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TECHNIQUES FOR THE DECOMPOSITION OF CARTAN'S CURVATURE TENSOR IN COMPLEX FINSLER MANIFOLDS

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Abstract. A Finsler metric of a manifold or vector bundle is defined as a smooth assignment for each base point, a norm on each fiber space and thus the class of Finsler metrics contains Riemannian metrics as a special subclass. The geometry of complex Finsler manifold has been developed by [7]. In complex Finsler manifolds, the study of theory of curvatures has been an active field of research over past few decades. In the present article, our main purpose is to discuss some techniques of decomposition for the well known Cartan's first curvature tensor S^i_{jkh}. Moreover, we attempted to establish few significant results that may produce vital connections between complex Finsler and complex Einstein's manifolds. Also, by adopting the techniques of decomposition, various cases and conditions have been developed and their advantages in the study of theory of relativity & cosmology have been pursued.

1. Introduction

Since the explanation of various physical systems and Mathematical devices is much more concerned with the utilization of numerous algebraic quantities involved in the illustration of geometrical phenomenon and states in which they occur, it is mandatory to put forward the ideas of such basic algebraic quantities. Certain types of quantities are commonly identified as scalar, vector and tensors. Among these the tensors are quite crucial and tedious geometric structures as these are the only quantities involving three meaningful aspects altogether. That is, a tensor can suppose to be a tool having direction, magnitude and orientation dependency.

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Different types of tensors, according to their nature are applied to discuss different systems, their states and properties. But from geometric point of view, the geometry from its origin up to its recent extent is all about the curvature tensors.

Among all curvature tensor, yet known to us, Cartan's-curvature tensors are very surprising because of having special classes of differentiation obtained with the help of Cartan's postulates, which are most popularly used in Finsler geometry. Although, the Cartan's curvature tensor is capable to determine the properties of geometric configurations in Finsler manifolds, but sometimes it seems to be necessarily convenient to decompose it in order to study the basic aspects of the manifolds under consideration. Here are few basic concepts of complex Finsler manifolds and Cartan's-curvature tensor, which we seem necessary to study the proposed research topic.

The basic ideas of Finsler manifolds may be traced back to the famous lecture of Riemann "Über die hypotheses, welche der Geometrie zugrunde leigen". In this lecture of 1854, Riemann has discussed several new methods by means of which an n-dimensional manifold can be intimated with a special kind of distance or metric function. It is supposed that this special metric function should have three properties in common, namely; the function must be positive definite, its first order differential must be homogeneous and convex in latter. The distance function **ds** between two points whose coordinates are given by z^i & $z^i + dz^i$, in a complex Finsler manifold is defined by some function $F(z^i, dz^i)$, i.e.,

$$ds = F(z^i, dz^i) \ \ \forall i = 1, 2, \cdots, n.$$

If $\mathbb{D}$ be the domain (i.e., an open connected set) of an n-dimensional manifold V_n, which is completely covered by a coordinate system, such that any point of $\mathbb{D}$ is represented by a set of n independent variables $z^i (i = 1, 2, \cdots, n)$, then the set of points of $\mathbb{D}$, whose co-ordinates are expressible as the function of single parameter t is regarded as a curve of V_n. Thus, the equation

$$z^i = z^i(t), \tag{1}$$

represents a curve C in V_n. If the equations (1) are of class C^1, we shall regard the expression whose components are given by

$$\dot{z}^i = \frac{dz^i}{dt} \tag{2}$$

as the tangent vector to C.

Next, suppose that we are given a function $F(z^i, \dot{z}^i)$ of the line element $(z^i, \dot{z}^i)$ of the curve defined by C in $\mathbb{D}$, then we have the following conditions:

Condition (a): The function $F(z^i, \dot{z}^i)$ is positively homogeneous of degree one in the $\dot{z}^i$, i.e.,

$$F(z^i, k\dot{z}^i) = kF(z^i, \dot{z}^i) \text{ with } k > 0. \tag{3}$$

Condition (b): The function $F(z^i, \dot{z}^i)$ is positive if, not all $\dot{z}^i$ vanish simultaneously, i.e.,

$$F(z^i, \dot{z}^i) > 0 \text{ with } \sum_i (z^i)^2 \neq 0. \tag{4}$$

Condition (c): The function $F(z^i, \dot{z}^i)$ is convex in z^i. It follows from a well known theorem on complex function that $f(z^i)$ on n variables $u^1, u^2, \cdots, u^n$

of a domain $\mathbb{D}$ in the manifold is said to be convex, if it contains the whole segment of a straight line which connects any two of its points. The function $f(u^i)$ is said to be convex in $\mathbb{D}$, if it is defined in $\mathbb{D}$ and if the inequality

$$f\left(\frac{u^i_{(1)}+u^i_{(2)}}{2}\right) \leq \frac{1}{2}\left[f(u^i_{(1)})+f(u^i_{(2)})\right] \tag{5}$$

is satisfied for all pairs of the points $u^i_{(1)}$ and $u^i_{(2)}$ of $\mathbb{D}$.

Eventually, if in the complex manifold C_n we introduce the fundamental function $F(z^i,\dot{z}^i)$ which is positively homogeneous of degree one with respect to the variables z^i and $\dot{z}^i$, then the function $F(z^i,\dot{z}^i)\geq 0$ is such that

$$F(z^i,\dot{z}^i)=|k|F(z^i,\dot{z}^i).$$

The arc length of the arc $z^i=z^i(t)$ for $t_1\leq t\leq t_2$ is defined by;

$$s=\int_{t_1}^{t_2} F(z^i,\dot{z}^i). \tag{6}$$

Such manifold is called *complex Finsler manifold* [17] and is symbolized by F^C_n. Moreover, the function $F(z^i,\dot{z}^i)$ is assumed to be invariant under coordinate transformations.

Now, putting $F(z^i,\dot{z}^i)\overset{\text{def}}{=}F^2(z^i,\dot{z}^i)$ and $g_{ij}(z^i,\dot{z}^i)\overset{\text{def}}{=}\frac{\partial^2F^2(z^i,\dot{z}^i)}{\partial\dot{z}^i\partial\dot{z}^j}$, we observe that g_{ij} is a symmetric covariant tensor and that

$$F^2(z^i,\dot{z}^i)=g_{ij}(z,\dot{z})\dot{z}^i\dot{z}^j. \tag{7}$$

We shall assume that the fundamental tensor g_{ij} of the complex Finsler manifold has a rank n and we use g_{ij} and its associate g^{ij} to lower an raise the indices.

1.1. **Fundamental Postulates of E. Cartan**[8]**.** The theory of E-Cartan which treats the Finsler manifolds from an entirely different point of view has played the most prominent role in the development of Finsler Geometry. In this subsection, we shall take a brief look on Cartan's monograph in which he discussed his postulates, which he defined by means of special classes of covariant derivatives.

In order to be able to endow the Finsler manifold $F^{(C)}_n$ with a so-called "Euclidean connection", Cartan considered the manifold X_{2n-1} of the line elements $(z^i,\dot{z}^i)$ which is $(2n-1)$ dimensional, since only the ration of the $\dot{z}^i$ are necessary to define a direction in the tangent manifold $T_n(z^i)$. The coordinates are referring to the centre of the line element $(z^i,\dot{z}^i)$. All quantities such as tensors are to be defined by means of the functions of line elements.

In the manifold $F^{(C)}_n$, a metric is defined by means of a function $F(z^i,\dot{z}^i)$ satisfying the three conditions of Finsler manifold, but the manifold X_{2n-1} is said to be endowed with Euclidean connection if the following construction is imposed on X_{2n-1}.

I: A metric with symmetric components $g_{ij}(z^i,\dot{z}^i)$ is given such that the square of the distance between the centres z^i and z^i+dz^i of the neighboring elements $(z^i,\dot{z}^i)$ and $(z^i+dz^i,\dot{z}^i)+d\dot{z}^i$ is given by the expression

$$g_{ij}(z,\dot{z})dz^idz^j. \tag{8}$$

Because, the dz^i form the components of a contravariant vector, it follows that the square of the length of an arbitrary contravariant vector will be

defined by

$$g_{ij}(z,\dot{z})X^iX^j. \tag{9}$$

II: An analytic expression would came into existence, which would represent the variation of the vector X^i when its element of support $(z^i,\dot{z}^i)$ experiences an infinitesimal small change and becomes $(z^i+dz^i,\dot{z}^i+d\dot{z}^i)$. This variation of X^i will be represented by means of a covariant (or absolute) differential:

$$DX^i = dX^i + C^i_{kh}(z,\dot{z})X^k d\dot{z}^h + \Gamma^i_{kh}(z,\dot{z})X^k dz^h, \tag{10}$$

where the coefficients C^i_{kh} and Γ^i_{kh} are the functions of the element of support. Naturally, the first postulate can be applied to these by proceeding like below:
If a vector X^i is transposed from $(z^i,\dot{z}^i)$ to $(z^i+dz^i,\dot{z}^i+d\dot{z}^i)$ by parallel displacement, i.e., if the actual change dX^i in X^i is in accordance with the equation;

$$DX^i = 0 \text{ or } dX^i = -C^i_{kh}X^h d\dot{z}^i - \Gamma^i_{kh}X^k dz^h, \tag{11}$$

then the length of X^i as given equation (8) remains invariant.

