$ classes in $L_G(a,b,c)$.*
   (b) *One lift orbit $Tr(\pm\alpha,\pm\beta,\pm\gamma)$ consists of non-singular triples and the other does not. The set $L_G(a,b,c)$ contains a single $PGL_2(q)$ class of non-singular triples. All the other triples generate proper affine subgroups.*
5. *Suppose that (A,B,C) is hyperbolic, non-singular and $\alpha\beta\gamma = 0$ so that there are only four disjoint sets among the $Tr(\pm\alpha,\pm\beta,\pm\gamma)$. Then, assuming that $l \le m \le n$, we have $l = 2$, $\alpha = 0$, and $DQ(\alpha,\beta,\gamma) = \beta^2+\gamma^2-4 \neq 0$. All four triple sets $Tr(\pm\alpha,\pm\beta,\pm\gamma)$ comprise a lift orbit, and $L_G(a,b,c)$ is a single $PGL_2(q)$ class of non-singular triples.*

Proposition 19. *Suppose that q is even. Let $G = PSL_2(q) = SL_2(q)$ and $L = PGL_2(q)$, considered as automorphism groups of $PSL_2(q)$. Let (A,B,C) in $SL_2(q)$ be an (l,m,n)-triple.*

1. *Suppose that (A,B,C) has a hyperbolic signature. Then, at most one of α,β,γ is zero.*
2. *Suppose that (A,B,C) is hyperbolic and non-singular. $Tr(\alpha,\beta,\gamma)$ is a single $PGL_2(q)$ class of triples. If $\langle A,B,C\rangle$ is a proper subgroup of $SL_2(q)$, then the signature must occur in Table 3.3.*

Proof: For Proposition 18 we argue as follows.

Statement 1. This follows from the discussion on Schur covers in Section 2.4.

Statement 2. As noted in Remark 11, at most one of α,β,γ can be zero for a hyperbolic signature. Next let us show that the projection $(A,B,C) \to (a,b,c)$ is 1-1 when restricted to $Tr(\alpha,\beta,\gamma)$. Any cover (A',B',C') of (a,b,c) must satisfy $A' = \pm A$, $B' = \pm B$, $C' = \pm C$. If, for instance, $A' = -A$, then $\alpha = \text{trace}(A') = -\text{trace}(A) = -\alpha$, and so $\alpha = 0$. Since $\beta,\gamma \neq 0$ then $B' = B$ and $C' = C$. But $I = A'B'C' = -ABC = -I$, a contradiction.

Statement 3. This is Corollary 14.

Statements 4 and 5. As noted in Remark 11, the transformation

$$(A,B,C) \to (-A,-B,C)$$

carries $Tr(\alpha,\beta,\gamma)$ to $Tr(-\alpha,-\beta,\gamma)$ and preserves the form $DQ(\alpha,\beta,\gamma) = \alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4$. With similar arguments, we see that the entire lift orbit consists of non-singular triples if (α, β, γ) is non-singular. The entire lift orbit determines the same $PGL_2(q)$ class $L_G(a, b, c)$. If the other lift orbit is non-singular, we get a second $PGL_2(q)$ in $L_G(a, b, c)$ disjoint from the first, according to Remark 11. If $\alpha\beta\gamma \neq 0$ then we cannot have both $DQ(\alpha, \beta, \gamma)$ and $DQ(-\alpha, \beta, \gamma)$ equal to 0. For then $2\alpha\beta\gamma = DQ(\alpha, \beta, \gamma) - DQ(-\alpha, \beta, \gamma) = 0$. The rest of the statements follow easily.

For Proposition 19 the arguments are similar. ∎

Theorem 20. *Every quasi-platonic action of $PSL_2(p)$ on a surface of genus $\sigma \geq 2$ is determined by a unique lift orbit representative of a trace triple (α, β, γ) satisfying the following conditions:*

1. *the corresponding signature (l, m, n) is hyperbolic, non-singular, and $l \leq m \leq n$;*
2. *there is an (A, B, C) in $Tr(\alpha, \beta, \gamma)$ such that $|\langle A, B, C\rangle| = |PSL_2(q)|$; and,*
3. *the generating triple for the action is (a, b, c), the projection of (A, B, C) into $PSL_2(p)$.*

To formulate the theorem for composite q, we need the following proposition.

Proposition 21. *The action of $Gal(\mathbb{F}_q)$ on the generating non-singular hyperbolic trace triples has no fixed points.*

Proof: The action of $\mathrm{Gal}(\mathbb{F}_q)$ on trace triple sets is

$$Tr(\alpha, \beta, \gamma) \to Tr(\rho(\alpha), \rho(\beta), \rho(\gamma))$$

for $\rho \in \mathrm{Gal}(\mathbb{F}_q)$. Thus, $\mathrm{Gal}(\mathbb{F}_q)$ permutes the trace triple sets. Also

$$DQ(\rho(\alpha), \rho(\beta), \rho(\gamma)) = \rho\,(DQ(\alpha, \beta, \gamma))$$

so that $Tr(\alpha, \beta, \gamma)$ permutes the non-singular trace triple sets. Additionally, $\mathrm{Gal}(\mathbb{F}_q)$ normalizes the $PGL_2(q)$ conjugation action so that $\mathrm{Gal}(\mathbb{F}_q)$ permutes the $PGL_2(q)$ orbits of triples (A, B, C). Now suppose that $(\rho(\alpha), \rho(\beta), \rho(\gamma)) = (\alpha, \beta, \gamma)$ for a non-singular triple. Then $\rho \cdot (A, B, C) = (\rho(A), \rho(B), \rho(C)) \in Tr(\alpha, \beta, \gamma)$ and so there is a $U \in GL_2(q)$ such that $\rho \cdot (A, B, C) = Ad_U \cdot (A, B, C)$. Then $\rho \circ Ad_{U^{-1}}$ fixes (A, B, C), so $\rho = Ad_U$ and thus $\rho = Ad_U = 1$. ∎

Let $J \leq \langle \pm I\rangle^3$ be the subgroup defining the lift orbits. Then $\mathrm{Gal}(\mathbb{F}_q) \times J$ acts without fixed points on non-singular hyperbolic trace triples.

