ALBANIAN JOURNAL
OF MATHEMATICS
Volume 5, Number 2, Pages 87–94
ISSN 1930-1235: (2011)

ON γ-SEMI-θ-CLOSED SETS AND LOCALLY γ-S-REGULAR SPACES

Sabir Hussain
Department of Mathematics,
Islamia University
Bahawalpur, Pakistan.
Email: sabiriub@yahoo.com

Bashir Ahmad
Department of Mathematics,
King Abdul Aziz University,
PO Box 80203, Jeddah 21589, Saudi Arabia.
Email: drbashir9@gmail.com

Abstract. We continue studying the properties and characterizations of γ-semi-θ-closed sets, γ-s-closed relative to a space X and locally γ-s-regular spaces defined and discussed by B. Ahmad and S. Hussain in 2008 and S. Hussain and B. Ahmad in 2007 and 2009.

1. Introduction

In 1969, Signal and Arya [22] defined a new separation axiom called almost regularity which is weaker than regularity. It has been shown in [17], that for Hausdorff spaces, this axiom occupies a position between Urysohn's separation axiom and T_3-axiom. Maheswari and Prasad [18] have defined another axiom called s-regularity which is weaker than regularity (without T_2). In 1982, C. Dorsett [8] defined and investigated a new separation axiom called semi-regular space. It is shown that s-regularity is weaker than semi-regularity. In 1979, S. Kasahara [14] defined an operation α on topological spaces. B. Ahmad and M. Khan [7] defined and study locally s-regular spaces. It is interesting to mention that class of s-regular spaces is a proper subclass of locally s-regular spaces. In 1992 (1993) , B. Ahmad and F. U. Rehman [1] [21] introduced the notions of γ-interior, γ-boundary and γ-exterior points in topological spaces. B. Ahmad and S. Hussain further studied the properties of γ-operations in topological spaces in [2] [3]. Recently S. Hussain, B. Ahmad and T. Noiri [12] introduced and discussed γ-semi-open sets in topological spaces. Further B. Ahmad and S. Hussain [5] explored the characterizations of γ-semi-open (closed), γ-semi-closure (interior) and γ-semi-continuous functions. They defined and discussed γ-s-closed spaces and subspaces([4], [10]) using γ-semi-closure. It is

2010 *Mathematics Subject Classification.* 54A05, 54A10, 54D10.
Key words and phrases. γ-closed(open), γ-closure, γ-regular(open), γ^*-semi-closed (open), γ-semi-closure(interior), γ-s-regular space, γ-semi-θ-closed (open), γ-s-closed relative, γ-semi-θ-perfect, γ-regular-open(closed).

known [10] that the concept of γ-s-closed space is a generalization of s-closed space [16]. They defined and discussed the characterizations and properties of γ-semi-θ-closed sets [10], γ-s-closed relative to a space X [4] and Locally γ-s-Regular spaces [4].

In this paper, we continue studying the properties and characterizations of γ-semi-θ-closed sets, γ-s-closed relative to a space X and locally γ-s-regular spaces defined and discussed in [4], [10], [11].

2. Preliminaries

Hereafter X will be represented as a topological space and we shall write a space in place of a topological space for our convenience.

Now we recall some notions defined in [5],[11], [12], [14] and [15]. In [14] an operation $\gamma : \tau \to$ P(X) is defined as a function from τ to the power set of X such that V $\subseteq V^{\gamma}$, for each V $\in \tau$, where V^{γ} denotes the value of γ at V. The operations defined by γ(G) = G, γ(G) = cl(G) and γ(G) = intcl(G) are examples of operation γ. Let A be a subset of space X . A point $x \in A$ is said to be γ-interior point [15] of A, if there exists an open nbd N of x such that $N^{\gamma} \subseteq$ A. We denote the set of all such points by int_{γ}(A). Thus

$$int_{\gamma}\ (\mathrm{A}) = \{x \in A : x \in N \in \tau \text{ and } N^{\gamma} \subseteq A\}.$$