III: The third postulate of E-Cartan contains the following four logics:

A: If the direction of a vector X^i coincides with that of its elements of support $(z^i,\dot{z}^i)$. its length is to be equal to $F(z^i,X^i)$.

B: Let X^i andY^i represent two vectors with a common element of support $(z^k,\dot{z}^k)$. When the latter undergoes an infinitesimal rotation about its own centre z^k and becomes $(z^k,\dot{z}^k+d\dot{z}^k)$, while the components X^i and Y^i remain invariant, then their corresponding covariant differentials (10) will be DX^i and DY^i and the following symmetric condition will hold good:

$$g_{ij}(z,\dot{z})X^iDY^j = g_{ij}(z,\dot{z})XD^iY^j. \tag{12}$$

C: If the direction of a vector with fixed components X^i coincides with that of its element of support, then its covariant differential given by equation (10) corresponding to an infinitesimal rotation of its element of support about its own centre vanishes identically.

D: When the displacement of a vector is such that the element of support is transported parallel to itself from z^k to z^k+dz^k, the coefficients like Γ^i_{kh} which appear in the covariant differential (10) will be symmetric in their lower indices h and k. In view of these conditions, we now conclude that following analytic aspect, which may be very useful to study the Cartan's curvature tensor and their covariant differentiation.

In view of equation (9), condition (A) obviously yields

$$F^2(z^i,\dot{z}^i) = g_{ij}\dot{z}^i\dot{z}^j. \tag{13}$$

Under the above conditions, [8] also gave a new form of equation (9) as below:

$$DX^i = dX^i + \Gamma^{*i}_{kj}X^k dz^j, \tag{14}$$

where he put the expression

$$\Gamma^{*i}_{kj} = \Gamma^i_{kj} - C^i_{kh}\Gamma^h_{rj}\dot{z}^r. \tag{15}$$

In view of the above expression, we now outline some properties of the covariant differentiation as discussed by [8]:

1.2. **Properties of covariant differentiation.** In view of the formula given by equation (10), we have the extended form of derivation of a tensor of any rank as below:

$$DT^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} = dT^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} + \sum_{\mu=1}^{r} T^{i_1 i_2 \cdots i_{\mu-1} i_{\mu+1} \cdots i_r}_{j_1 j_2 \cdots j_s} \times \tag{16}$$
$$(C^{i_\mu}_{kh} d\dot{z}^h + \Gamma^{i_\mu}_{kh} dz^h) - \sum_{\theta=1}^{s} T^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_{\mu-1} j_{\mu+1} \cdots j_s} (C^k_{j_\mu h} d\dot{z}^h + \Gamma^k_{j_\mu h} dz^h).$$

In fact, there is no ambiguity that the tensor T of type (r, s) is a function of the element of support $(z, \dot{z})$. That is why the term dT involves the variation of the latter. Evidently it is quite clear that if we take this fact into account, we may easily observe that this differentiation technique obeys the usual laws of covariant differentiation viz. differential of the sum is equal to the sum of the differential and the product of ordinary differentiation.

[8] also considered the covariant differential of a contravariant vector $X^i = X^i(z, \dot{z})$ with respect to the unit vector $\dot{z}^h$ in the direction of element of support as follows:

$$DX^i = \left(F \frac{\partial X^i}{\partial \dot{z}^h} + A^i_{kh} X^k \right) Dl^h + X^i_{|h} dz^h, \tag{17}$$

where he used $X^i_{|h} = \frac{\partial X^i}{\partial z^h} - \frac{\partial X^i}{\partial \dot{z}^k} \frac{\partial G^k}{\partial \dot{z}^k} + \Gamma^{*i}_{kh} X^k$.

Now, in order to aim our purpose of study, we briefly discuss Cartan's curvature tensor as given by [8]. We take into account a commutation formula arising from the covariant derivatives as given in equation (17). Evidently, there exist two different processes of partial derivation, namely the process $X^i_{|h}$ which is defined in equation (17) and the process;

$$X^i|_h \equiv \left(F \frac{\partial X^i}{\partial \dot{z}^h} + A^i_{kh} X^k \right) \tag{18}$$

and hence in view of these differentiation processes, the equation (17) can be written as

$$DX^i = X^i|_h Dl^h + X^i_{|h} dz^h. \tag{19}$$

Further, we consider the commutation formula given in equation (18) corresponding to repeated application on indices as

$$X^i|_{hk} - X^i|_{kh} = F \left(F_{\dot{z}^k} \frac{\partial X^i}{\partial \dot{z}^h} - F_{\dot{z}^h} \frac{\partial X^i}{\partial \dot{z}^k} \right) + \tag{20}$$
$$+ X^r \left\{ F \left(\frac{\partial A^i_{rh}}{\partial \dot{z}^k} - \frac{\partial A^i_{rk}}{\partial \dot{z}^h} \right) + A^i_{km} A^m_{rh} - A^i_{mh} A^m_{rk} \right\}.$$

Using the result $F \left(\frac{\partial A^i_{rh}}{\partial \dot{z}^k} - \frac{\partial A^i_{rk}}{\partial \dot{z}^h} \right) = F_{\dot{z}^k} A^i_{rh} - F_{\dot{z}^h} A^i_{rk}$ in equation (20) and re minding equation (18), we have

$$X^i|_{hk} - X^i|_{kh} = \left\{ F_{\dot{z}^k} X^i|_h - F_{\dot{z}^h} X^i|_k \right\} + S^i_{jkh} X^j, \tag{21}$$

where we have written [5]:

$$S^i_{jkh} = A^i_{rk}A^r_{jh} - A^i_{rh}A^r_{jk}. \tag{22}$$

This tensor is called the *Cartan's first curvature tensor* and we shall consider this one for our decomposition studies. Here are few identities, which the Cartan's first curvature tensor satisfies:

$$S^i_{jkh} = -S^i_{jhk}, \tag{23}$$

i.e., Cartan's first curvature tensor is anti-symmetric in its last index pair.

If we lower the index of this curvature tensor by writing $S_{ijkh} = g_{rj}S^r_{ikh}$, we have

$$S_{ijkh} = -S_{jikh}. \tag{24}$$

2. Techniques for the Decomposition of Cartan's I-Curvature Tensor field

General 2nd order tensors in the three dimensional manifolds contain nine independent components, but sometimes it is desirable to reduce the dimensionality of such tensor fields in a meaningful way as this process may let us know the physical state represented by a tensor. Various techniques for tensor decompositions are available to reduce the dimensionality or to transform the tensor in such a way that describes important aspects about those for which they are standing for. In order to decompose the Cartan's I-curvature tensor in complex Finsler manifolds, we shall make use of the following four techniques:

- **1:** Technique of symmetric-antisymmetric Decomposition [22, 18, 3, 19, 1, 16, 11, 10]
- **2:** Technique of eigenvector-eigenvalues Decomposition [22, 18, 13, 14, 23, 21, 15, 12]
- **3:** Technique of isotropic-deviator Decomposition [22, 18, 24, 4]
- **4:** Technique of singular value decomposition (SVD)[22, 18, 19, 20, 2, 1, 16, 6]

To use the aforementioned decomposition techniques, for the sake of feasibility, we first factorize the Cartan's I-curvature tensor in terms of the outer/open product of two second order tensors, each having 3^2-components in a three dimensional complex Finsler manifold as follows [3]:

$$S^i_{jkh} = G^i_j f_k. \tag{25}$$

Here, for the exhibition of physical significance due to such factorization, we would consider the first mixed tensor lying at the right side of equation (25) as an Einstein tensor and the second one covariant tensor as to describe the degree of curvature of a surface given by the function $F(z^i, \dot{z}^i)$. Also, a powerful reason behind this kind of assumption arises from one of the feature of Finsler manifold that the fundamental function intimated with $F^{(C)}_n$ is homogeneous in its first order differential.

It is remarkable that for the right hand side mixed tensor of rank 2, the property of symmetry and skew-symmetry is not an intrinsic one, as it is evident from the well known transformation law of mixed tensor's symmetry/anti-symmetry that the property of symmetry/anti-symmetry of a mixed tensor between a pair of dissimilar indices (one covariant and other contravariant) is not invariant under the transformation.