Theorem 22. *Let notation be as in Theorem 22 except that we consider $PSL_2(q)$ actions. Then the conclusion of Theorem 22 holds except that we consider $Gal(\mathbb{F}_q) \times J$ orbit representatives on non-singular hyperbolic trace triples.*

4. Sample quasi-platonic actions of $PSL_2(q)$

In this section, we determine all actions for $q = 7, 8$; give interesting partial results for some other small primes; and give a table of the number of actions for values of $q \leq 50$ and $q = 2^6, q = 3^4$. Finally, we determine all $(2, 3, n)$, $(2, 4, n)$, $(2, 6, n)$, and $(3, 3, n)$ actions as these capture almost all genus actions and correspond to families with the small tiling polygons and large actions.

All calculations, except the families, can be completed using MAGMA [19], following these steps.

(1) Find all orders of elements of $PSL_2(q)$ and then all possible hyperbolic signatures, (l, m, n) with $l \leq m \leq n$.
(2) Determine the trace order map $\mathbb{F}_q \to \{PSL_2(q) \text{ orders}\}$.
(3) For each (l, m, n) in Step 1, construct the set of associated trace triples, using the map constructed in Step 2.
(4) For odd q, select one trace triple from each lift orbit.
(5) Eliminate all singular trace triples.
(6) For each trace triple remaining, construct a triple (A, B, C) and compute the size of $\langle A, B, C\rangle$. Reject those triples for which $|\langle A, B, C\rangle| \neq |SL_2(q)|$.

Example 23. *Let $G = PSL_2(7)$ The orders of elements and the corresponding traces of covering elements are given in the order-trace table following.*

order	2	3	4	7
traces	0	± 1	± 3	± 2

The table of actions follows. Each line gives the signature, a representative trace triple, and the genus for each action class. The notes column describes situations when the number of actions is less than expected for a given signature.

(l, m, n)	(α, β, γ)	*genus*	*notes*
$(2, 3, 7)$	$(0, 1, 2)$	3	
$(2, 4, 7)$	$(0, 3, 2)$	10	
$(2, 7, 7)$	$(0, 2, 2)$	19	
$(3, 3, 4)$	$(1, 1, 3), (1, 1, 4)$	8	
$(3, 3, 7)$	$(1, 1, -2)$	17	$(1, 1, 2)$ *is singular*
$(3, 4, 4)$	$(-1, 3, 3)$	15	$(1, 3, 3)$ *yields* Σ_4
$(3, 4, 7)$	$(1, 3, 2), (1, 3, -2)$	24	
$(3, 7, 7)$	$(1, 2, 2), (-1, 2, 2)$	33	
$(4, 4, 4)$	$(3, 3, 3), (3, 3, 4)$	22	
$(4, 4, 7)$	$(3, 3, -2)$	31	$(3, 3, 2)$ *is singular*
$(4, 7, 7)$	$(3, 2, 2), (-3, 2, 2)$	40	
$(7, 7, 7)$	$(2, 2, -2)$	49	$(2, 2, 2)$ *is singular*

Table 4.1 $PSL_2(7)$ actions

Remark 24. *We observe from the preceding example that the signature $(3, 3, 7)$ has half of its trace triples singular and the other half non-singular. This holds for all $PSL_2(p)$. For, the two trace triples are $(1, 1, 2)$ and $(1, 1, -2)$. The first is singular and the second is non-singular. For the non-singular triple the only possible proper subgroups with an element of order p are the parabolic affine subgroups and the full $PSL_2(p)$. So the group must be $PSL_2(p)$. Similar remarks apply to the (p, p, p) signature and the trace triples $(2, 2, 2)$ and $(2, 2, -2)$.*

Example 25. *Let $G = PSL_2(8)$ The orders of elements and the corresponding traces of covering elements are given. We write $\mathbb{F}_8 = \mathbb{F}_2[w]$ where w is a generator of the cyclic group $\mathbb{F}_8^*$, and construct the order-trace table.*

order	2	3	7	9
traces	0	1	w^3, w^5, w^6	w, w^2, w^4

The table of actions follows. We do not have to worry about the sign action on triples, though $\mathrm{Out}(G)$ is now generated by the Frobenius action $z \to z^2$ on $\mathbb{F}_8$.

The table organization is as in Table 4.1. There are no projective or exceptional subgroups with hyperbolic signatures.