Note that A is γ-open [15] iff A $=int_{\gamma}$(A). A is called γ- closed [21] iff $X - A$ is γ-open. A point x $\in$ X is called a γ-closure point [15] of A , if $U^{\gamma} \cap A \neq \phi$, for each open nbd U of x. The set of all γ-closure points of A is called γ-closure of A and is denoted by $cl_{\gamma}(A)$. A is called γ-closed, if $cl_{\gamma}(A) \subseteq$A. Note that cl_{γ}(A) is contained in every γ-closed superset of A. An operation γ on τ is said be regular [15], if for any open nbds U, V of $x \in X$, there exists an open nbd W of x such that $U^{\gamma} \cap V^{\gamma} \supseteq W^{\gamma}$. An operation γ on τ is said to be open [15], if for every nbd U of each $x \in X$, there exists γ-open set B such that $x \in B$ and $U^{\gamma} \subseteq B$. A subset A of a space X is said to be a γ-semi-open set [12], if there exists a γ-open set O such that $O \subseteq A \subseteq cl_{\gamma}(O)$. The set of all γ-semi-open sets is denoted by $SO_{\gamma}(X)$. A is γ-semi-closed iff $X - A$ is γ-semi-open in X. Note that A is γ-semi-closed iff $int_{\gamma}(cl_{\gamma}(A)) \subseteq A$. The intersection of all γ-semi-closed sets containing A is called γ-semi-closure [5] of A and is denoted by $scl_{\gamma}(A)$. Note that A is γ-semi-closed iff $scl_{\gamma}(A) = A$. The union of γ-semi-open subsets of A is called γ-semi-interior [5] of A and is denoted by $sint_{\gamma}(A)$. A is γ-semi-regular [5], if it is both γ-semi-open and γ-semi-closed. The class of all γ-semi-regular sets of X is denoted by $SR_{\gamma}(A)$. Note that if γ is a regular operation, then the union of γ-semi-regular sets is γ-semi-regular. A space X is said to be γ-s-regular [11], if for any γ-semi-regular set A and $x \notin A$, there exist disjoint γ-open sets U and V such that $A \subseteq U$ and $x \in V$.

3. γ-semi-θ-closed Sets.

In [10] a point x of a space X is defined as the γ-semi-θ-closure point of a subset A of X, if $A \cap scl_{\gamma}(U) \neq \phi$, for every $U \in SO_{\gamma}(X)$ containing x. The set of all γ-semi-θ-closure points of A is called γ-semi-θ-closure of A and is denoted by $s_{\gamma}cl_{\theta}(A)$. A subset A is said to be γ-semi-θ-closed, if $A = s_{\gamma}cl_{\theta}(A)$. The complement of a γ-semi-θ-closed set is said to be γ-semi-θ-open.

Now we define:

Definition 3.1. *A function $f : X \to Y$ is said to be a γ-semi-θ-closed, if $f(K)$ is γ-semi-θ-closed in Y, for every γ-semi-θ-closed set K of X.*

Theorem 3.2. *A function $f : X \to Y$ is γ-semi-θ-closed iff $s_\gamma cl_\theta(f(A)) \subseteq f(s_\gamma cl_\theta(A))$, for every subset A of X, where γ is open.*

Proof. Necessity: let f be γ-semi-θ-closed and A any subset of X. Then $f(s_\gamma cl_\theta(A))$ is γ-semi-θ-closed. But $f(A) \subseteq f(s_\gamma cl_\theta(A))$ implies that $s_\gamma cl_\theta(f(A)) \subseteq f(s_\gamma cl_\theta(A))$ [10].

Sufficiency: let $s_\gamma cl_\theta(f(A)) \subseteq f(s_\gamma cl_\theta(A))$, for every subset A of X. Let B be a γ-semi-θ-closed set of X. Then $s_\gamma cl_\theta(f(B)) \subseteq f(s_\gamma cl_\theta(B)) = f(B)$. But $f(B) \subseteq s_\gamma cl_\theta(f(B))$. This proves that $s_\gamma cl_\theta(f(B)) = f(B)$. This gives that $f(B)$ is γ-semi-θ-closed. Hence the proof. □

Theorem 3.3. *Let γ be an open operation. Then the following are equivalent for a function $f : X \to Y$:*

(1) f is γ-semi-θ-closed.

(2) $s_\gamma cl_\theta(f(A)) \subseteq f(s_\gamma cl_\theta(A))$, for every subset A of X.