2.1. **Technique 1: Decomposition of S^i_{jkh} using technique first.** Let us decompose the two tensors of equation (25) one by one as follows:

Theorem 2.1. *Under the technique (1), the tensor G^i_j of Eq.(25) (which is preassumed to be an Einstein one) produces the following decomposition form:*

$$\|G^i_j\| = \frac{1}{2}[\|G^p_q\| + \|G^q_p\|]_{p,q=\alpha,\beta,\gamma} + \frac{1}{2}[\|G^p_q\| - \|G^q_p\|]_{p,q=\alpha,\beta,\gamma} \tag{26}$$

Proof. One can straightforwardly demonstrate this theorem by keeping in mind that "any tensor can be expressed as a sum of its symmetric and antisymmetric parts" [22, 18, 3, 19, 1, 16, 10, 11]. Thus under this assumption, the Einstein tensor G^i_j can be expressed as follows:

$$\underbrace{[\frac{1}{2}(G^i_j + G^j_i)]}_{Symmetric\,part} + \underbrace{[\frac{1}{2}(G^i_j - G^j_i)]}_{Antisymmetric\,part}, \tag{27}$$

where $G^i_j(z,\dot{z}) := R^i_j - \frac{R}{2}\delta^i_j$ is an Einstein tensor in terms of Ricci tensor and curvature scalar.
Also, the purely covariant form of this Einstein tensor can be found as

$$G_{ik} = g_{in}G^n_k = g_{in}(R^n_k - \frac{R}{2}\delta^n_k) = R_{ik} - \frac{R}{2}g_{ik}. \tag{28}$$

Now, Eq. (27) in matrix form can be written as

$$\begin{aligned}\|G^i_j\| = \frac{1}{2}\left\{\begin{pmatrix} G^\alpha_\alpha & G^\alpha_\beta & G^\alpha_\gamma \\ G^\beta_\alpha & G^\beta_\beta & G^\beta_\gamma \\ G^\gamma_\alpha & G^\gamma_\beta & G^\gamma_\gamma \end{pmatrix} + \begin{pmatrix} G^\alpha_\alpha & G^\alpha_\beta & G^\alpha_\gamma \\ G^\beta_\alpha & G^\beta_\beta & G^\beta_\gamma \\ G^\gamma_\alpha & G^\gamma_\beta & G^\gamma_\gamma \end{pmatrix}\right\} + \\ + \frac{1}{2}\left\{\begin{pmatrix} G^\alpha_\alpha & G^\alpha_\beta & G^\alpha_\gamma \\ G^\beta_\alpha & G^\beta_\beta & G^\beta_\gamma \\ G^\gamma_\alpha & G^\gamma_\beta & G^\gamma_\gamma \end{pmatrix} - \begin{pmatrix} G^\alpha_\alpha & G^\alpha_\beta & G^\alpha_\gamma \\ G^\beta_\alpha & G^\beta_\beta & G^\beta_\gamma \\ G^\gamma_\alpha & G^\gamma_\beta & G^\gamma_\gamma \end{pmatrix}\right\},\end{aligned} \tag{29}$$

where $G^\alpha_\alpha, G^\alpha_\beta, G^\alpha_\gamma \ldots$ ect. are the components of G^i_j in 3-dimensional complex Finsler manifold.
For the sake of convenience, we write the Eq. (29) as in the following notations:

$$\|G^i_j\| = \frac{1}{2}[\|G^p_q\| + \|G^q_p\|]_{p,q=\alpha,\beta,\gamma} + \frac{1}{2}[\|G^p_q\| - \|G^q_p\|]_{p,q=\alpha,\beta,\gamma}$$

□

Theorem 2.2. *Covariant differentiation of Eq.(2.2) yields an analytic expression which represents a relation between variation in Einstein tensor G^i_j and functions of element of support $C^i_{kh}(z,\dot{z})$ & $\Gamma^i_{jh}(z,\dot{z})$.*

Proof. Making use of the concept of covariant differentiation given by Eq. (16), we now differentiate Eq.(2.2) with respect to z^l as below:

$$D\|G^i_j\| = D[\frac{1}{2}(\text{Symmetric part})] + D[\frac{1}{2}(\text{Antisymmetric part})] \text{ of Eq. (29).}$$

Differentiation of Eq. (2.2) yields a lengthy but straightforward relation as below:

$$
\begin{aligned}
(30)\quad D\|G^i_j\| = \frac{1}{2}[\{d\|G^p_q\| + \|C^p_{rs}\|(z,\dot z)\|G^r_q\|d\dot z^s + \Gamma^p_{rs}(z,\dot z)\|G^r_q\|dz^s - \\
- \|C^r_{qs}\|(z,\dot z)\|G^p_r\|d\dot z^s - \Gamma^r_{qs}(z,\dot z)\|G^p_r\|dz^s\} + \{d\|G^q_p\| + \|C^q_{rs}\|(z,\dot z)\|G^r_p\|d\dot z^s + \\
+ \Gamma^q_{rs}(z,\dot z)\|G^r_p\|dz^s - \|C^r_{ps}\|(z,\dot z)\|G^q_r\|d\dot z^s - \Gamma^r_{ps}(z,\dot z)\|G^q_r\|dz^s\}] + \frac{1}{2}[\{d\|G^p_q\| + \\
+ \|C^p_{rs}\|(z,\dot z)\|G^r_q\|d\dot z^s + \Gamma^p_{rs}(z,\dot z)\|G^r_q\|dz^s - \|C^r_{qs}\|(z,\dot z)\|G^p_r\|d\dot z^s - \Gamma^r_{qs}(z,\dot z)\times \\
\|G^p_r\|dz^s\} - \{d\|G^q_p\| + \|C^q_{rs}\|(z,\dot z)\|G^r_p\|d\dot z^s + \Gamma^q_{rs}(z,\dot z)\|G^r_p\|dz^s - \|C^r_{ps}\|(z,\dot z)\times \\
\|G^q_r\|d\dot z^s - \Gamma^r_{ps}(z,\dot z)\|G^q_r\|dz^s\}].
\end{aligned}
$$

Most probably, the Eq. (30) predict an analytic expression which connects the variation of Einstein tensor with various components of tensorial and non-tensorial quantities such as $C^p_{rs}(z,\dot z)$ and $\Gamma^p_{rs}(z,\dot z)$ etc., which are themselves the functions of element of support $(z^i, \dot z^i)$.
This analytic expression may be of great geometrical as well as physical significance in the study of various properties of Einstein's manifolds when treated with Finsler Geometry. □

In order to study some possible/probable connections between Einstein and Finsler manifolds, we consider a special case where the covariant differentiation given by Eq. (30) vanishes.

Theorem 2.3. *The vanishing of the analytic expression (30) i.e.,* $D\|G^i_j\| = 0$ *implies the existence of an Einstein's field equation of the form:*

$$\|R^i_j\|(z,\dot z) = \gamma[\|T^i_j\| + \frac{1}{2}\delta^i_j\|T\|](z,\dot z) = c \text{ (Stationary value)},$$

where T^i_j *is the well known energy-momentum tensor and* T *is its trace.*

Proof. Let us use the usual Einstein's tensor $G^i_j = R^i_j - \frac{R}{2}\delta^i_j$, where R^i_j being the Ricci tensor as well as R being the curvature scalar of complex Finsler manifold. If we assume the vanishing of an analytic expression (30), we have

$$D\|G^i_j\| = D\|R^i_j - \frac{R}{2}\delta^i_j\| = 0. \tag{31}$$

Now to derive the Einstein's field equations, we introduce a field tensor which is of the same rank and type with symmetric properties as the Ricci tensor R^i_j. Hence, in our case, we introduce an energy-momentum tensor T^i_j which is such that

$$D\|T^i_j\|(z,\dot z) = 0. \tag{32}$$

The equality of Eq. (31) and Eq. (32) implies

$$D\|G^i_j\| = D\|R^i_j - \frac{R}{2}\delta^i_j\| \propto D\|T^i_j\|, \tag{33}$$

or

$$D\|G^i_j\| = D\|R^i_j - \frac{R}{2}\delta^i_j\| = \gamma\|T^i_j\| = 0, \tag{34}$$

where γ is a constant of proportionality and in the study of theory of relativity, will be called cosmological constant. Thus, if we integrate Eq. (34) over the complex

Finsler manifold $F_n^{(C)}$ with respect to some coordinate z^l, we would have a well known Einstein's field equation in th complex Finsler manifold as below:

$$\int_{F_n^{(C)}} [D\|G^i_j\|(z,\dot{z})]dz^l = \int_{F_n^{(C)}} [D\|R^i_j - \frac{R}{2}\delta^i_j\|(z,\dot{z})]dz^l = \tag{35}$$
$$= \int_{F_n^{(C)}} [\gamma D\|T^i_j\|(z,\dot{z})]dz^l = \text{some constant of integration.}$$

The above expression is due to the well known Euler's condition $\int_{t_0}^{t_1} F(z^i,\dot{z}^i)dt =$ stationary, where $\dot{z}^i = \frac{dz^i}{dt}$. The expression (35) on simplification yields

$$\|G^i_j\|(z,\dot{z}) = \|R^i_j - \frac{R}{2}\delta^i_j\|(z,\dot{z}) = \gamma\|T^i_j\|(z,\dot{z}) = \tag{36}$$
$$= c \text{ (stationary value).}$$

Evidently, Eq. (36) stands for the Einstein's field equation in local component form. Moreover, if we contract Eq. (36) with respect to indices i and j, we have

$$\|G^i_i\|(z,\dot{z}) = \|R - \frac{R}{2}\|(z,\dot{z}) = \gamma\|T^i_i\|(z,\dot{z}) = c \text{ (stationary value).}$$

Thus with the help of expression mentioned just above, our field equation (36) implies

$$\|R^i_j\|(z,\dot{z}) = \gamma[\|T^i_j\| + \frac{1}{2}\delta^i_j\|T\|](z,\dot{z}) = c \text{ (stationary value),} \tag{37}$$

which is again the field equation used for analytical purposes. □

Definition 2.4. *There is a special case when the complex Finsler manifold is empty. Then in such case, the energy-momentum tensor must vanish, i.e.,* $T^i_j = 0$. *Hence from Eq. (37)* $\|R^i_j\|(z,\dot{z}) = 0$. *This condition gives rise to a special Finsler manifold which is Ricci flat and thereby called "Ricci flat complex Finsler manifold".*

In order to discuss the Eq. (25) completely, we now proceed to decompose the tensor f_{kh} using technique (1). Further, from the standpoint of physical significance, we assume that this second rank covariant tensor describes the degree of curvature of the Finsler surface given by the fundamental function $F(z^i,\dot{z}^i)$.

