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STABILITY OF CUBIC FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN $\mathcal{L}$-FUZZY NORMED SPACES

N. EGHBALI

ABSTRACT. We determine some stability results concerning the cubic functional equation in non-Archimedean fuzzy normed spaces. Our result can be regarded as a generalization of the stability phenomenon in the framework of $\mathcal{L}$-fuzzy normed spaces.

1. Introduction

The stability of functional equations is an interesting area of research for mathematicians, but it can be also of importance to persons who work outside of the realm of pure mathematics.

It seems that the stability problem of functional equations had been first raised by Ulam [18]. Moreover the approximated mappings have been studied extensively in several papers. (See for instance [14], [6], [3], and [4]).

Fuzzy notion introduced firstly by Zadeh [19] that has been widely involved in different subjects of mathematics. Zadeh's definition of a fuzzy set characterized by a function from a nonempty set X to $[0, 1]$. Goguen in [5] generalized the notion of a fuzzy subset of X to that of an $\mathcal{L}$-fuzzy subset, namely a function from X to a lattice L.

Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in the field of science and engineering.

Later in 1984, Katsaras [9] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space.

With [10] and by modifying the definition of a fuzzy normed space in [2], Mirmostafaee and Moslehian in [12] introduced a notion of a non-Archimedean fuzzy normed space. Shekari et al. ([16]) considered the quadratic functional equation in $\mathcal{L}$-fuzzy normed space. Also Saadati and Park considered the $f(lx+y)+f(lx-y) = 2l^2f(x) + 2f(y)$ and proved the Hyers-Ulam-Rasssias stability of this equation in $\mathcal{L}$-fuzzy normed spaces ([17]).

The stability problem for the cubic functional equation was proved by Jun and Kim [7] for mappings $f : X \to Y$, where X is a real normed space and Y is a Banach space. Later on, in ([8],[11],[13]) the problem of stability of some cubic equation were discussed.

Defining the class of approximate solutions of a given functional equation one can ask whether every mapping from this class can be somehow approximated

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by an exact solution of the considered equation in the non-Archimedean $\mathcal{L}$-fuzzy normed spaces. To answer this question, we establish a non-Archimedean $\mathcal{L}$-fuzzy Hyers-Ulam-Rassias stability of the cubic functional equation $f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)$. Then we define the non-Archimedean fuzzy continuity of the cubic mappings and we investigate the continuity of approximate cubic mappings.

2. **Preliminaries**

In this section, we provide a collection of definitions and related results which are essential and used in the next discussions.

Definition 2.1. Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $t, s \in \mathbb{R}$,
(N1) $N(x,c) = 0$ for $c \leq 0$;
(N2) $x = 0$ if and only if $N(x,c) = 1$ for all $c > 0$;
(N3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y, s+t) \geq min\{N(x,s), N(y,t)\}$;
(N5) $N(x,.)$ is a non-decreasing function on $\mathbb{R}$ and $lim_{t\to\infty} N(x,t) = 1$;
(N6) for $x \neq 0$, $N(x,.)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed linear space.

Definition 2.2. Let (X,N) be a fuzzy normed linear space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - lim_{n\to\infty} x_n = x$.

Definition 2.3. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is known that every convergent sequence in a fuzzy normed space is Cauchy and if each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and furthermore the fuzzy normed space is called a fuzzy Banach space.

Definition 2.4. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a t-norm if it satisfies the following conditions:
($*1$) $*$ is associative,
($*2$) $*$ is commutative,
($*3$) $a * 1 = a$ for all $a \in [0,1]$ and
($*4$) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Definition 2.5. ([5]). Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice and let U be a non-empty set called the universe. An $\mathcal{L}$-fuzzy set in U is defined as a mapping $\mathcal{A} : U \to L$. For each u in U, $\mathcal{A}(u)$ represents the degree (in L) to which u is an element of $\mathcal{A}$.