(3) For every subset B of Y and every γ-semi-θ-open set U of X containing $f^{-1}(B)$, there exists a γ-semi-θ-open set V of Y containing B such that $f^{-1}(V) \subseteq U$.

(4) For every point $y \in Y$ and every γ-semi-θ-open set U of X containing $f^{-1}(y)$, there exists a γ-semi-θ-open set V of Y containing y such that $f^{-1}(V) \subseteq U$.

Proof. The proof is similar to that of Theorem 5.2 [6] and is thus omitted. □

Recall [Lemma 1 [4]] that a subset A of a space X is γ-s-closed relative to X iff every cover of A by γ-semi-θ-open sets of X has a finite subcover.

Theorem 3.4. *Let $f : X \to Y$ be a γ-semi-θ-closed function such that $f^{-1}(y)$ is γ-s-closed relative to X, for each point y of Y. If K is γ-s-closed relative to Y, then $f^{-1}(K)$ is γ-s-closed relative to X, where γ is a regular operation.*

Proof. Let $\{U_\alpha : \alpha \in I\}$ be any cover of $f^{-1}(K)$ by γ-semi-θ-open sets of X. For each $y \in K$, $f^{-1}(y)$ is γ-s-closed relative to X and by Lemma 1 [4], there exists a finite subset $I(y)$ of I such that $f^{-1}(y) \subseteq \cup\{U_\alpha : \alpha \in I(y)\}$. Let $U(y) = \cup\{U_\alpha : \alpha \in I(y)\}$, then U(y) is γ-semi-θ-open in X [4]. Since f is γ-semi-θ-closed, by Theorem 3.4, there exists a γ-semi-θ-open set V(y) containing y such that $f^{-1}(V(y)) \subseteq U(y)$, since $\{V(y) : y \in K\}$ is a γ-semi-θ-open cover of K. By Lemma 1 [4], there exists a finite subset K_0 of K such that $K \subseteq \cup\{V(y) : y \in K_0\}$. Therefore, we obtain

$$f^{-1}(K) \subseteq \cup\{f^{-1}(V(y)) : y \in K_0\} \subseteq \cup\{U_\alpha(y) : \alpha \in I(y), y \in K_0\}.$$

This shows that $f^{-1}(K)$ is γ-s-closed relative to X. This completes the proof. □

Corollary 1. *Let $f : X \to Y$ be a γ-semi-θ-closed surjection such that $f^{-1}(y)$ is γ-s-closed relative to X for each point $y \in Y$. If Y is γ-s-closed, then X is γ-s-closed, where γ is a regular operation.*

In [4], we defined that a filterbase $\Im$ on X is said to γ-SR-converge to $x \in X$, if for each $V \in SR_\gamma(X)$, there exists $F \in \Im$ such that $F \subseteq V$.

Definition 3.5. *A point $x \in X$ is called γ-semi-θ-adherent point of a filterbase $\Im$ in X, if $x \in [s_\gamma ad]_\theta(\Im) = \bigcap\{s_\gamma cl_\theta(F) : F \in \Im\}$.*

Definition 3.6. *A filterbase $\Im$ is said to be γ-semi-θ-directed towards $S \subseteq X$, if every filterbase subordinate to $\Im$ has a γ-semi-θ-adherent point in S.*

Definition 3.7. *A function $f : X \to Y$ is said to be γ-semi-θ-perfect, if for every filterbase $\Im$ in f(X) γ-SR-converges to $y \in Y$, $f^{-1}(\Im)$ is γ-semi-θ-directed towards $f^{-1}(y)$.*

Theorem 3.8. *Every γ-semi-θ-perfect function is γ-semi-θ-closed, where γ is open.*

Proof. Let $f : X \to Y$ be a γ-semi-θ-perfect function and A any subset of X. Let $y \in s_\gamma cl_\theta(f(A))$. Then there exists a filterbase $\Im$ on $f(A)$ which γ-SR-convergrs to y. Put $\xi = \{f^{-1}(F) \cap A : F \in \Im\}$. Then ξ is a γ-filterbase on X subordinate to the filterbase $f^{-1}(\Im)$. Since $f^{-1}(\Im)$ is γ-semi-θ-directed towards $f^{-1}(y)$, we have $f^{-1}(y) \cap [s_\gamma ad]_\theta(\xi) \neq \phi$. Therefore, we obtain $y \in f(s_\gamma cl_\theta A)$. By Theorem 3.2, f is γ-semi-θ-closed. Hence the proof. □