Theorem 2.5. *Under the technique (1), the splitting of tensor* f_{kh} *of Eq.(25) (which is preassumed to be a degree of curvature of Finsler surface) produces the following decomposition form:*

$$\|f_{kh}\| = \frac{1}{2}[\|f_{pq}\| + \|f_{qp}\|]_{p,q=\alpha,\beta,\gamma} + \frac{1}{2}[\|f_{pq}\| - \|f_{qp}\|]_{p,q=\alpha,\beta,\gamma} \tag{38}$$

Proof. We can straightforwardly proof this theorem by keeping in mind that "any tensor can be expressed as a sum of its symmetric and antisymmetric parts" [22, 18, 3, 19, 1, 16, 10, 11]. Thus under this assumption, the tensor f_{kh} can be expressed as follows:

$$\underbrace{[\frac{1}{2}(f_{kh} + f_{hk})]}_{Symmetric\,part} + \underbrace{[\frac{1}{2}(f_{kh} - f_{hk})]}_{Antisymmetric\,part}, \tag{39}$$

which in matrix form can be written as

$$\|f_{kh}\| = \frac{1}{2}\left\{\begin{pmatrix} f_{\alpha\alpha} & f_{\alpha\beta} & f_{\alpha\gamma} \\ f_{\beta\alpha} & f_{\beta\beta} & f_{\beta\gamma} \\ f_{\gamma_\alpha} & f_{\gamma\beta} & f_{\gamma\gamma} \end{pmatrix} + \begin{pmatrix} f_{\alpha\alpha} & f_{\alpha\beta} & f_{\alpha\gamma} \\ f_{\beta\alpha} & f_{\beta\beta} & f_{\beta\gamma} \\ f_{\gamma_\alpha} & f_{\gamma\beta} & f_{\gamma\gamma} \end{pmatrix}\right\} + \tag{40}$$
$$+ \frac{1}{2}\left\{\begin{pmatrix} f_{\alpha\alpha} & f_{\alpha\beta} & f_{\alpha\gamma} \\ f_{\beta\alpha} & f_{\beta\beta} & f_{\beta\gamma} \\ f_{\gamma_\alpha} & f_{\gamma\beta} & f_{\gamma\gamma} \end{pmatrix} - \begin{pmatrix} f_{\alpha\alpha} & f_{\alpha\beta} & f_{\alpha\gamma} \\ f_{\beta\alpha} & f_{\beta\beta} & f_{\beta\gamma} \\ f_{\gamma_\alpha} & f_{\gamma\beta} & f_{\gamma\gamma} \end{pmatrix}\right\},$$

where $f_{\alpha\alpha}, f_{\alpha\beta} \ldots$ etc. are the components f_{kh} in 3-dimensional complex Finsler manifold.
Now, for the feasibility, we can write the Eq. (40) as in the following notations:

$$\|f_{kh}\| = \frac{1}{2}[\|f_{pq}\| + \|f_{qp}\|]_{p,q=\alpha,\beta,\gamma} + \frac{1}{2}[\|f_{pq}\| - \|f_{qp}\|]_{p,q=\alpha,\beta,\gamma}$$

□

Theorem 2.6. *Covariant differentiation of Eq.(2.14) yields an analytic expression which represents a relation between variation in degree of curvature f_{kh} of a Finsler surface $F(z^i, \dot{z}^i)$ and functions of element of support $C^i_{kh}(z, \dot{z})$ & $\Gamma^i_{jh}(z, \dot{z})$.*

Proof. We again make use of the concept of covariant differentiation given by (16) for the Eq. (38) as follows:
Differentiating Eq. (38) covariantly with respect to z^l we obtain

$$\begin{aligned} D\|f_{kh}\| = \frac{1}{2}[\{d\|f_{pq}\| - \|f_{rq}\|(\|C^r_{ps}\|(z,\dot{z})d\dot{z}^s + \Gamma^p_{rs}(z,\dot{z})dz^s) - \\ - \|f_{pr}\|(\|C^r_{qs}\|(z,\dot{z})d\dot{z}^s + \Gamma^r_{qs}(z,\dot{z})dz^s)\} + \{d\|f_{qp}\| - \|f_{rp}\|(\|C^r_{qs}\|(z,\dot{z})d\dot{z}^s + \\ + \Gamma^r_{qs}(z,\dot{z})dz^s) - \|f_{qr}\|(\|C^r_{ps}\|(z,\dot{z})d\dot{z}^s + \Gamma^r_{ps}(z,\dot{z})dz^s)\}] + \frac{1}{2}[\{d\|f_{pq}\| - \|f_{rq}\times \\ \|(\|C^r_{ps}\|(z,\dot{z})d\dot{z}^s + \Gamma^p_{rs}(z,\dot{z})dz^s) - \\ - \|f_{pr}\|(\|C^r_{qs}\|(z,\dot{z})d\dot{z}^s + \Gamma^r_{qs}(z,\dot{z})dz^s)\} - \{d\|f_{qp}\| - \|f_{rp}\|(\|C^r_{qs}\|(z,\dot{z})d\dot{z}^s + \\ + \Gamma^r_{qs}(z,\dot{z})dz^s) - \|f_{qr}\|(\|C^r_{ps}\|(z,\dot{z})d\dot{z}^s + \Gamma^r_{ps}(z,\dot{z})dz^s)\}] \end{aligned} \tag{41}$$

□

Theorem 2.7. *With the following trivial assumption*

- **A1:** *Introducing a fundamental tensor g_{ij} in Eq. (38) which is of same rank and type as f_{kh}, such that $Dg_{ij} = 0$.*
- **A2:** *To illustrate connection between the metric $ds^2 = F(z, \dot{z})$ and the Cartan's curvature tensor of the Finsler continuum, we assume $z_k = F(z, \dot{z})$ to be a two dimensional smooth curved surface in 3-dimensional complex Finsler manifold. Where the assumption of smoothness is in the sense that at each point, surface possesses a tangent plane.*

we can explore that the second rank covariant tensor f_{kh} describes theory of various curvatures, viz. principal curvatures of a Monge's surface and many more.

Proof. In view of the assumptions A1 and A2, the equation of the surface in complex Finsler manifold in Monge's form can be written as $z_k = F(z, \dot{z})$, where z_k being some complex coordinate. The function $F(z, \dot{z})$ is supposed to be differentiable as many times as desirable. Since the fundamental function $F(z, \dot{z})$ is a positive

homogeneous function of degree one in $\dot{z}^i$, thereby using the well known Euler's theorem on homogeneous function, we have

$$F_{\dot{z}^i}(z,\dot{z}) = F(z,\dot{z}), \tag{42}$$

$$F_{\dot{z}^i\dot{z}^j}(z,\dot{z})\dot{z}^i = 0, \tag{43}$$

where the notations $F_{\dot{z}^i}$ and $F_{\dot{z}^i\dot{z}^j}$ are used to denote the derivatives of $F(z,\dot{z})$ with respect to $\dot{z}^i$ and $\dot{z}^i\dot{z}^j$ respectively.