Definition 2.6. ([1]). A t-norm on $\mathcal{L}$ is a mapping $*_L : L^2 \to L$ satisfying the following conditions:
(i) $(\forall x \in L)(x *_L 1_{\mathcal{L}} = x)$(: boundary condition);

(ii) $(\forall (x,y) \in L^2)(x *_L y = y *_L x)$(: commutativity);
(iii) $(\forall (x,y,z) \in L^3)(x *_L (y *_L z)) = ((x *_L y) *_L z)$(: associativity);
(iv) $(\forall (x,y,z,w) \in L^4)(x \leq_L x' \text{ and } y \leq_L y' \Rightarrow x *_L y \leq_L x' *_L y')$(: monotonicity).

A t-norm $*_L$ on $\mathcal{L}$ is said to be continuous if, for any $x, y \in \mathcal{L}$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converges to x and y, respectively, $lim_{n\to\infty}(x_n *_L y_n) = x *_L y$.

Definition 2.7. The triple $(V, \mathcal{P}, *_L)$ is said to be an $\mathcal{L}$-fuzzy normed space if V is vector space, $*_L$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{P}$ is an $\mathcal{L}$-fuzzy set on $V \times (0, \infty)$ satisfying the following conditions:
for all $x, y \in V$ and $t, s \in (0, \infty)$,
(a) $\mathcal{P}(x,t) >_L 0_{\mathcal{L}}$;
(b) $\mathcal{P}(x,t) = 1_{\mathcal{L}}$ if and only if $x = 0$;
(c) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
(d) $\mathcal{P}(x,t) *_L \mathcal{P}(y,s) \leq_L \mathcal{P}(x+y, t+s)$;
(e) $\mathcal{P}(x,t) : (0,\infty) \to L$ is continuous;
(f) $lim_{t\to 0}\mathcal{P}(x,t) = 0_{\mathcal{L}}$ and $lim_{t\to\infty}\mathcal{P}(x,t) = 1_{\mathcal{L}}$.
In this case, $\mathcal{P}$ is called an $\mathcal{L}$-fuzzy norm.

Definition 2.8. A negator on $\mathcal{L}$ is any decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$.

Definition 2.9. If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L$, then $\mathcal{N}$ is called an involutive negator.

In this paper, the involutive negator $\mathcal{N}$ is fixed.

Definition 2.10. A sequence (x_n) in an $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, *_L)$ is called a Cauchy sequence if, for each $\varepsilon \in L - \{0_{\mathcal{L}}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$, $\mathcal{P}(x_n - x_m, t) >_L \mathcal{N}(\varepsilon)$, where $\mathcal{N}$ is a negator on $\mathcal{L}$.

A sequence (x_n) is said to be convergent to $x \in V$ in the $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, *_L)$, if $\mathcal{P}(x_n - x, t) \to 1_{\mathcal{L}}$, whenever $n \to +\infty$ for all $t > 0$.

An $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, *_L)$ is said to be complete if and only if every Cauchy sequence in V is convergent.

Definition 2.11. Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $|.| : \mathbb{K} \to \mathbb{R}$ such that for any $a, b \in \mathbb{K}$ we have
(1) $|a| \geq 0$ and equality holds if and only if $a = 0$,
(2) $|ab| = |a||b|$,
(3) $|a + b| \leq max\{|a|, |b|\}$.

Note that $|n| \leq 1$ for each integer n. We always assume, in addition, that $|.|$ is non-tivial, i.e., there exists an $a_0 \in \mathbb{K}$ such that $|a_0| \neq 0, 1$.

Definition 2.12. A non-Archimedean $\mathcal{L}$-fuzzy normed space is a triple $(V, \mathcal{P}, *_L)$, where V is a vector space, $*_L$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{P}$ is an $\mathcal{L}$-fuzzy set on $V \times (0, +\infty)$ satisfying the following conditions:
for all $x, y \in V$ and $t, s \in (0, \infty)$,
(a) $\mathcal{P}(x,t) >_L 0_{\mathcal{L}}$;
(b) $\mathcal{P}(x,t) = 1_{\mathcal{L}}$ if and only if $x = 0$;
(c) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$;
(d) $\mathcal{P}(x,t) *_L \mathcal{P}(y,s) \leq_L \mathcal{P}(x+y, max\{t,s\})$;

(e) $\mathcal{P}(x,.) : (0,\infty) \to L$ is continuous;
(f) $lim_{t\to 0}\mathcal{P}(x,t) = 0_{\mathcal{L}}$ and $lim_{t\to\infty}\mathcal{P}(x,t) = 1_{\mathcal{L}}$.