(l,m,n)	(α,β,γ)	*genus*	*notes*
$(2,3,7)$	$(0,1,w^3)$	7	
$(2,3,9)$	$(0,1,w^2)$	15	
$(2,7,7)$	$(0,w^3,w^5),(0,w^3,w^6)$	55	$(0,w^3,w^3)$ *is singular*
$(2,7,9)$	$(0,w^3,w),(0,w^3,w^2),(0,w^3,w^4)$	63	$(0,w,w)$ *is singular*
$(2,9,9)$	$(0,w,w^2),(0,w,w^4)$	71	$(0,w,w)$ *is singular*
$(3,3,7)$	$(1,1,w^3)$	41	
$(3,3,9)$	$(1,1,w)$	57	
$(3,7,7)$	$(1,w^3,w^3),(1,w^3,w^5),(1,w^3,w^6)$	97	
$(3,7,9)$	$(1,w^3,w),(1,w^3,w^2),(1,w^3,w^4)$	105	
$(3,9,9)$	$(1,w,w)$	113	*6 singular classes*
$(7,7,7)$	$(w^3,w^3,w^5),(w^3,w^3,w^6),$ $(w^3,w^5,w^3),(w^3,w^6,w^3)$	145	*5 singular classes*
$(7,7,9)$	$(w^a,w^b,w), a,b=3,5,6$	153	
$(7,9,9)$	$(w^3,w^a,w^b), a,b=1,2,4$	161	
$(9,9,9)$	$(w,w,w),(w,w,w^4),\ (w,w^2,w^2)$ $(w,w^2,w^4),(w,w^4,w),\ (w,w^4,w^2)$	169	*3 singular classes*

Table 4.2 $PSL_2(8)$ *actions*

Example 26. *Let* G *be one of* $PSL_2(11)$, $PSL_2(13)$, $PSL_2(32)$, $PSL_2(47)$. *The orders of elements and the corresponding traces of covering elements are given below.*

$q=11,$

order	2	3	5	6	11
traces	0	± 1	$\pm 3, \pm 4$	± 5	± 2

$q=13,$

order	2	3	6	7	13
traces	0	± 1	± 4	$\pm 3, \pm 5, \pm 6$	± 2

$q=32,$

order	2	3	11	31	33
traces	0	1	5 vals	15 vals	10 vals

$q=47,$

order	2	3	4	6	8	12	23	24	47
traces	0	± 1	± 7	± 14	4 vals	4 vals	22 vals	8 vals	± 2

$q=49,$

order	2	3	4	5	6	7	8	12	24	25
traces	0	± 1	± 3	4 vals	2 vals	± 2	4 vals	4 vals	8 vals	20 vals

There are too many hyperbolic, non-singular trace triples to list, but we write down a few of interest. Of special interest are the signatures with proper subgroup actions *and equilateral signatures ($l = m = n$) with many actions.*

q	$\|PSL_2(q)\|$	$(\mathbf{l}, \mathbf{m}, \mathbf{n})$	*# admissible trace triples*	*number of actions*	*proper subgroup*	*genus*
11	660	$(2,5,5)$	16	2	2 (A_5)	34
11	660	$(3,3,5)$	16	2	2 (A_5)	45
11	660	$(3,5,5)$	32	4	4 (A_5)	89
11	660	$(5,5,5)$	40	8	2 (A_5)	133
13	1092	$(7,7,7)$	156	39	0	313
32	32736	$(31,31,31)$	2940	518	0	14785
47	51888	$(3,3,4)$	8	1	1 (Σ_4)	4325
47	51888	$(23,23,23)$	9724	2431	0	22561
49	58800	$(2,3,7)$	4	0	1 ($PSL_2(7)$)	701
49	58800	$(2,3,8)$	8	0	2 ($PGL_2(7)$)	701

Table 4.3 Other Sample Actions

Example 27. *Next, we give a summary table for all q in the range $5 \leq q \leq 49$ and the prime powers $64 = 2^6$ and $81 = 3^4$.*

q	$\|PSL_2(q)\|$	orders	signatures	actions	min genus	max genus
5	60	3	4	5	4	13
7	168	4	12	17	3	49
$8 = 2^3$	504	4	14	46	7	169
$9 = 3^2$	360	4	10	18	10	73
11	660	5	26	72	26	241
13	1092	5	27	162	14	421
$16 = 2^4$	4080	5	27	341	205	1681
17	2448	6	46	329	52	1009
19	3420	6	47	441	96	1441
23	6072	7	72	901	231	2641
$25 = 5^2$	7800	7	71	618	326	3001
$27 = 3^3$	9828	5	28	542	118	3862
29	12180	7	74	1578	146	5461
31	14880	8	107	1897	311	6721
$32 = 2^5$	32736	5	28	2370	1241	14881
37	25308	7	74	4302	704	11629
41	34440	9	151	4385	411	15961
43	39732	7	75	5517	474	18481
47	51888	9	151	8443	1082	24289
$49 = 7^2$	58880	10	175	4247	1471	25873
$64 = 2^6$	262080	9	135	13332	11761	124993
$81 = 3^4$	265680	9	122	11672	15499	123121

Table 4.3 Enumeration of actions for selected $PSL_2(q)$

Example 28. *Let us determine $(2,3,n)$, $(2,4,n)$, $(2,6,n)$, and $(3,3,n)$ actions. The discussion that follows may need small adjustments for even q. Using the table* *in the proof of Proposition 15, the trace triples (α, β, γ) may be assumed to have the following form by selecting an appropriate lift orbit representative. In the last row we assume that $n \neq p$.*