Clearly, $[F(z,\dot{z})]_{(z,\dot{z}=0)}$ & $[F_{\dot{z}^i}(z,\dot{z})]_{(z,\dot{z}=0)} = 0$. Also, $[F_{\dot{z}^i\dot{z}^j}(z,\dot{z})]_{(z,\dot{z}=0)} = 0$. Thus by definition of fundamental function $F(z,\dot{z})$, we have

$$F(z^i,\dot{z}^i) = [g_{ij}(z^k)dz^i dz^j]^{1/2}, \tag{44}$$

where $g_{ij}(z^k)$ are the coefficients independent of z^i. This metric defined by Eq. (44) is the metric in Riemannian manifold. Moreover, since

$$F(z,\dot{z}) := \frac{1}{2}F^2(z,\dot{z}) \text{ and } g_{ij}(z,\dot{z}) = \frac{\partial^2 F^2(z,\dot{z})}{\partial\dot{z}^i\partial\dot{z}^j},$$

then from Eq. (7), the surface equation can be written as

$$F(z,\dot{z}) = \frac{1}{2}F^2(z,\dot{z}) = \frac{1}{2}g_{ij}(z,\dot{z})d\dot{z}^i d\dot{z}^j = z_k. \tag{45}$$

Now, by our assumption A1, we introduce f_{ij} in place of g_{ij} in Eq. (45), which yields

$$F(z,\dot{z}) = \frac{1}{2}f_{ij}(z,\dot{z})d\dot{z}^i d\dot{z}^j = z_k, \tag{46}$$

where this $f_{ij}(z,\dot{z})$ is equal to $\frac{\partial^2 F^2(z,\dot{z})}{\partial\dot{z}^i\partial\dot{z}^j}$ and hence able to determine the degree of surface at any point P of the manifold.

Further, as the normal section of the Monge's surface $z_k = F(z,\dot{z})$ of given complex Finsler manifold must have greatest and the least curvatures which will be called the principal curvatures. Thus to determine the principal curvatures say κ_a and κ_b, we should determine the eigenvalues (latent roots) of $\|f_{ij}\|(z,\dot{z})$. The latent root equation of the matrix $\|f_{ij}\|(z,\dot{z})$ is written as

$$|f_{ij} - \lambda\delta_{ij}| = 0, \tag{47}$$

where λ is called indeterminate.
Eq. (47) on expansion in usual way yields the following:

$$\begin{aligned}\lambda^3 + \lambda^2(f_{\alpha\alpha}f_{\beta\beta}f_{\gamma\gamma}) + \lambda(f_{\alpha\gamma}f_{\gamma\alpha} + f_{\alpha\beta}f_{\beta\gamma} + f_{\gamma\beta}f_{\beta\gamma} - f_{\beta\beta}f_{\gamma\gamma} - \\ - f_{\alpha\alpha}f_{\gamma\gamma} - f_{\alpha\alpha}f_{\beta\beta}) + (f_{\alpha\alpha}f_{\beta\beta}f_{\gamma\gamma} - f_{\alpha\alpha}f_{\beta\gamma}f_{\gamma\beta} - f_{\alpha\beta}f_{\beta\gamma}f_{\gamma\gamma} + f_{\alpha\beta}f_{\gamma\alpha}f_{\beta\gamma} + \\ + f_{\alpha\gamma}f_{\beta\gamma}f_{\gamma\beta} - f_{\alpha\gamma}f_{\gamma\alpha}f_{\beta\beta}) = 0.\end{aligned} \tag{48}$$

But, as we have given the logic that for our Monge's surface, there may be only two principal curvatures κ_a and κ_b. Then if we vanish one of the index say γ, Eq. (48) will be reduced to the following form:

$$\lambda^2 - \lambda(f_{\alpha\alpha} + f_{\beta\beta}) + (f_{\alpha\alpha}f_{\beta\beta} - f_{\alpha\beta}f_{\beta\alpha}) = 0. \tag{49}$$

This expression produces the following facts:

$$\kappa_a + \kappa_b = (f_{\alpha\alpha} + f_{\beta\beta}) \text{ and } \kappa_a\kappa_b = (f_{\alpha\alpha}f_{\beta\beta} - f_{\alpha\beta}f_{\beta\alpha}) = |f_{ij}|. \tag{50}$$

If the coordinates $d\dot{z}^i$ and $d\dot{z}^j$ involved in the Eq. (46) are measured in the direction of the principal axis, we would have

$$\|f_{ij}\| = \begin{pmatrix} \kappa_a & 0 \\ 0 & \kappa_b \end{pmatrix},$$

and thus

$$z_k = F(z, \dot{z}) = \frac{1}{2}\kappa_a (d\ddot{z}^i)^2 + \frac{1}{2}\kappa_b (d\dot{z}^j)^2 + \cdots \tag{51}$$

Also, in a general direction, the Gaussian curvature would be weighted mean of the minimal and maximal values of κ_a and κ_b. Hence

$$[\mathbb{K}]_P = \kappa_a \kappa_b = |f_{ij}|(z, \dot{z}), \tag{52}$$

while the mean curvature

$$\mu = \frac{1}{2}(\kappa_a + \kappa_b). \tag{53}$$

Thus, we have shown that how a second rank covariant tensor describes the theory of various curvatures. □

We, now, proceed to discuss a special case which would provide a methodology to let us know that whether the underlying complex Finsler manifold is an Einstein one or not.

Theorem 2.8. *The constancy of Gaussian curvature tensor yields the necessary and sufficient condition for a complex Finsler manifold to be an Einstein one.*

Proof. As it is known to us that a manifold with constant Riemannian curvature tensor is an Einstein one. In the similar way, we now check for the constancy of Gaussian curvature Eq. (52) for Finsler manifold. If the Gaussian curvature of the surface $z_k = F(z, \dot{z})$ becomes constant at any point P of the Finsler manifold, we shall say that the Finsler manifold is an Einstein one. For this purpose, we use scalar representation of Gaussian curvature given by Eq. (52) as follows:
The Gaussian curvature of the Finsler manifold defined at any point with respect to a two directions $(z, \dot{z})$ is given by

$$R(z, \dot{z}, Z) = \frac{[K_{ijkh}(z, \dot{z})\dot{z}^i \dot{z}^h Z^j Z^k]}{[g_{ih}(z, \dot{z})g_{jk}(z, \dot{z}) - g_{ij}(z, \dot{z})g_{kh}(z, \dot{z})]\dot{z}^i \dot{z}^h Z^j Z^k}. \tag{54}$$

But Eq. (52) also stands for the Gaussian curvature deduced from the Cartan's first curvature tensor, thereby from Eq. (52) and Eq. (54), we have

$$[\mathbb{K}]_P = [R(z, \dot{z}, Z)]_P = |f_{ij}|(z, \dot{z}) = \frac{[K_{ijhk}(z, \dot{z})]}{[g_{ih}(z, \dot{z})g_{jk}(z, \dot{z}) - g_{ij}(z, \dot{z})g_{kh}(z, \dot{z})]} \tag{55a}$$

$$[\mathbb{K}]_p [g_{ih}(z, \dot{z})g_{jk}(z, \dot{z}) - g_{ij}(z, \dot{z})g_{kh}(z, \dot{z})] = k_{ijhk}(z, \dot{z}) \equiv |f_{ij}|(z, \dot{z}). \tag{55b}$$

Transvecting Eq. (55) with g^{hk} and summing over i and k from 1 to N, we obtain

$$[\mathbb{K}]_p [\delta_i^k g_{jk}(z, \dot{z}) - N g_{ij}(z, \dot{z})] = g^{hk}(z, \dot{z}) K_{ijhk}(z, \dot{z}) \equiv |f_{ij}|(z, \dot{z}),$$

or

$$[\mathbb{K}]_P g_{ij}(z, \dot{z})[1 - N] = K_{ij}(z, \dot{z}) \equiv |f_{ij}|(z, \dot{z}). \tag{56}$$

Again transvecting with g^{ij} and summing over i and j from 1 to N, we get

$$[\mathbb{K}]_P g_{ij}(z, \dot{z}) g^{ij}(z, \dot{z})[1 - N] = K_{ij}(z, \dot{z}) g^{ij}(z, \dot{z}) \equiv |f_{ij}|(z, \dot{z}),$$

which implies

$$N(1-N)[\mathbb{K}]_P = K(z,\dot{z}) \equiv |f_{ij}|(z,\dot{z}).$$

Substituting $[\mathbb{K}]_P = \frac{K(z,\dot{z})}{N(1-N)} \equiv |f_{ij}|(z,\dot{z})$ in Eq. (56), we obtain

$$|f_{ij}|(z,\dot{z}) \equiv K_{ij}(z,\dot{z}) = \frac{K(z,\dot{z})}{N} g_{ij}(z,\dot{z}), \tag{57}$$

which is the necessary and sufficient condition for a complex Finsler manifold to be an Einstein one. □

2.2. **Technique 2: Decomposition of Cartan's I-curvature Tensor Field by means of Eigenvalue-Eigenvector Method.** As from the standpoint of various mathematical and engineering applications, eigenvalue problems are among the most crucial problems in connection with matrices and tensors. Also, the study of such latentroot problems in quantum mechanics is highly insisted due to having *spectrum* (a set of eigenvalues) and the *spectral radii* (the largest of the absolute values of latentroots) of any tensor field. Further it is known that a set of special vectors and scalar values, customarily called eigenvectors and eigenvalues are associated with second rank tensors. Various analysis and visualization techniques use such sets of latentroots and latentvectors and are particularly crucial in the visualization of topological structures of and tensor field.