Theorem 3.9. *A function $f : X \to Y$ is γ-semi-θ-perfect iff $[s_\gamma ad]_\theta f(\Im) \subset f([s_\gamma ad]_\theta(\Im))$, for every filterbase $\Im$ in X.*

Proof. Necessity: suppose that $f : X \to Y$ is γ-semi-θ-perfect. Let $\Im$ be a filterbase in X and $y \in [s_\gamma ad]_\theta f(\Im)$. Then there exists a filterbase ξ in $f(X)$ which is subordinate to $f(\Im)$ and γ-SR-converges to y. Put $H = \{f^{-1}(G) \cap F : F \in \Im, G \in \xi\}$. Then H is filterbase in X subordinate to $f^{-1}(\xi)$. Since f is γ-semi-θ-perfect, $f^{-1}(\xi)$ is γ-semi-θ-directed towards $f^{-1}(y)$. Therefore we have $f^{-1}(y) \cap [s_\gamma ad]_\theta(H) \neq \phi$ and hence $y \in f([s_\gamma ad]_\theta(\Im))$. This proves that $[s_\gamma ad]_\theta f(\Im) \subset f([s_\gamma ad]_\theta \Im)$. This proves necessity.

Sufficiency: suppose that $[s_\gamma ad]_\theta f(\Im) \subset f([s_\gamma ad]_\theta(\Im))$, for every filterbase $\Im$ in X. We prove that f is γ-semi-θ-perfect. Assume contrary that f is not γ-semi-θ-perfect, then there exists a filterbase $\Im$ in $f(X)$ such that $\Im$ γ-SR-converges to a point $y \in Y$. But $f^{-1}(\Im)$ is not γ-semi-θ-directed towards $f^{-1}(y)$. Thus there exists a filterbase ξ in X which is subordinate to $f^{-1}(\Im)$ and $f^{-1}(y) \cap [s_\gamma ad]_\theta(\xi) = \phi$. Therefore we have $y \notin [s_\gamma ad]_\theta f(\xi)$ and hence $y \notin s_\gamma cl_\theta(f(G_1))$, for some $G_1 \in \xi$. Then there exists a γ-semi-open set V containing y such that $scl_\gamma(V) \cap f(G_1) = \phi$. Since $\Im$ γ-SR-converges to y and ξ is subordinate to $f^{-1}(\Im)$, there exists a $G_2 \in \xi$ such that $f(G_2) \subset scl_\gamma(V)$. Consequently, we obtain $G_1 \cap G_2 = \phi$. This contradicts that ξ is a filterbase. This proves that f is γ-semi-θ-perfect. Hence the proof. □

4. γ-s-closed Relative to a Space.

In [10] a subset A of a space X is defined to be γ-s-closed relative to X, if for every cover of A by γ-semi-open sets of X, there exists a finite subset I_0 of I such that $A \subseteq \bigcup_{\alpha \in I_0} scl_\gamma(V_\alpha)$.

Theorem 4.1. *A subset K of a space X is γ-s-closed relative to X iff $K \cap [s_\gamma ad]_\theta(\Im) \neq \phi$, for every filterbase $\Im$ in K.*

Proof. Necessity: suppose that K is γ-s-closed relative to X. Assume that $\Im$ is a filterbase in K such that $K \cap [s_\gamma ad]_\theta(\Im) = \phi$. Then for each $x \in K$, there exists a γ-semi-open set U_x containing x and $F_x \in \Im$ such that $F_x \cap scl_\gamma(U_x) = \phi$. Since K is γ-s-closed relative to X, there exists a finite number of points $x_1, x_2, ..., x_n$ in K such that $K \subset \bigcup\{scl_\gamma(U_{x_j}) : j = 1, 2, ..., n\}$. Put $F = \bigcap\{F_{x_j} : j = 1, 2, ..., n\}$, then we obtain $F \cap K = \phi$. This contradicts that $\Im$ is a filterbase in K. Therefore $K \cap [s_\gamma ad]_\theta(\Im) \neq \phi$. This completes necessity.