(l, m, n)	$(2, 3, n)$	$(2, 4, n)$	$(2, 6, n)$	$(3, 3, n)$
condition on n	$n \geq 7$	$n \geq 5$	$n \geq 4$	$n \geq 4$
(α, β, γ)	$(0, 1, \gamma)$	$(0, \sqrt{2}, \gamma)$	$(0, \sqrt{3}, \gamma)$	$(1, 1, \gamma)$
$QD(\alpha, \beta, \gamma)$	$\gamma^2 - 3$	$\gamma^2 - 2$	$\gamma^2 - 1$	$(\gamma - 2)(\gamma + 1)$
#projective action classes	$\frac{\phi(n)}{2e}$	$\frac{\phi(n)}{2e}$	$\frac{\phi(n)}{2e}$	$\frac{\phi(n)}{e}$

There are no singular triples that yield hyperbolic signatures. If $n = p$, then $QD(\alpha, \beta, \gamma) \neq 0$ except in the case $(3, 3, p)$. There is exactly one automorphism class in each case. The split nature of the $(3, 3, p)$ actions was noted in the $PSL_2(7)$ discussion.

Now suppose that $n \neq p$, so that c is semi-simple. Recall that $PSL_2(q)$ has a semi-simple element of order n iff $q^2 = 1 \mod 2n$. There are $\phi(n)$ possible traces for elements of that project to elements of order n. However, the trace triple sets $Tr(0, \beta, \gamma)$ and $Tr(0, \beta, -\gamma)$ project to the same projective class, whereas $Tr(1, 1, \gamma)$ and $Tr(1, 1, -\gamma)$ project to different projective classes. Thus we get $\phi(n)/2$ projectively inequivalent actions in the first three cases and $\phi(n)$ projectively inequivalent $(3, 3, n)$ actions. Furthermore, all of these traces belong to the same minimal field $\mathbb{F}_{p^e}$. So all the actions occur only for $PSL_2(p^e)$. After accounting for the Galois action we obtain the number of actions listed in the table. This example extends the well known result that $PSL_2(p)$ has three inequivalent Hurwitz actions when $p = \pm 1 \bmod 7$ and exactly $PSL_2(p^3)$ action when $p \neq \pm 1 \bmod 7$.

5. Galois action on $PSL_2(q)$ dessins

5.1. **The Galois action in the general case.** Given a quasi-platonic G action on a surface S there is a projection $\pi_G : S \to S/G = \widehat{\mathbb{C}}$ which we may assume is branched over $\{0, 1, \infty\}$. Any such map $\beta : S \to \widehat{\mathbb{C}}$ branched over $\{0, 1, \infty\}$ is called a Belyi function; it is called regular if $\beta = \pi_G$ is induced by a group action of some group G. According to Belyi's theorem, [1], S can be defined over a number field as long as there is a Belyi function of any type. The intersection of all such defining fields is called the *moduli field* of S. It can be shown that S has a defining equation over its moduli field (see [11]), and we shall assume that S is defined over its moduli field for the remainder of the paper.

If $\psi \in \mathrm{Gal}(\mathbb{C})$, then we define S^ψ to be the Riemann surface obtained by applying ψ to the coefficients of the defining equation(s) of S. There is an induced map, still denoted ψ, $\psi : S \to S^\psi$ by applying ψ coordinatewise. The map is a bijection, but definitely not a morphism. Since S is defined over a number field, the surface S^ψ only depends on the action of ψ on the algebraic closure of $\mathbb{Q}$. For any two affine or projective varieties X, Y and map $f : X \to Y$, we define X^ψ, Y^ψ, $\psi : X \to X^\psi$, and $\psi : Y \to Y^\psi$ in a similar fashion. The map $f^\psi : X^\psi \to Y^\psi$ is defined by $f^\psi(\psi(x)) = \psi(f(x)), x \in X$ or $f^\psi = \psi f \psi^{-1}$. Consequently, for every automorphism g of S, g^ψ is an automorphism of S^ψ, and $g \to \psi g \psi^{-1}$ is an isomorphism of $\mathrm{Aut}(S)$ to $\mathrm{Aut}(S^\psi)$. If $\epsilon : G \to \mathrm{Aut}(S)$ defines the G-action, then $\epsilon^\psi : g \to \epsilon(g)^\psi$ is a G-action on S^ψ, the quotient map $\pi_G^\psi : S^\psi \to S^\psi/G$, is branched over $\{0, 1, \infty\}$ and the following diagram commutes.

$$\begin{array}{ccc} S & \stackrel{\psi}{\to} & S^{\psi} \\ \downarrow \pi_G & & \downarrow \pi_G^{\psi} \\ \widehat{\mathbb{C}} & \stackrel{\psi}{\to} & \widehat{\mathbb{C}} \end{array} \tag{37}$$

Observe that ψ fixes $0, 1, \infty \in \widehat{\mathbb{C}}$ so that ψ maps the G ramification points on S to those on S^{ψ}. Specifically,

$$\begin{aligned} \pi_G^{-1}(0) &\stackrel{\psi}{\to} \left(\pi_G^{\psi}\right)^{-1}(0) \\ \pi_G^{-1}(1) &\stackrel{\psi}{\to} \left(\pi_G^{\psi}\right)^{-1}(1) \\ \pi_G^{-1}(\infty) &\stackrel{\psi}{\to} \left(\pi_G^{\psi}\right)^{-1}(\infty) \end{aligned} \tag{38}$$

are bijections. We will call ϵ^{ψ} the ψ Galois transform of ϵ.