The eigenvectors of a tensor have the property that when the inner product of the original tensor and an eigenvector is taken, the consequence will be a vector which is a scalar multiple of the original eigenvector. That is if T is any tensor and X is its eigenvector then $TX = \lambda X$, where λ are the solutions of this equation and are the eigenvalues of T.

Here we again consider the open product given by Eq. (25) and use the technique (2)[22, 18, 13, 14, 23, 21, 15, 12] separately for each of the tensor residing to the right hand side in Eq. (25).

In case of 3-dimensional complex Finsler manifold, we have the Einstein's tensor $G^i_j(z,\dot{z})$ in matrix form as follows:

$$\|G^i_j\|(z,\dot{z}) = \begin{pmatrix} G^\alpha_\alpha & G^\alpha_\beta & G^\alpha_\gamma \\ G^\beta_\alpha & G^\beta_\beta & G^\beta_\gamma \\ G^\gamma_\alpha & G^\gamma_\beta & G^\gamma_\gamma \end{pmatrix}_{i,j=\alpha,\beta,\gamma} \tag{58}$$

Here, because of the manifold under consideration being complex Finsler, it is evident that each of the nine components of the above matrix will be the functions of so called element of support $(z,\dot{z})$ and hence will be the complex entries.
Now, by definition of latentroot-latentvector approach, we consider an eigenvector X_i having the components $X_\alpha, X_\beta, X_\gamma$ such that the characteristic equation for the given Einstein's tensor becomes

$$G^i_j X_i = \lambda X_j \text{ or,} \tag{59a}$$

$$(G^i_j - \lambda\delta^i_j) \tag{59b}$$

If we omit the null vector $X_i = 0$, the Eq. (59) implies

$$|G^i_j - \lambda\delta^i_j| = 0, \tag{60}$$

which for a 3-dimensional complex Finsler-Einstein's tensor expands to

$$D(\lambda) = \begin{pmatrix} G^\alpha_\alpha - \lambda & G^\alpha_\beta & G^\alpha_\gamma \\ G^\beta_\alpha & G^\beta_\beta - \lambda & G^\beta_\gamma \\ G^\gamma_\alpha & G^\gamma_\beta & G^\gamma_\gamma - \lambda \end{pmatrix} = 0, \tag{61}$$

where $D(\lambda)$ stands for the characteristic determinant.
Now, using Crammer's rule and simplifying the Eq. (61), we obtain

$$D(\lambda) = \lambda^3 - I^1\lambda^2 + I^2\lambda - I^3 = 0, \tag{62}$$

where I^1, I^2 and I^3 are the invariants defined by relations

$$I^1 = G^i_i = (G^\alpha_\alpha + G^\beta_\beta + G^\gamma_\gamma) \tag{63a}$$

$$I^2 = \frac{1}{2}(G^i_i G^j_j - G^i_j G^j_i) = \left(\begin{vmatrix} G^\alpha_\alpha & G^\alpha_\beta \\ G^\beta_\alpha & G^\beta_\beta \end{vmatrix} + \begin{vmatrix} G^\alpha_\alpha & G^\alpha_\gamma \\ G^\gamma_\alpha & G^\gamma_\gamma \end{vmatrix} + \begin{vmatrix} G^\beta_\beta & G^\beta_\gamma \\ G^\gamma_\beta & G^\gamma_\gamma \end{vmatrix} \right), \tag{63b}$$

$$I^3 = e_{ijk} G^i_i G^j_j G^k_k = \begin{vmatrix} G^\alpha_\alpha & G^\alpha_\beta & G^\alpha_\gamma \\ G^\beta_\alpha & G^\beta_\beta & G^\beta_\gamma \\ G^\gamma_\alpha & G^\gamma_\beta & G^\gamma_\gamma \end{vmatrix} \tag{63c}$$

In view of the above decomposition formulae, we now proceed to discuss an important theorem as below:

Theorem 2.9. *The characteristic equation (62) of Einstein's tensor G^i_j will have*

- **i:** *mixed type (i.e. purely real as well as complex) latentroots if the Einstein's tensor is self-conjugate*
- **ii:** *purely real latentroots if the Einstein's tensor is pure*
- **iii:** *purely complex latentroots if the Einstein's tensor is hybrid*

Proof. If we employ the basic feature of self-conjugacy of Einstein's tensor $G^i_j(z, \dot{z})$, i.e. the this tensor is self-conjugate if [9]:

$$G^i_j(z, \dot{z}) = \begin{pmatrix} G^\mu_\nu & G^{\bar{\mu}}_{\bar{\nu}} \\ G^\mu_{\bar{\nu}} & G^{\bar{\mu}}_{\bar{\nu}} \end{pmatrix}, \tag{64}$$

which implies

$$\bar{G}^i_j(z, \dot{z}) = G^{\bar{i}}_{\bar{j}}(z, \dot{z}). \tag{65}$$

Then each components of the matrix given by Eq. (58) will satisfy the relation (65).

Applying the above methodology to our characteristic equation (62), we have

$$\begin{aligned} D(\lambda) = \lambda^3 - G^i_i(z, \dot{z})\lambda^2 + \frac{\lambda}{2}[G^i_i(z, \dot{z})G^j_j(z, \dot{z}) - G^i_j(z, \dot{z})G^j_i(z, \dot{z})] \\ - e_{ijk} G^i_i(z, \dot{z}) G^j_j(z, \dot{z}) G^k_k(z, \dot{z}) = 0, \end{aligned} \tag{66}$$

where we have substituted the values of invariants I^1, I^2 and I^3 from Eq. (63a), Eq. (63b) and Eq. (63c) respectively.

Applying Eq. (64) to Eq. (66), we obtain

$$(67)\quad D(\lambda) = \lambda^3 - \lambda^2 \begin{pmatrix} G^{\mu}_{\mu} & G^{\bar{\mu}}_{\mu} \\ G^{\mu}_{\bar{\mu}} & G^{\bar{\mu}}_{\bar{\mu}} \end{pmatrix} + \frac{\lambda}{2}[\begin{pmatrix} G^{\mu}_{\mu} & G^{\bar{\mu}}_{\mu} \\ G^{\mu}_{\bar{\mu}} & G^{\bar{\mu}}_{\bar{\mu}} \end{pmatrix} \begin{pmatrix} G^{\nu}_{\nu} & G^{\bar{\nu}}_{\nu} \\ G^{\nu}_{\bar{\nu}} & G^{\bar{\nu}}_{\bar{\nu}} \end{pmatrix}$$
$$- \begin{pmatrix} G^{\mu}_{\nu} & G^{\bar{\mu}}_{\nu} \\ G^{\mu}_{\bar{\nu}} & G^{\bar{\mu}}_{\bar{\nu}} \end{pmatrix} \begin{pmatrix} G^{\nu}_{\mu} & G^{\bar{\nu}}_{\mu} \\ G^{\nu}_{\bar{\mu}} & G^{\bar{\nu}}_{\bar{\mu}} \end{pmatrix}] - e_{ijk}(z,\dot{z})[\begin{pmatrix} G^{\mu}_{\mu} & G^{\bar{\mu}}_{\mu} \\ G^{\mu}_{\bar{\mu}} & G^{\bar{\mu}}_{\bar{\mu}} \end{pmatrix} \begin{pmatrix} G^{\nu}_{\nu} & G^{\bar{\nu}}_{\nu} \\ G^{\nu}_{\bar{\nu}} & G^{\bar{\nu}}_{\bar{\nu}} \end{pmatrix} \times$$
$$\begin{pmatrix} G^{\xi}_{\xi} & G^{\bar{\xi}}_{\xi} \\ G^{\xi}_{\bar{\xi}} & G^{\bar{\xi}}_{\bar{\xi}} \end{pmatrix}] = 0.$$

Simplifying Eq. (67) by keeping Eq. (65) in mind, we have

$$(68)\quad D(\lambda) = \lambda^3 - \lambda^2(G^{\mu}_{\mu}\bar{G}^{\mu}_{\mu} - G^{\bar{\mu}}_{\mu}G^{\mu}_{\bar{\mu}}) + \frac{\lambda}{2}[(G^{\mu}_{\mu}\bar{G}^{\mu}_{\mu} - G^{\bar{\mu}}_{\mu}G^{\mu}_{\bar{\mu}})$$
$$\times (G^{\nu}_{\nu}\bar{G}^{\nu}_{\nu} - G^{\bar{\nu}}_{\nu}G^{\mu}_{\bar{\nu}}) - (G^{\mu}_{\nu}\bar{G}^{\nu}_{\nu} - G^{\bar{\mu}}_{\nu}G^{\mu}_{\bar{\nu}})(G^{\nu}_{\mu}\bar{G}^{\nu}_{\mu} - G^{\bar{\nu}}_{\mu}G^{\nu}_{\bar{\mu}})] - e_{ijk}(z,\dot{z})\times$$
$$[(G^{\mu}_{\mu}\bar{G}^{\mu}_{\mu} - G^{\bar{\mu}}_{\mu}G^{\mu}_{\bar{\mu}})(G^{\nu}_{\nu}\bar{G}^{\nu}_{\nu} - G^{\bar{\nu}}_{\nu}G^{\nu}_{\bar{\nu}})(G^{\xi}_{\xi}\bar{G}^{\xi}_{\xi} - G^{\bar{\xi}}_{\xi}G^{\xi}_{\bar{\xi}})] = 0.$$

From Eq. (68), one can immediately conclude that the eigenvalues and hence the corresponding eigenvectors of the Einstein's tensor $G^i_j(z,\dot{z})$ will be of mixed type, i.e. some of them will be purely real and some will be complex.