Sufficiency: suppose that $K \cap [s_\gamma ad]_\theta(\Im) \neq \phi$, for every filterbase $\Im$ in K. Assume that K is not γ-s-closed relative to X. Then there exists a cover $\{U_\alpha : \alpha \in I\}$ of K be γ-semi-open sets of X such that $K \nsubseteq \bigcup\{scl_\gamma(U_\alpha) : \alpha \in I_o\}$ for every $I_0 \in J(I)$, where $J(I)$ denotes the family of all finite subsets of I. Put $F_{I_o} = \bigcap\{K - scl_\gamma(U_\alpha) : \alpha \in I_o\}$, for each $I_o \in J(I)$. Then $\Im = \{F_{I_o} : I_o \in J(I)\}$ is a filterbase in K and $K \cap [s_\gamma ad]_\theta(\Im) = \phi$. This is a contradiction. Therefore, K is γ-s-closed relative to X. This completes the proof. □

Theorem 4.2. *If $f : X \to Y$ is a γ-semi-θ-perfect function, then $f^{-1}(K)$ is γ-s-closed relative to X, for every $K \subseteq Y$, γ-s-closed relative to Y. Where γ is open.*

Proof. Let $f : X \to Y$ be a γ-semi-θ-perfect function and $K \subset Y$ γ-s-closed relative to Y. We prove that $f^{-1}(K)$ is γ-s-closed relative to X. Let $\Im$ be a filterbase in X. Then $\xi = \{f(F) \cap K : F \in \Im\}$ is a filterbase in K subordinate to the filterbase $f(\Im)$. By Theorem 4.1, $K \cap [s_\gamma ad]_\theta(\xi) \neq \phi$ and hence we obtain $K \cap [s_\gamma ad]_\theta f(\Im) \neq \phi$. Since f is γ-semi-θ-perfect, therefore by Theorem 3.8, we obtain $K \cap f([s_\gamma ad]_\theta(\Im)) \neq \phi$. This gives $f^{-1}(K) \cap [s_\gamma ad]_\theta(\Im) \neq \phi$. Hence by Theorem 4.1, we conclude that $f^{-1}(K)$ is γ-s-closed relative to X. Hence the proof. □

Theorem 4.3. *Let γ be an open operation. A function $f : X \to Y$ is γ-semi-θ-perfect iff*

(1) f is γ-semi-θ-closed, and

(2) $f^{-1}(y)$ is γ-s-closed relative to X, for each $y \in Y$.

Proof. Necessity: let $f : X \to Y$ be a γ-semi-θ-perfect function. Then (1) follows from Theorem 3.8 and (2) follow from Theorem 4.2, because each singleton set in Y is γ-s-closed relative to Y [10].

Sufficiency: we suppose on the contrary that conditions (1) and (2) hold but f is not γ-semi-θ-perfect. Then there exists a filterbase $\Im$ in $f(X)$ such that $\Im$ γ-SR-converges to a point $y \in Y$. But $f^{-1}(\Im)$ is not γ-semi-θ-directed towards $f^{-1}(y)$. Thus there exists a filterbase ξ in X which is subordinate to $f^{-1}(\Im)$ and $f^{-1}(y) \cap [s_\gamma ad]_\theta(\xi) = \phi$. Therefore we have $y \notin [s_\gamma ad]_\theta f(\xi)$ and hence $y \notin s_\gamma cl_\theta(f(G_1))$, for some $G_1 \in \xi$. Then there exists a γ-semi-open set V containing y such that $scl_\gamma(V) \cap f(G_1) = \phi$. Since $\Im$ γ-SR-converges to y and ξ is subordinate to $f^{-1}(\Im)$, therefore, there exists a $G_2 \in \xi$ such that $f(G_2) \subset scl_\gamma(V)$. Consequently, we obtain $G_1 \cap G_2 = \phi$. This contradicts that ξ is filterbase. This proves that f is γ-semi-θ-perfect. This completes the proof. □

5. Locally γ-s-Regular Spaces .

In [10] a subset A of a space X is called γ-regular-open, if $A = int_\gamma(cl_\gamma(A))$. The set of all γ-regular-open sets in X is denoted by $RO_\gamma(X, \tau)$. Note that $RO_\gamma(X, \tau) \subseteq \tau_\gamma \subseteq \tau$ [10].