Remark 29. *The diagram 37 and the equations 38 hold with π_G replaced by any Belyi function β. A new dessin or bipartite graph is created between the isomorphic images $\psi(\beta^{-1}(0))$ and $\psi(\beta^{-1}(1))$ by removing and reconnecting the arcs of the dessin according to the new Belyi function β^{ψ}. The new dessin captures the geometry of S^{ψ}. This approach is needed when there is no group action. We shall phrase everything in terms of the group actions and tilings on S and S^{ψ}. There is always a cover $S' \to S$ which carries a regular dessin.*

Without actually knowing the equations of S and S^{ψ}, we can determine the action of ψ on rotation numbers, information we shall use shortly. Let a non-trivial automorphism $g \in \mathrm{Aut}(S)$ fix the point $x_0 \in S$ and let f be any function that vanishes at x_0 to order 1. Then, $f \circ g = \mathrm{rot}(g, x_0) f + k$ where k vanishes at x_0 with order 2 or greater. Now apply ψ to get

$$g^{\psi}(\psi(x_0)) = \psi g \psi^{-1}(\psi(x_0)) = \psi(x_0),$$

and

$$f^{\psi} \circ g^{\psi} = \psi\left(\mathrm{rot}(g, x_0)\right) f^{\psi} + k^{\psi},$$

and k^{ψ} vanishes at $\psi(x_0)$ with order 2 or greater. We see that g^{ψ} fixes $\psi(x_0)$ and

$$\mathrm{rot}(g^{\psi}, \psi(x_0)) = \psi(\mathrm{rot}(g, x_0)). \tag{39}$$

Now let $N > 1$ be any integer and $\zeta = \exp(2\pi i/N)$. The cyclotomic field $\mathbb{Q}[\zeta]$ is a normal subfield of $\mathbb{C}$, and the action of ψ on $\mathbb{Q}[\zeta]$ is given by $\zeta \to \zeta^s$ for some number s relatively prime to N. Applying equation 39 to $h = \epsilon(g)$ we get

$$\mathrm{rot}(\epsilon^{\psi}(g), \psi(x_0)) = (\mathrm{rot}(\epsilon(g), x_0))^s. \tag{40}$$

Now we work out the Galois action on epimorphisms. There are epimorphisms

$$\begin{aligned} \eta_1 : T_{l,m,n} \to G,\ A \to a_1,\ B \to b_1,\ C \to c_1 \\ \eta_2 : T_{l,m,n} \to G,\ A \to a_2,\ B \to b_2,\ C \to c_2, \end{aligned}$$

such that η_1 uniformizes the ϵ action on S, with generating vector (a_1, b_1, c_1); and η_2 uniformizes the ϵ^{ψ} action on S^{ψ} with generating vector (a_2, b_2, c_2). Following the discussion in Section 2; let $\triangle\overline{DEF}$ be a triangle in S determining the triple (a_1, b_1, c_1) and let $\triangle\overline{D'E'F'}$ be a triangle in S^{ψ} determining (a_2, b_2, c_2). Since

$G\cdot\overline{D}=\pi_G^{-1}(0)$ and $G\cdot\overline{D}'=\left(\pi_G^{\psi}\right)^{-1}(0)$, (equation 38) then there is a $u\in G$, so that $\psi(u\overline{D})=\overline{D'}$, and the stabilizer of $\overline{D}'$ is $u\langle a_1\rangle u^{-1}$. To determine the element $a_2\in u\langle a_1\rangle u^{-1}$ we work with rotation numbers. To this end, let $N=\operatorname{lcm}(l,m,n)$ and $\zeta=\exp(2\pi i/N)$ as above. Then the cyclotomic field $\mathbb{Q}[\zeta]$ contains the rotation numbers of a_1,b_1,c_1. Let t be an integer such that $st=1 \bmod N$. Then

$$\begin{aligned}\operatorname{rot}(\epsilon^{\psi}(ua_1^tu^{-1}),\overline{D'}) &= (\operatorname{rot}(\epsilon(ua_1^tu^{-1}),u\overline{D}))^s\\ &=(\operatorname{rot}(\epsilon(a_1),\overline{D}))^{st}\\ &=\exp\left(\frac{2\pi i}{l}\right);\end{aligned}$$

and similarly $\operatorname{rot}(\epsilon^{\psi}(vb_1^tv^{-1}),\overline{E'})=\exp\left(\frac{2\pi i}{m}\right)$, $\operatorname{rot}(\epsilon^{\psi}(wc_1^tw^{-1}),\overline{E'})=\exp\left(\frac{2\pi i}{m}\right)$. It follows that $(a_2,b_2,c_2)=(ua_1^tu^{-1},vb_1^tv^{-1},wc_1^tw^{-1})$. We summarize the preceding discussion by the following theorem. It is known as the branch cycle argument and a proof is given as Lemma 2.8 in [21].

Theorem 30. *Let $\eta_1:T_{l,m,n}\to G,\ A\to a_1,\ B\to b_1,\ C\to c_1$ be an epimorphism defining an (l,m,n) G-action on the Riemann surface S. Let $\psi\in\operatorname{Gal}(\mathbb{C})$, $N=\operatorname{lcm}(l,m,n)$, $\zeta=\exp(2\pi i/N)$, and suppose that $\psi(\zeta)=\zeta^s$. Select t so that $st=1 \bmod N$. Then the ϵ^{ψ}, the ψ Galois transform action on S^{ψ}, is induced by $\eta_2: T_{l,m,n}\to G,\ A\to a_2,\ B\to b_2,\ C\to c_2$, such that*

$$(a_2,b_2,c_2)=(ua_1^tu^{-1},vb_1^tv^{-1},wc_1^tw^{-1}) \tag{41}$$

for some $u,v,w\in G$. Moreover, $a_2b_2c_2=1$ and $G=\langle a_2,b_2,c_2\rangle$.