We, now, consider the case (ii) when Einstein's tensor is *pure*, i.e. the tensor $G^i_j(z,\dot{z})$ possesses the components of the form:

$$(69)\qquad G^i_j(z,\dot{z}) = \begin{pmatrix} G^{\mu}_{\nu} & 0 \\ 0 & G^{\bar{\mu}}_{\bar{\nu}} \end{pmatrix}$$

Then under this condition, the characteristic Eq. (66) for Einstein's tensor yields

$$(70)\quad D(\lambda) = \lambda^3 - \lambda^2 \begin{pmatrix} G^{\mu}_{\mu} & 0 \\ 0 & G^{\bar{\mu}}_{\bar{\mu}} \end{pmatrix} + \frac{\lambda}{2}[\begin{pmatrix} G^{\mu}_{\mu} & 0 \\ 0 & G^{\bar{\mu}}_{\bar{\mu}} \end{pmatrix} \begin{pmatrix} G^{\nu}_{\nu} & 0 \\ 0 & G^{\bar{\nu}}_{\bar{\nu}} \end{pmatrix}$$
$$- \begin{pmatrix} G^{\mu}_{\nu} & 0 \\ 0 & G^{\bar{\mu}}_{\bar{\nu}} \end{pmatrix} \begin{pmatrix} G^{\nu}_{\mu} & 0 \\ 0 & G^{\bar{\nu}}_{\bar{\mu}} \end{pmatrix}] - e_{ijk}(z,\dot{z})[\begin{pmatrix} G^{\mu}_{\mu} & 0 \\ 0 & G^{\bar{\mu}}_{\bar{\mu}} \end{pmatrix} \begin{pmatrix} G^{\nu}_{\nu} & 0 \\ 0 & G^{\bar{\nu}}_{\bar{\nu}} \end{pmatrix} \times$$
$$\begin{pmatrix} G^{\xi}_{\xi} & 0 \\ 0 & G^{\bar{\xi}}_{\bar{\xi}} \end{pmatrix}] = 0.$$

Again in view of the self-conjugacy condition (65), the Eq. (70) implies

$$(71)\quad D(\lambda) = \lambda^3 - \lambda^2(G^{\mu}_{\mu}\bar{G}^{\mu}_{\mu}) + \frac{\lambda}{2}[(G^{\mu}_{\mu}\bar{G}^{\mu}_{\mu})(G^{\nu}_{\nu}\bar{G}^{\nu}_{\nu}) - (G^{\mu}_{\nu}\bar{G}^{\mu}_{\nu})\times$$
$$(G^{\nu}_{\mu}\bar{G}^{\nu}_{\mu})] - e_{ijk}(z,\dot{z})[(G^{\mu}_{\mu}\bar{G}^{\mu}_{\mu})(G^{\nu}_{\nu}\bar{G}^{\nu}_{\nu})(G^{\xi}_{\xi}\bar{G}^{\xi}_{\xi})] = 0,$$

which evidently shows that under purity condition of $G^i_j(z,\dot{z})$, it's latentroots and hence the corresponding latentvectors will be purely real.

Taking account of the case (iii), when components of Einstein's tensor become hybrid, i.e. when

$$(72)\qquad G^i_j(z,\dot{z}) = \begin{pmatrix} 0 & G^{\bar{\mu}}_{\nu} \\ G^{\mu}_{\bar{\nu}} & 0 \end{pmatrix}.$$

Under the *hybrid* nature Eq. (72), the characteristic determinant Eq. (66) will take the form:

$$(73)\quad D(\lambda) = \lambda^3 - \lambda^2 \begin{pmatrix} 0 & G^{\bar{\mu}}_{\mu} \\ G^{\mu}_{\bar{\mu}} & 0 \end{pmatrix} + \frac{\lambda}{2}[\begin{pmatrix} 0 & G^{\bar{\mu}}_{\mu} \\ G^{\mu}_{\bar{\mu}} & 0 \end{pmatrix} \begin{pmatrix} 0 & G^{\bar{\nu}}_{\nu} \\ G^{\nu}_{\bar{\nu}} & 0 \end{pmatrix} - \begin{pmatrix} 0 & G^{\bar{\mu}}_{\nu} \\ G^{\mu}_{\bar{\nu}} & 0 \end{pmatrix} \begin{pmatrix} 0 & G^{\bar{\nu}}_{\mu} \\ G^{\nu}_{\bar{\mu}} & 0 \end{pmatrix}] - e_{ijk}(z,\dot{z})[\begin{pmatrix} 0 & G^{\bar{\mu}}_{\mu} \\ G^{\mu}_{\bar{\mu}} & 0 \end{pmatrix} \begin{pmatrix} 0 & G^{\bar{\nu}}_{\nu} \\ G^{\nu}_{\bar{\nu}} & 0 \end{pmatrix} \times \begin{pmatrix} 0 & G^{\bar{\xi}}_{\xi} \\ G^{\xi}_{\bar{\xi}} & 0 \end{pmatrix}] = 0.$$

In view of self-conjugacy, the Eq. (73) yields

$$(74)\quad D(\lambda) = \lambda^3 - \lambda^2(G^{\bar{\mu}}_{\mu}G^{\mu}_{\bar{\mu}}) + \frac{\lambda}{2}[(G^{\bar{\mu}}_{\mu}G^{\mu}_{\bar{\mu}})(G^{\bar{\nu}}_{\nu}G^{\mu}_{\bar{\nu}}) - (G^{\bar{\mu}}_{\nu}G^{\mu}_{\bar{\nu}})\times (G^{\bar{\nu}}_{\mu}G^{\nu}_{\bar{\mu}})] - e_{ijk}(z,\dot{z})[(G^{\bar{\mu}}_{\mu}G^{\mu}_{\bar{\mu}})(G^{\bar{\nu}}_{\nu}G^{\nu}_{\bar{\nu}})(G^{\bar{\xi}}_{\xi}G^{\xi}_{\bar{\xi}})] = 0.$$

From Eq. (74), its obvious that the latentroots as well as the corresponding latentvectors of Einstein's tensor will be purely complex. □

Now, by observing the proof of Theorem (2.9), we can easily estimate the ticklishness of the process of checking pure and hybrid nature of Einstein's tensor. The process of checking seems to be quite lengthy, because under this process one would need to check the pure and hybrid nature for each of the components of matrix given by Eq. (58). Therfore, for such a purpose, we shall utilize the two well known operators O^{sh}_{ir} and $*O^{sh}_{ir}$ defined by [9] (page 133). The O and $*O$ operators are defined as follows:

$$(75a)\qquad O^{sh}_{ir} = \frac{1}{2}(\delta^s_i\delta^h_r - F^s_iF^h_r),$$

$$(75b)\qquad *O^{sh}_{ir} = \frac{1}{2}(\delta^s_i\delta^h_r + F^s_iF^h_r).$$

Moreover the operators O and $*O$ satisfy the following relations:

$$(76a)\qquad O + *O = A, \text{ where A being the identity operator,}$$

$$(76b)\qquad O.O = O,$$

$$(76c)\qquad O.*O = O,$$

$$(76d)\qquad *O.O = O,$$

$$(76e)\qquad *O.*O = *O.$$

To omit the ambiguity regarding pure or hybrid nature of Einstein's tensor, we now use the above cited properties of operators O and $*O$.

Definition 2.10. *The Einstein,s tensor $G^i_j(z,\dot{z})$ is declared to be pure or hybrid according to the following facts:*

1: *$G^i_j(z,\dot{z})$ is pure if $OG^i_j(z,\dot{z}) = G^i_j(z,\dot{z})$ or $*OG^i_j(z,\dot{z}) = 0$,*
2: *$G^i_j(z,\dot{z})$ is hybrid if $OG^i_j(z,\dot{z}) = 0$ or $*OG^i_j(z,\dot{z}) = G^i_j(z,\dot{z})$*

.