A space X is a γ-extremally disconnected space [10], if $cl_\gamma(U)$ is a γ-open set, for every γ-open set U in X. A space X is said to be locally γ-s-regular [11], if for each point of X has a γ-regular-open nbd which is γ-s-regular subspace of X.

The following Theorem shows that locally γ-s-regularity is a γ-regular-open hereditary property:

Theorem 5.1. *Every γ-regular-open subspace of a locally γ-s-regular space is locally γ-s-regular, where γ is a regular operation.*

Proof. Let Y be a γ-regular-open subspace of a locally γ-s-regular space X and $x \in Y$. Since X is locally γ-s-regular, therefore $x \in X$ has a γ-regular-open set V containing x which is γ-s-regular subspace of X. Since $int_{\gamma_Y}((cl_{\gamma_Y})(A)) = Y \cap int_{\gamma_X}((cl_{\gamma_X})(A))$ [2], for any γ-open set Y of X and any set A of Y. Therefore, there exists a γ-regular-open set U of Y such that $U = Y \cap V$ and $x \in U$. Now we show that U is γ-s-regular subspace of Y. Since V is γ-s-regular, by Theorem 3.4 [11], for γ-semi-regular set V_1 of V containing x, there exists γ-open set W such that $x \in W \subset cl_{\gamma_V}(W) \subset V_1$. Since $V \in RO_\gamma(X)$ and $W \subseteq V$, therefore we have $x \in W \subseteq cl_{\gamma_X}(W) \cap V \subseteq V_1$ [11]. But $cl_{\gamma_X}(W) \subset cl_{\gamma_X}(V_1) = V_1$. This gives $x \in W \subset cl_{\gamma_X}(W) \subset V_1$. Thus we have $x \in W \cap U \subset cl_{\gamma_V}(W) \cap U \subset V_1 \cap U$. Since Y is γ-semi-closed in X [5], therefore we obtain $cl_{\gamma_X}(W \cap U) \subset cl_{\gamma_X}(W) \cap cl_{\gamma_X}(U) = cl_{\gamma_X}(W) \cap U$ or $cl_{\gamma_X}(W \cap U) \subset cl_{\gamma_X}(W) \cap U$. This gives that $x \in W \cap U \subset cl_{\gamma_X}(W \cap U) \subseteq cl_{\gamma_X}(W) \cap U \subseteq V_1 \cap U$ or $x \in W \cap U \subset cl_{\gamma_U}(W \cap U) \subset V_1 \cap U$, where $W \cap U$ is γ-open and $V_1 \cap U$ is γ-semi-regular in U, since γ is regular. Thus, by Theorem 3.4 [11], U is γ-s-regular. This proves that Y is γ-s-regular. This completes the proof. □

Corollary 2. *Every γ-clopen subspace of a locally γ-s-regular space is locally γ-s-regular, where γ is a regular operation.*

Corollary 3. *Every γ-open subspace of a locally γ-s-regular space is locally γ-s-regular, if X is γ-extremally disconnected, where γ is a regular operation.*

Next we characterize locally γ-s-regular spaces as:

Theorem 5.2. *In a space X, the following are equivalent:*

(1) X is locally γ-s-regular.

(2) Every point $x \in X$ has a γ-regular-open set containing x which is a γ-s-reguar subspace of X.

(3) Every point $x \in X$ has a γ-open nbd U of x such that $int_{\gamma_X}(cl_{\gamma_X}(U))$ is a γ-s-regular subspace of X.

(4) Every point $x \in X$ has a γ-open set U containing x such that $scl_{\gamma_X}(U)$ is a γ-s-regular subspace of X.

(5) Every point $x \in X$ has a γ-open nbd U such that $s_\gamma cl_\theta(U)$ is a γ-s-regular subspace of X.

Proof. (1) $\Rightarrow$ (2). This is straightforward.

(2) $\Rightarrow$ (3). Let U be γ-open nbd of $x \in X$. Then $int_{\gamma_X}(cl_{\gamma_X}(U))$ is a γ-regular-open set containing x. By (2), $int_{\gamma_X}(cl_{\gamma_X}(U))$ is a γ-s-regular subspace of X. This proves (3).