Remark 31. *As discussed in Section 2,we have $(a_1,b_1,c_1)\in K_G^{\circ}(a_1,b_1,c_1)$ and $(a_2,b_2,c_2)\in K_G^{\circ}(a_1^t,b_1^t,c_1^t)$. We shall see below that both $K_G^{\circ}(a_1,b_1,c_1)$ and also $K_G^{\circ}(a_1^t,b_1^t,c_1^t)$ have exactly the same number of elements. However, because of the possible presence of companion orbits in $K_G^{\circ}(a_1^t,b_1^t,c_1^t)$ the action determined by equation 41 may not be uniquely identifiable in $K_G^{\circ}(a_1^t,b_1^t,c_1^t)$. So we shall call $K_G^{\circ}(a_1^t,b_1^t,c_1^t)$ a* Galois $t-$target. *We can use covers of $SL_2(q)$ to resolve the indeterminacy.*

There is no simple formula for (a_2,b_2,c_2) known to the author and the triple needs to be found computationally. Indeed, it is not immediately apparent that the Galois t-target $K_G^{\circ}(a_1^t,b_1^t,c_1^t)$ is non-empty from simple group theoretic considerations. We demonstrate that $K_G^{\circ}(a_1^t,b_1^t,c_1^t)$ is non-empty in a non-constructive way using character theory. The field automorphism $\psi^{-1}\in\operatorname{Gal}(\mathbb{C})$ act by $\zeta\to\zeta^t$ on the primitive Nth roots of unity. For any representation $\rho:G\to GL_k(\mathbb{C})$ the eigenvalues of the matrices $\rho(a),\rho(b),\rho(c)$ are Nth roots of unity, and, so if χ is the character of ρ then.

$$\chi(a^t)=\psi^{-1}(\chi(a)),\chi(b^t)=\psi^{-1}(\chi(b)),\chi(c^t)=\psi^{-1}(\chi(c)).$$

Applying this to equation 21 it follows then that

$$\begin{aligned}\left|K_G(a^t,b^t,c^t)\right| &= \frac{|G|^2}{|\mathrm{Cent}(a^t)|\cdot|\mathrm{Cent}(b^t)|\cdot|\mathrm{Cent}(c^t)|}\sum_\chi \frac{\chi(a^t)\chi(b^t)\chi(c^t)}{\chi(1)}\\ &= \frac{|G|^2}{|\mathrm{Cent}(a)|\cdot|\mathrm{Cent}(b)|\cdot|\mathrm{Cent}(c)|}\sum_\chi \frac{\psi^{-1}\left(\chi(a)\chi(b)\chi(c)\right)}{\chi(1)}\\ &= \psi^{-1}\left(\frac{|G|^2}{|\mathrm{Cent}(a)|\cdot|\mathrm{Cent}(b)|\cdot|\mathrm{Cent}(c)|}\sum_\chi \frac{\chi(a)\chi(b)\chi(c)}{\chi(1)}\right)\\ &= \psi^{-1}\left(|K_G(a,b,c)|\right) = |K_G(a,b,c)|\,.\end{aligned}$$

Using formulas similar to equation 18 we deduce

$$\left|K_G^\circ(a^t,b^t,c^t)\right| = |K_G^\circ(a,b,c)|\,. \tag{42}$$

Separating companion actions with a Schur cover. Now we use lifts to a Schur cover discussed in Section 2.4 to resolve Galois t-target indeterminacy. Let $\widetilde{S}$ be a surface upon which $\widetilde{G}$ acts with signature $(\widetilde{l},\widetilde{m},\widetilde{n})$, and generating vector $(\widetilde{a},\widetilde{b},\widetilde{c})$. Then $G=\widetilde{G}/Z$ has a natural (l,m,n) action on $S=\widetilde{S}/Z$ with generating vector (a,b,c). We have the following diagram

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\psi} & \widetilde{S}^\psi \\ \downarrow \pi_Z & & \downarrow \pi_Z^\psi \\ S & \xrightarrow{\psi} & S^\psi \\ \downarrow \pi_G & & \downarrow \pi_G^\psi \\ \widehat{\mathbb{C}} & \xrightarrow{\psi} & \widehat{\mathbb{C}} \end{array} \tag{43}$$

where the composite maps on the left and right columns are $\pi_{\widetilde{G}}$ and $\pi_{\widetilde{G}}^\psi$.