3. Stability of cubic equation in $\mathcal{L}$-fuzzy normed spaces

Let $\mathbb{K}$ be a non-Archimedean field, X a vector space over $\mathbb{K}$ and $(Y, \mathcal{P}, *_L)$ a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathbb{K}$.

In this section we investigate the cubic functional equation. We define an $\mathcal{L}$-fuzzy approximately cubic mapping. Let Ψ be an $\mathcal{L}$-fuzzy set on $X \times X \times [0,\infty)$ such that $\Psi(x, y, .)$ is nondecreasing,

$$\Psi(cx, cx, t) \geq_L \Psi(x, x, \tfrac{t}{|c|}),\ \forall x \in X,\ c \neq 0$$

and

$$lim_{t\to\infty}\Psi(x, y, t) = 1_{\mathcal{L}},\ \forall x, y \in X,\ t > 0.$$

Troughs this paper, we show the $a_1 *_L a_2 *_L ... *_L a_n$ by $\prod_{j=1}^{n} a_j$.

Definition 3.1. A mapping $f : X \to Y$ is said to be Ψ-approximately cubic if

$$\mathcal{P}(f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x), t) \geq_L \Psi(x, y, t),\ \forall x, y \in X, t > 0. \tag{3.1}$$

Theorem 3.2. *Let $\mathbb{K}$ be a non-Archimedean field, X a vector space over $\mathbb{K}$ and $(Y, \mathcal{P}, *_L)$ a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathbb{K}$. Let $f : X \to Y$ be a Ψ-approximately cubic mapping. Suppose that $f(0) = 0$. If there exist an $\alpha \in \mathbb{R}$ $(\alpha > 0)$ and an integer k, $k \geq 2$ with $|2^k| < \alpha$ and $|2| \neq 0$ such that*

$$\Psi(2^{-k}x, 2^{-k}y, t) \geq_L \Psi(x, y, \alpha t),\ \forall x \in X, t > 0, \tag{3.2}$$

and

$$lim_{n\to\infty}\prod_{j=n}^{\infty}\Psi(x, \tfrac{\alpha^j t}{|2|^{kj}}) = 1_{\mathcal{L}},\ \forall x \in X, t > 0,$$

then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mathcal{P}(f(x) - C(x), t) \geq_L \prod_{i=1}^{\infty}\mathcal{M}(x, \frac{\alpha^{i+1}t}{|2^{ki}|}),\ \forall x \in X, t > 0, \tag{3.3}$$

where

$$\mathcal{M}(x, t) = \prod_{i=1}^{\infty}\Psi(2^{i-1}x, 0, t/4).$$

Proof. First, we show, by induction on j, that, for all $x \in X$, $t > 0$ and $j \geq 1$,

$$\mathcal{P}(f(2^j x) - 4^j f(x), t) \geq_L \mathcal{M}_j(x, t). \tag{3.4}$$

Put $y = 0$ in (3.1). Then for all $x \in X$ and $t > 0$

$$\mathcal{P}(f(2x) - 4f(x), t) \geq_L \Psi(x, 0, t/4). \tag{3.5}$$

This proves (3.4) for $j = 1$. Let (3.4) holds for some $j > 1$. Replacing x by $2^j x$ in (3.5), we get