(3) $\Rightarrow$ (4). Let U be γ-open nbd of a point $x \in X$. Then by Lemma 3.5 [10], $scl_{\gamma_X}(U) = int_{\gamma_X}(cl_{\gamma_X}(U))$. This shows that $scl_{\gamma_X}(U)$ is a γ-regular-open set of X containing x. By (3), $scl_{\gamma_X}(U)$ is γ-s-regular subspace of X. This proves (4).

(4) $\Rightarrow$ (5). Let U be γ-open nbd of x in X. Then by Proposition 3.9 [10], $s_\gamma cl_{\theta_X}(U) = scl_{\gamma_X}(U)$. By (4), $s_\gamma cl_{\theta_X}(U)$ is a γ-s-regular subspace of x. This proves (5).

(5) $\Rightarrow$ (1). Let $x \in X$ and U a γ-regular-open nbd of x. Then by (5), $s_\gamma cl_{\theta_X}(U) = scl_{\gamma_X}(U) = int_{\gamma_X}(cl_{\gamma_X}(U)) = U$ is γ-s-regular subspace of X. This proves that X is locally γ-s-regular. This completes the proof. □

Theorem 5.3. *A space X is locally γ-s-regular iff for each point $x \in X$, there exists a γ-regular-open set A of X such that $x \in A$ and A is locally γ-s-regular, where γ is regular.*

Proof. Necessity follows from Theorem 5.1.

Sufficiency: Let $x \in X$ and A a γ-regular-open set of X such that $x \in A$ and A is locally γ-s-regular. This gives that there exists a γ-regular-open set U in A such that $x \in U$ and U is γ-s-regular subspace of A. Hence by Lemma 3.5 [10], $U = scl_{\gamma_X}(U) = int_{\gamma_X}(cl_{\gamma_X}(U))$. Thus for each $x \in X$, there exists a γ-regular-open set U in X such that $x \in U$ and U is a γ-s-regular subspace of X. This proves that X is locally γ-s-regular. This completes the proof. □

In [5] a function $f : X \to Y$ is defined to be γ-semi-continuous, if for any γ-open B of Y, $f^{-1}(B)$ is γ-semi-open in X. A function $f : X \to Y$ is defined to be γ-semi-open (respt. γ-semi-closed) [6], if for each γ-open (respt. γ-closed) set U in X, $f(U)$ is γ-semi-open (respt. γ-semi-closed) in Y.

Theorem 5.4. *Let $f : X \to Y$ be γ-semi-continuous, γ-semi-open and γ-semi-closed preserving surjection. If X is γ-s-regular, then Y is γ-s-regular, where γ is an open operation.*

Proof. Let U be a γ-regular-open set in Y and $y \in U$. Let $x \in f^{-1}(y)$. Then $f^{-1}(U)$ is γ-semi-open in X and $x \in f^{-1}(U)$. Since X is γ-s-regular , therefore by Theorem 3.4 [11], there exists a γ-open set V such that $x \in V \subset cl_\gamma(V) \subset f^{-1}(U)$ or $y \in f(V) \subset f(cl_\gamma(V)) \subset f(f^{-1}(U)) \subset U$ or $y \in f(V) \subset f(cl_\gamma(V)) \subset U$, where $f(V)$ is γ-semi-open and $f(cl_\gamma(V))$ is γ-semi-closed [5]. Therefore $cl_\gamma(f(V)) \subset f(cl_\gamma(V))$ and $y \in f(V) \subset cl_\gamma(f(V)) \subset f(cl_\gamma(V)) \subset U$ or $y \in f(V) \subset cl_\gamma(f(V)) \subset U$. This proves that Y is γ-s-regular [11]. Hence the proof. □

Theorem 5.5. *Let $f : X \to Y$ be γ-semi-continuous, γ-semi-open preserving bijection. If X is γ-s-regular, then Y is locally γ-s-regular, where γ is a regular operation.*