In view of the Definition (2.10), we now claim the following:

Proposition 2.11. *Einstein's tensor $G^i_j(z,\dot{z})$ is hybrid in nature.*

Proof. The proof of the proposition follows directly from the Definition (2.10) and the relations (75a) and (75b).
Applying Eq. (75a) and Eq. (75b) to $G^i_j(z,\dot{z})$ one by one, we have

$$O^{si}_{jr}G^r_s(z,\dot{z}) = \frac{1}{2}(\delta^s_j\delta^i_r - F^s_jF^i_r)G^r_s(z,\dot{z}) = \frac{1}{2}(\delta^s_j\delta^i_rG^r_s - \\ - F^s_jF^i_rG^r_s)(z,\dot{z}) = \frac{1}{2}(G^i_j - G^i_j)(z,\dot{z}) = 0 \tag{77}$$

and

$$*\,O^{si}_{jr}G^r_s(z,\dot{z}) = \frac{1}{2}(\delta^s_j\delta^i_r + F^s_jF^i_r)G^r_s(z,\dot{z}) = \frac{1}{2}(\delta^s_j\delta^i_rG^r_s + \\ + F^s_jF^i_rG^r_s)(z,\dot{z}) = \frac{1}{2}(G^i_j + G^i_j)(z,\dot{z}) = G^i_j(z,\dot{z}). \tag{78}$$

Now, comparing Eq. (77) and Eq. (78) with Definition (2.10), we conclude that Einstein's tensor is hybrid.

Moreover, we can predict the hybrid nature of Einstein's tensor by considering the work of [9], who has been verified that the fundamental metric tensor g_{ij} and the Ricci tensor R_{ij} both are hybrid in i and j. As the Einstein's tensor is the composition of Ricci tensor and metric tensor, so evidently it is hybrid and hence will have purely complex latentroots as well as latentvectors. □

Eventually, it remains to discuss eigenvalue-eigenvector decomposition for the second tensor $f_{kh}(z,\dot{z})$ (which is preassumed to be the degree of curvature) of Cartan's I-curvature tensor given by Eq. (25). Applying similar cases and conditions which have been applied for the Einstein's tensor in the preceeding sections, we can easily decompose $f_{kh}(z,\dot{z})$ and we can lucidly observe that the decomposition of this tensor has almost similar results as calculated for Einstein's tensor.

The eigenvalue-eigenvector decomposition of $f_{kh}(z,\dot{z})$ yields some great geometrical significances. Some of the significances are discussed as follows:

2.3. **Geometric configuration of $f_{kh}(z,\dot{z})$ and its latentroots-latentvectors.** Likewise the planar geometry of vectors, the second rank covariant, contravariant or mixed tensors have the natural geometries in the form of quadric surfaces. Generally the surface which is represented by general equation of second degree in x, y and z is called quadric surface or conicoid and is defined as

$$ax + by + cz + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + \\ + d = 0, \tag{79}$$

which can be reduced to any standard form like ellipsoid, hyperboloid of one sheet and two sheets and elliptic paraboloid etc.
In the similar way, we can write the quadratic surface of $f_{kh}(z,\dot{z})$ as

$$[r_k.f_{kh}.r_h] = 1, \quad \text{with} \quad \|f_{kh}(z,\dot{z})\| = \begin{pmatrix} f_{\alpha\alpha} & f_{\alpha\beta} & f_{\alpha\gamma} \\ f_{\beta\alpha} & f_{\beta\beta} & f_{\beta\gamma} \\ f_{\gamma\alpha} & f_{\gamma\beta} & f_{\gamma\gamma} \end{pmatrix}_{k,h=\alpha,\beta,\gamma}. \tag{80}$$

Expansion of the determinant of $f_{kh}(z,\dot{z})$ yields

$$f_{\alpha\alpha}z^2_\alpha + f_{\beta\beta}z^2_\beta + f_{\gamma\gamma}z^2_\gamma + (f_{\alpha\beta} + f_{\beta\alpha})z_\alpha z_\beta + (f_{\alpha\gamma} + f_{\gamma\alpha})z_\alpha z_\gamma + \\ + (f_{\beta\gamma} + f_{\gamma\beta})z_\beta z_\gamma = 1, \tag{81}$$

which are a subset of quadric surface and can be reduced to ellipsoid, real or imaginary elliptic cylinders, hyperboloid etc. If we consider only the symmetry of $f_{kh}(z,\dot{z})$, then $\exists$ a $1-1$ correspondence between $f_{kh}(z,\dot{z})$ and the quadric surface. Even for a general tensor, there is a symmetric plus a family of non-symmetric tensors which produce the same surface. This fact is due to the components $f_{\alpha\beta}+f_{\beta\alpha}$, $f_{\alpha\gamma}+f_{\gamma\alpha}$ and $f_{\beta\gamma}+f_{\gamma\beta}$ involved in Eq. (81).

If in any coordinate system $f_{kh}(z,\dot{z})$ becomes diagonalized, the Eq. (81) of quadric surface reduces to the form:

$$\lambda_1 z_\alpha^2+\lambda_2 z_\beta^2+\lambda_3 z_\gamma^2=1, \tag{82}$$

which is again a quadric surface having its geometric axes aligned with that specific coordinate system and of course the eigenvectors.

If all the latentroots are distinct, then clearly latentvectors will be orthogonal. If the latentroots are positive as well as distinct, the tensor $f_{kh}(z,\dot{z})$ is geometrically an ellipsoid with a circular cross section with two of its axes equal in length.

If all the eigenvalues are equal, the tensor $f_{kh}(z,\dot{z})$ is geometrically a sphere. Also, there is a relation among the invariants I^1, I^2 and I^3 of $f_{kh}(z,\dot{z})$ and its eigenvalues say λ_1,λ_2 and λ_3 given by

$$I^1=f_{kk}(z,\dot{z})=(f_{\alpha\alpha}+f_{\beta\beta}+f_{\gamma\gamma})(z,\dot{z})=\lambda_1+\lambda_2+\lambda_3, \tag{83a}$$

$$I^2=\frac{1}{2}(f_{kk}f_{hh}-f_{kh}f_{hk})=\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1, \tag{83b}$$

$$I^3=e_{khl}(f_{kk}f_{hh}f_{ll})=\lambda_1\lambda_2\lambda_3. \tag{83c}$$

These are called principal invariants of $f_{kh}(z,\dot{z})$ and geometrically concerned with the quadric surface given by Eq. (80).

3. Results and discussions

Here is the brief discussion over some vital results obtained from our article written in favor of decomposition techniques.

- Whatever be the rank of tensor, it can be firstly factorize into arbitrary number of tensors by means of open product so that each individual tensor could be decomposed with simplest decomposition techniques without seeking the higher order SVD techniques. Though the process may go on quite lengthy, but with this, checking actual geometric configurations of original tensor could seem rather convenient. As in our case the factorization of Cartan's first curvature tensor evolves two very surprising components namely Einstein's tensor and degree of curvature and both of them are widely used in the analysis of geometry of gravitation and differential geometry of curved surfaces.
- By adopting the process of covariant differentiation given by equation (16), for the symmetric and anti-symmetric part of Einstein's tensor, we have developed an expression, which most probably predicts some complicated relations among the components of Einstein's tensor and various tensorial and non-tensorial quantities as given by equation (30). Of course, this relation is vitally important as it describes direct or indirect correlation between components of complex Einstein's tensor and Christoffel's second kind bracket symbol, Ricci tensor, scalar curvature, Riemann metric tensor as well as a third order tensor $C^p_{rs}(z,\dot{z})$ etc.

- By introducing an energy-momentum tensor and after then Euler's well known stationary integrability condition, we have derived Einstein's field equation. Thus by first decomposition technique, we have shown that Cartan's curvature tensor is able producing Einstein's field equation and hence applicable to induce features of complex Finsler manifolds in Einstein's manifold.
- There is given a special case which evokes that if the energy momentum tensor and hence the components of Ricci's tensor vanishes the complex Finsler manifold reduces to a special manifold which should be called Ricci flat complex Finsler manifold.
- We exposed the second factorized part of Cartan's first curvature tensor in such a way that it describes the degree of curvature of smooth complex Finsler surface. The surface is assumed to be Monge's surface which will have greatest and least curvatures called principal curvatures. By decomposing second factorized part, we have calculated such principal curvatures for our Monge's surface.
- After calculating the latentroots of second factorized component of Cartan's first curvature tensor, we have calculated the Gaussian and mean curvatures of the Monge's surface. Thus we have shown that it is possible to derive the Gaussian, mean and principal curvatures from the given Cartan's curvature tensor.
- We have illustrated that if the Gaussian curvature tensor at any point of the Monge's surface becomes constant, then the complex Finsler manifold will turn into complex Einstein's manifold.
- In order to check the nature of latentroots/latentvector of Cartan's first curvature tensor using technique 2 we have introduced the case of self conjugacy, case of purity and hybridness of Einstein's tensor. Thus by merely checking the nature of latent roots/latent vectors, one can easily discuss the metric signatures and hence the complete characteristics of complex Finsler manifolds.
- For the feasibility and to avoid the heavy calculations, we have introduced the Yano's O and $*O$ operators and shown that what would be the effects of pure and hybrid nature of any tensor over its latentroots/latentvectors.
- We have given the geometric significance of our second factorized component of Cartans I-curvature tensor using technique 2. Here we have introduces an equation for quadric surface involving second factorized component and after checking its latent roots/latent vectors have discussed the geometric configurations generated by this quadric surface.

Remark 3.1. *The rest two techniques for decomposition will be studied in the next manuscript.*

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