$$\mathcal{P}(f(2^{j+1}x) - 4f(2^j x), t) \geq_L \Psi(2^j x, 0, t/4).$$

Since $|4| \leq 1$, it follows that

$$\mathcal{P}(f(2^{j+1}x)-4^{j+1}f(x),t) \geq_L \mathcal{P}(f(2^{j+1}x)-4f(2^jx),t)*_L\mathcal{P}(4f(2^jx)-4^{j+1}f(x),t) = \mathcal{P}(f(2^{j+1}x)-4f(2^jx),t) *_L \mathcal{P}(f(2^jx)-4^jf(x),t/|4|) \geq_L \mathcal{P}(f(2^{j+1}x)-4f(2^jx),t) *_L \mathcal{P}(f(2^jx)-4^jf(x),t) \geq_L \mathcal{M}_j(x,t) *_L \Psi(2^jx,0,t/4) = \mathcal{M}_{j+1}(x,t).$$

Thus (3.4) holds for all $j \geq 1$. In particular, we have

$$\mathcal{P}(f(2^kx) - 4^kf(x),t) \geq_L \mathcal{M}(x,t). \tag{3.6}$$

Replacing x by $2^{-(kn+k)}x$ in (3.6) and using the inequality (3.2), we obtain

$$\mathcal{P}(f(\tfrac{x}{2^{kn}}) - 4^kf(\tfrac{x}{2^{kn+k}}),t) \geq_L \mathcal{M}(\tfrac{x}{2^{kn+k}},t) \geq_L \mathcal{M}(x,\alpha^{n+1}t).$$

and so

$$\mathcal{P}(2^{2kn}f(\tfrac{x}{2^{kn}}) - 2^{2k(n+1)}f(\tfrac{x}{2^{k(n+1)}}),t) \geq_L \mathcal{M}(x,\tfrac{\alpha^{n+1}t}{2^{2kn}}) \geq_L \mathcal{M}(\tfrac{\alpha^{n+1}t}{2^{kn}})$$

Hence it follows that

$$\mathcal{P}(2^{2kn}f(\tfrac{x}{2^{kn}}) - 2^{2k(n+p)}f(\tfrac{x}{2^{k(n+p)}}),t) \geq_L \prod_{j=n}^{n+p}(\mathcal{P}(2^{2kj}f(\tfrac{x}{2^{kj}}) - 2^{2k(j+1)}f(\tfrac{x}{2^{k(j+1)}}),t) \geq_L \prod_{j=n}^{n+p}\mathcal{M}(x,\tfrac{\alpha^{j+1}t}{2^{kj}}).$$

Since $lim_{n\to\infty}\prod_{j=n}^{\infty}\mathcal{M}(x,\frac{\alpha^{j+1}t}{2^{kj}}) = 1_{\mathcal{L}}$ for all $x \in X$ and $t > 0$, $\{2^{2kn}f(\frac{x}{2^{kn}})\}$ is a Cauchy sequence in the non-Archimedean $\mathcal{L}$-fuzzy Banach space $(Y,\mathcal{P},*_L)$. Hence we can define a mapping $C : X \to Y$ such that

$$lim_{n\to\infty}\mathcal{P}(2^{2kn}f(\frac{x}{2^{kn}}) - C(x),t) = 1_{\mathcal{L}}. \tag{3.7}$$

Next, for all $n \geq 1$, $x \in X$ and $t > 0$, we have

$$\mathcal{P}(f(x) - 2^{2kn}f(\tfrac{x}{2^{kn}}),t) = \mathcal{P}(\textstyle\sum_{i=0}^{n-1}2^{2ki}f(\tfrac{x}{2^{ki}}) - 2^{2k(i+1)}f(\tfrac{x}{2^{k(i+1)}}),t) \geq_L \prod_{i=0}^{n-1}\mathcal{P}(2^{2ki}f(\tfrac{x}{2^{ki}}) - 2^{2k(i+1)}f(\tfrac{x}{2^{k(i+1)}}),t) \geq_L \prod_{i=0}^{n-1}\mathcal{M}(x,\tfrac{\alpha^{i+1}t}{|2^{ki}|})$$

and so

$$\mathcal{P}(f(x)-C(x),t) \geq_L \mathcal{P}(f(x)-2^{2kn}f(\frac{x}{2^{kn}}),t)*_L\mathcal{P}(2^{2kn}f(\frac{x}{2^{kn}})-C(x),t) \geq_L \prod_{i=0}^{n-1}\mathcal{M}(x,\tfrac{\alpha^{i+1}t}{|2^{ki}|}) *_L \mathcal{P}(2^{2kn}f(\tfrac{x}{2^{kn}}) - C(x),t). \tag{3.8}$$