Proof. Let U be γ-regular open set in Y such that $y \in U$. Let F be γ-closed set of U such that $y \notin F$. Then there exists a γ-closed set G of Y such that $F = U \cap G$ [2] and $y \notin G$. Then $f^{-1}(G)$ is γ-semi-closed in X and $f^{-1}(y) \notin f^{-1}(G)$. Since X is γ-s-regular, therefore there exists disjoint γ-open sets U_1, U_2 in X such that $f^{-1}(y) \in U_1$, $f^{-1}(G) \subset U_2$ or $y \in f(U_1)$, $G \subset f(U_2)$, where $f(U_1)$ and $f(U_2)$ are γ-semi-open in Y. Clearly $y \in U \cap f(U_1)$ and $F = U \cap G \subseteq U \cap f(U_2)$, where $U \cap f(U_1)$ and $U \cap f(U_2)$ are disjoint γ-semi-open sets in U [5]. This proves that Y is locally γ-s-regular. This completes the proof. □

References

[1] B. Ahmad and F. U. Rehman: *Operations on Topological Spaces-II, Math. Today,* 11 (1993),13-20.

[2] B. Ahmad and S. Hussain: *Properties of γ-Operations in Topological Spaces, Aligarh Bull. Math.,* 22 (1) (2003),45-51.

[3] B. Ahmad and S. Hussain: *γ^*- Regular and γ-Normal Spaces, Math. Today,* 22 (1)(2006), 37-44.

[4] B. Ahmad and S. Hussain: *On γ-s-Closed Subspaces , Far East Jr. Math. Sci.,* 31 (2)(2008), 279-291.

[5] B. Ahmad and S. Hussain: *γ-Semi-Open Sets in Topological Spaces-II , Southeast Asian Bull. Maths.,* 34(6)(2010), 997-1008.

[6] B. Ahmad, S. Hussain and T. Noiri: *On Some Mappings in Topological Spaces , Eur. J. Pure Appl. Math.,* 1(2008),22-29.

[7] B. Ahmad, M. Khan and T. Noiri: *On Locally s-Regular Spaces, Indian Jr. Pure Appl. Math.,* 27(1996), 1078-1092.

[8] C. Dorsett: *Semi-Regular Spaces, Soochow Jr. Math,* 8(1982), 45-53.

[9] C. Dorsett: *s-Regular and s-Normal Spaces, Math. Nachr.,* 115(1984), 265-270.

[10] S. Hussain and B. Ahmad : *On γ-s-Closed Spaces, Sci. Magna Jr.,* 3(4)(2007), 89-93.

[11] S. Hussain and B. Ahmad : *On γ-s^*-Regular Spaces and Almost γ-s-Continuous Functions, Lobachevskii J. Math.,* 30(4) (2009), 263-268. DOI:10.1134/S1995080209040039.

[12] S. Hussain, B. Ahmad and T. Noiri: *γ-Semi-Open Sets in Topological Spaces, Asian Eur. J. Math.,* 3(3) (2010), 427-433. DOI: 10.1142/S1793557110000337

[13] D. S. Jankovic: *On Functions with Closed Graphs, Glasnik Mat.,* 18(1983), 141-146.

[14] S. Kasahara: *Operation-Compact Spaces, Math. Japon.,* 24(1979), 97-105.

[15] H. Ogata: *Operations on Topological Spaces and Associated Topology, Math. Japon.,* 36 (1) (1991), 175-184.

[16] G. D. Maio and T. Noiri: *On s-Closed Spaces, Indian jr. Pure Appl. Math,* 18 (3) (1987), 226-233.

[17] G. D. Maio and T. Noiri: *Week and Strong Forms of Irresolute Functions, Third National Conference on Topology (Italian)(Trieste, 1986). Rend. Circ. Mat. palermo, (2) Suppl. No.,* 18 (1988), 255-273.

[18] S. N. Maheshwari and R. Prasad: *On s-Regular Spaces, Glasnik Mat.,* 10 (30) (1975), 347-350.

[19] T. Noiri: *On Semi-Continuous Mappings, Bull. Cal. Math. Soc.,* 65(1973), 197-201.

[20] T. Noiri, B. Ahmad amd M. Khan: *Almost S-Continuous Functions, Kyungpook Math. Jr.,* 35 (1995), 311-322.

[21] F.U. Rehman and B. Ahmad: *Operations on Topological Spaces-I, Math. Today,* 10(1992), 29-36.

[22] M. K. Singal and S. P. Arya: *On Almost-Regular Spaces, Glasnik Mat.,* 4 (24)(1969), 89-99.