Remark 32. *The map π_Z is a $|Z|$-fold branched cover of the $\widetilde{G}$ dessin to the G dessin. The cover is 1-1 on arcs and has ramification degrees $\widetilde{l}/l$, $\widetilde{m}/m$ and $\widetilde{n}/n$ over $\pi_G^{-1}(0)$, $\pi_G^{-1}(1), \pi_G^{-1}(\infty)$ respectively. Once this is understood combinatorially, the two dessins can be completed to surfaces by gluing in the appropriate polygons.*

Now suppose we are lucky enough to have $K_G^\circ(\widetilde{a},\widetilde{b},\widetilde{c})$ be a single $\widetilde{G}$ orbit. Define $\widetilde{N}$, $\widetilde{s},\widetilde{t}$ as in Theorem 30. Then the Galois transform of $(\widetilde{a},\widetilde{b},\widetilde{c})$ is

$$\left(\widetilde{u}\widetilde{a}^{\widetilde{t}}\widetilde{u}^{-1}, \widetilde{v}\widetilde{b}^{\widetilde{t}}\widetilde{v}^{-1}, \widetilde{w}\widetilde{c}^{\widetilde{t}}\widetilde{w}^{-1}\right)$$

for suitable elements. By equation 42, the Galois t target has a unique $\widetilde{G}$ orbit. We now just apply π to $\left(\widetilde{u}\widetilde{a}^{\widetilde{t}}\widetilde{u}^{-1}, \widetilde{v}\widetilde{b}^{\widetilde{t}}\widetilde{v}^{-1}, \widetilde{w}\widetilde{c}^{\widetilde{t}}\widetilde{w}^{-1}\right)$ to find the action of a unique class for $(ua^tu^{-1}, vb^tv^{-1}, wc^tw^{-1})$. We can now state a theorem for computing the Galois action on $PSL_2(q)$ quasi-platonic actions.

Theorem 33. *Let ϵ be a quasi-platonic action of $PSL_2(q)$ determined by a generating (l,m,n) triple $(a_1,b_1,c_1)$and $\psi\in \mathrm{Gal}(\mathbb{C})$. Then the Galois transform ϵ^ψ with generating vector (a_2,b_2,c_2) may be determined as follows.*

1. *Select a covering triple $(\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_1)$ in $SL_2(q)$ and the corresponding action $\widetilde{\epsilon}$.*
2. *Determine the ψ transform $(\widetilde{a}_2,\widetilde{b}_2,\widetilde{c}_2)$ of $(\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_1)$ by Theorem 30.*

(3) *Project the ψ transform $(\widetilde{a}_2, \widetilde{b}_2, \widetilde{c}_2)$ by π*

$$(a_2, b_2, c_2) = \pi(\widetilde{a}_2, \widetilde{b}_2, \widetilde{c}_2).$$

5.2. **Galois action examples for $PSL_2(q)$.** As in the classification of actions we will just give examples instead of a comprehensive theorem. For the $PSL_2(7)$, $PSL_2(8)$ we the use the same table format as Tables 4.1 and 4.2 except the last column now contains the list of Galois orbit sizes. Every orbit of size one corresponds to a surface with rational coefficients.

(l, m, n)	(α, β, γ)	genus	Orbit Sizes
$(2, 3, 7)$	$(0, 1, 2)$	3	$\{1\}$
$(2, 4, 7)$	$(0, 3, 2)$	10	$\{1\}$
$(2, 7, 7)$	$(0, 2, 2)$	19	$\{1\}$
$(3, 3, 4)$	$(1, 1, 3), (1, 1, 4)$	8	$\{2\}$
$(3, 3, 7)$	$(1, 1, -2)$	17	$\{1\}$
$(3, 4, 4)$	$(-1, 3, 3)$	15	$\{1\}$
$(3, 4, 7)$	$(1, 3, 2), (1, 3, -2)$	24	$\{1\}$
$(3, 7, 7)$	$(1, 2, 2), (-1, 2, 2)$	33	$\{1, 1\}$
$(4, 4, 4)$	$(3, 3, 3), (3, 3, 4)$	22	$\{2\}$
$(4, 4, 7)$	$(3, 3, -2)$	31	$\{1\}$
$(4, 7, 7)$	$(3, 2, 2), (-3, 2, 2)$	40	$\{2\}$
$(7, 7, 7)$	$(2, 2, -2)$	49	$\{1\}$

Table 5.1 Galois action on $PSL_2(7)$ actions

(l, m, n)	(α, β, γ)	genus	Orbit Sizes
$(2, 3, 7)$	$(0, 1, w^3)$	7	$\{1\}$
$(2, 3, 9)$	$(0, 1, w^2)$	15	$\{1\}$
$(2, 7, 7)$	$(0, w^3, w^5), (0, w^3, w^6)$	55	$\{1, 1\}$
$(2, 7, 9)$	$(0, w^3, w), (0, w^3, w^2), (0, w^3, w^4)$	63	$\{3\}$
$(2, 9, 9)$	$(0, w, w^2), (0, w, w^4)$	71	$\{1, 1\}$
$(3, 3, 7)$	$(1, 1, w^3)$	41	$\{1\}$
$(3, 3, 9)$	$(1, 1, w)$	57	$\{1\}$
$(3, 7, 7)$	$(1, w^3, w^3), (1, w^3, w^5), (1, w^3, w^6)$	97	$\{3\}$
$(3, 7, 9)$	$(1, w^3, w), (1, w^3, w^2), (1, w^3, w^4)$	105	$\{3\}$
$(3, 9, 9)$	$(1, w, w), (1, w^2, w^2), (1, w^4, w^4)$	113	$\{1\}$
$(7, 7, 7)$	$(w^3, w^3, w^5), (w^3, w^3, w^6),$ $(w^3, w^5, w^3), (w^3, w^6, w^3)$	145	$\{1, 1, 1, 1\}$
$(7, 7, 9)$	$(w^a, w^b, w), a, b = 3, 5, 6$	153	$\{3, 3, 3\}$
$(7, 9, 9)$	$(w^3, w^a, w^b), a, b = 1, 2, 4$	161	$\{3, 3, 3\}$
$(9, 9, 9)$	$(w, w, w), (w, w, w^4), (w, w^2, w^2)$ $(w, w^2, w^4), (w, w^4, w), (w, w^4, w^2)$	169	$\{1, 1, 1, 1, 1, 1\}$

Table 5.2 Galois action on $PSL_2(8)$ actions

Before proceeding with the remaining examples, we characterize the size of the orbits of the Galois action.