Taking the limit as $n \to \infty$ in (3.8), we obtain

$$\mathcal{P}(f(x) - C(x),t) \geq_L \prod_{i=1}^{\infty}\mathcal{M}(x,\frac{\alpha^{i+1}t}{|2^{ki}|}), \tag{3.9}$$

which proves (3.3). As $*_L$ is continuous, from a well known result in $\mathcal{L}$-fuzzy normed space(see[15], Chapter 12), it follows that

$$lim_{n\to\infty}\mathcal{P}(4^{kn}f(2^{-kn}(2x+y)) + 4^{kn}f(2^{-kn}(2x-y)) - 2.4^{kn}f(2^{-kn}(x+y)) - 2.4^{kn}f(2^{-kn}(x-y)) - 12.4^{kn}f(2^{-kn}x),t) = \mathcal{P}(C(2x+y) + C(2x-y) - 2C(x+y) - 2C(x-y) - 12C(x),t)$$

for almost all $t > 0$.

On the other hand, replacing x, y by $2^{-kn}x, 2^{-kn}y$ in (3.1) and (3.2), we get

$$\mathcal{P}(4^{kn}f(2^{-kn}(2x+y))+4^{kn}f(2^{-kn}(2x-y)-2.4^{kn}f(2^{-kn}(x+y))-2.4^{kn}f(2^{-kn}(x-y))-12.4^{kn}f(2^{-kn}x),t)=\mathcal{P}(C(2x+y)+C(2x-y)-2C(x+y)-2C(x-y)-12C(x),t)\geq_L \Psi(2^{-kn}x,2^{-kn}y,\frac{t}{|2^{2kn}|}\geq_L \Psi(2^{-kn}x,2^{-kn}y,\frac{t}{|2^{kn}|}).$$

Since $lim_{n\to\infty}\Psi(x,y,\frac{\alpha^n t}{|2^{kn}|})=1_{\mathcal{L}}$, we infer that C is a cubic mapping.
For the uniqueness of C, let $C':X\to Y$ be another cubic mapping such that

$$\mathcal{P}(C'(x)-f(x),t)\geq_L \mathcal{M}(x,t).$$

Then we have, for all $x,y\in X$ and $t>0$,

$$\mathcal{P}(C(x)-C'(x),t)\geq_L \mathcal{P}(C(x)-2^{2kn}f(\tfrac{x}{2^{kn}}),t)*_L \mathcal{P}(2^{2kn}f(\tfrac{x}{2^{kn}})-C'(x),t).$$

Therefore from (3.7), we have $C=C'$. □

Definition 3.3. Let $\mathbb{K}$ be a non-Archimedean field, X a normed space and $(Y,\mathcal{P},*_L)$ a non-Archimedean $\mathcal{L}$-fuzzy normed space over $\mathbb{K}$. Let $f:X\to Y$ be a Ψ-approximately cubic mapping. We say that $f:X\to Y$ is non-Archimedean $\mathcal{L}$-fuzzy continuous at a point $s_0\in X$ if for all $\varepsilon>0$ there exists $\delta>0$ such that for each s with $||s-s_0||<\delta$

$$lim_{t\to\infty}\mathcal{P}(T(s)-T(s_0),t\varepsilon)=1,$$

uniformly on X.

Theorem 3.4. *Let $\mathbb{K}$ be a non-Archimedean field, X a normed space and $(Y,\mathcal{P},*_L)$ a non-Archimedean $\mathcal{L}$-fuzzy normed space over $\mathbb{K}$. Let $f:X\to Y$ be a Ψ-approximately cubic mapping. If for some $x\in X$ and all $n\in\mathbb{N}$, the mapping $g:\mathbb{R}\to Y$ defined by $g(s)=2^{2kn}f(\frac{x}{2^{kn}})$ is non-Archimedean $\mathcal{L}$-fuzzy continuous. Then the mapping $s\mapsto C(sx)$ from $\mathbb{R}$ to Y is non-Archimedean $\mathcal{L}$-fuzzy continuous.*

Proof. Using Theorem (3.2) we deduce that, there exists a unique cubic mapping C such that

$$lim_{n\to\infty}\mathcal{P}(2^{2kn}f(\tfrac{x}{2^{kn}})-C(x),t)=1_{\mathcal{L}}.$$

By the non-Archimedean $\mathcal{L}$-fuzzy continuity of the mapping $t\mapsto 2^{2kn}f(\frac{tx}{2^{kn}})$, there exists δ such that for each s with $0<|s-s_0|<\delta$, we have

$$lim_{t\to\infty}\mathcal{P}(2^{2kn}f(\tfrac{sx}{2^{kn}})-2^{2kn}f(\tfrac{s_0x}{2^{kn}}),t\varepsilon)=1_L.$$

It follows that

$$lim_{t\to\infty}\mathcal{P}(C(sx)-C(s_0x),t\varepsilon)\geq_L lim_{t\to\infty}\mathcal{P}(C(sx)-2^{2kn}f(\tfrac{sx}{2kn}),t)*_L \mathcal{P}(2^{2kn}f(\tfrac{sx}{2kn})-2^{2kn}f(\tfrac{s_0x}{2kn}),t)*_L \mathcal{P}(C(s_0x)-2^{2kn}f(\tfrac{s_0x}{2kn}),t)=1_L$$

for each s with $0<|s-s_0|<\delta$. Hence, the mapping $s\mapsto C(sx)$ is non-Archimedean $\mathcal{L}$-fuzzy continuous.

□

Theorem 3.5. *Let $\mathbb{K}$ be a non-Archimedean field, X a normed space and $(Y,\mathcal{P},*_L)$ a non-Archimedean $\mathcal{L}$-fuzzy normed space over $\mathbb{K}$. Let $f:X\to Y$ be a Ψ-approximately cubic mapping. If for some $x\in X$ and all $n\in\mathbb{N}$, the mapping $g:\mathbb{R}\to Y$ defined by $g(s)=2^{2kn}f(\frac{x}{2^{kn}})$ is non-Archimedean $\mathcal{L}$-fuzzy continuous. Then $C(rx)=r^3C(x)$ for each $x\in X$ and $r\in\mathbb{R}$.*

Proof. For each $q \in \mathbb{Q}$, we have $C(qx) = q^3C(x)$. $\mathbb{Q}$ is a dense subset of $\mathbb{R}$. Fix $r \in \mathbb{R}$ and $t > 0$. Choose a rational sequence q_n such that $q_n \to r$. Then, there exists $\delta > 0$ such that

$$\mathcal{P}(C(rx) - r^3C(x), t) \geq_L$$
$$\mathcal{P}(C(rx) - C(q_nx), t) *_L \mathcal{P}(C(q_nx) - q_n^3C(x), t) *_L \mathcal{P}(q_n^3C(x) - r^3C(x), t).$$

By using the Theorem (3.4) for given $\varepsilon > 0$, we have

$$\mathcal{P}(C(rx) - r^3C(x), t) \geq_L 1 - \varepsilon *_L 1 *_L \mathcal{P}(q_n^3C(x) - r^3C(x), t).$$

By taking n tend to infinity,

$$\mathcal{P}(C(rx) - r^3C(x), t) \geq_L 1 - \varepsilon.$$

So the proof is complete. □

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Department of Mathematics, University of Mohaghegh Ardabili, P. O. Box 179, Ardabil, Iran

E-mail address: nasrineghbali@gmail.com