Proposition 34. *For the Galois action of* $Gal(\mathbb{C})$, *on the hyperbolic* (l, m, n) *actions of* $PSL_2(q)$ *all orbits have the same size.*

Proof: Let (A, B, C) be any lift to $\widetilde{G} = SL_2(q)$ of a generating (l, m, n) triple (a, b, c). Let $(\widetilde{l}, \widetilde{m}, \widetilde{n})$ be the signature of (A, B, C) and set $N = \mathrm{lcm}(l, m, n)$ and $\widetilde{N} = \mathrm{lcm}(\widetilde{l}, \widetilde{m}, \widetilde{n})$. Consider the set $X^{\circ}_{\widetilde{G}}(\widetilde{l}, \widetilde{m}, \widetilde{n})$. It can be partitioned into disjoint projective classes

$$X^{\circ}_{\widetilde{G}}(\widetilde{l}, \widetilde{m}, \widetilde{n}) = \bigcup_{(A',B',C')} L^{\circ}_{\widetilde{G}}(A', B', C')$$

where $\left|L^{\circ}_{\widetilde{G}}(A', B', C')\right| = |PGL_2(q)|$ for various $(\widetilde{l}, \widetilde{m}, \widetilde{n})$ triples (A', B', C'). Two elements $U, V \in SL_2(q)$ have the same order if an only if they are power conjugate, $V = WU^tW^{-1}$ for some $W \in GL_2(q)$. Thus

$$(A', B', C') = (W_1A^{t_1}W_2^{-1}, W_2B^{t_2}W_2^{-1}, W_3C^{t_3}W_3^{-1}),$$

for some selection of t_i and W_i. By definition

$$L^{\circ}_{\widetilde{G}}(A', B', C') = L^{\circ}_{\widetilde{G}}(W_1A^{t_1}W_2^{-1}, W_2B^{t_2}W_2^{-1}, W_3C^{t_3}W_3^{-1}) = L^{\circ}_{\widetilde{G}}(A^{t_1}, B^{t_2}, C^{t_3}).$$

Then we see that the abelian group $\left(\mathbb{Z}^*_{\widetilde{N}}\right)^3$ acts on the $PGL_2(q)$ classes in $X^{\circ}_{\widetilde{G}}(\widetilde{l}, \widetilde{m}, \widetilde{n})$ by $L^{\circ}_{\widetilde{G}}(A, B, C) \to L^{\circ}_{\widetilde{G}}(A^{t_1}, B^{t_2}, C^{t_3})$. The action has a kernel containing $\langle\pm 1\rangle^3$. The action of the absolute Galois group is simply the diagonal action of $\mathbb{Z}^*_{\widetilde{N}}$ on $X^{\circ}_{\widetilde{G}}(\widetilde{l}, \widetilde{m}, \widetilde{n})/PGL_2(q)$, namely $L^{\circ}_{\widetilde{G}}(A, B, C) \to L^{\circ}_{\widetilde{G}}(A^t, B^t, C^t)$. Since $\left(\mathbb{Z}^*_{\widetilde{N}}\right)^3$ acts transitively, then the Galois orbit space of $X^{\circ}_{\widetilde{G}}(\widetilde{l}, \widetilde{m}, \widetilde{n})/PGL_2(q)$ is the coset space of an appropriate homomorphic image of the pair $\left(\left(\mathbb{Z}^*_{\widetilde{N}}\right)^3, \mathbb{Z}^*_{\widetilde{N}}\right)$. The structure is independent of the type of lift chosen. The Galois orbit space structure maps $1-1$ onto a subset of the corresponding orbit space structure of $X^{\circ}_{G}(l, m, n)/PGL_2(q)$. ∎

Example 35. *We consider again* $(2, 3, n)$, $(2, 4, n)$, $(2, 6, n)$, *and* $(3, 3, n)$ *actions, with* $q \geq 7$.

(1) *Any* q, $n = p$. *There is only one action as previously discussed.*
(2) *Odd* q, $l = 2$, $n \neq p$. *There are* $\phi(n)/2$ *projective classes of triples - note that* $L^{\circ}_{\widetilde{G}}(A, B, C)$ *and* $L^{\circ}_{\widetilde{G}}(A^{-1}, B^{-1}, C^{-1})$ *are projectively equivalent. The Galois action consists of a single orbit of size* $\phi(n)/2$. *If* $q = p^e$ *is composite, we must further divide by the faithful action of* $Gal(\mathbb{F}_q)$.
(3) *Odd* q, $n \neq p$. *For the* $(3, 3, n)$ *signature we get two Galois orbits of size* $\phi(n)/2$. *The same remarks as above apply for composite* q.
(4) *Even* q, $n \neq p$. *There is a single Galois orbit with* $\phi(n)/2$ *projectively inequivalent actions.*

Example 36. *For* $PSL_2(47)$ *there are* 2431 $(23, 23, 23)$ *actions consisting of* 121 *Galois orbits each of size* 11. *For* $PSL_2(32)$ *there are* 2940 *projective classes of actions in* 196 *Galois orbits of size* 15 *each. Each Galois orbit provides* $3 = 15/5$ *inequivalent actions.*

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