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FIXED POINT PROPERTY FOR THE HYPERSPACES OF NON-METRIC CHAINABLE CONTINUA

IVAN LONČAR

Abstract. The main purpose of this paper is to prove that some hyperspaces of a non-metric chainable continuum have the fixed point property.

1. Introduction

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space X is denoted by $\mathrm{w}(X)$.

A *generalized arc* is a Hausdorff continuum with exactly two non-separating points (end points) x, y. Each separable arc is homeomorphic to the closed interval $\mathbb{I} = [0, 1]$.

We say that a space X is *arcwise connected* if for every pair x, y of points of X there exists a generalized arc L with end points x, y.

An inverse system [3, pp. 135-142] is denoted by $\mathbf{X} = \{X_a, p_{ab}, A\}$. Suppose that we have two inverse systems $\mathbf{X} = \{X_a, p_{ab}, A\}$ and $\mathbf{Y} = \{Y_b, q_{bc}, B\}$. A *morphism of the system* X *into the system* $\mathbf{Y}$ [1, p. 15] is a family $\{\varphi, \{f_b : b \in B\}\}$ consisting of a nondecreasing function $\varphi : B \to A$ such that $\varphi(B)$ is cofinal in A, and of maps $f_b : X_{\varphi(b)} \to Y_b$ defined for all $b \in B$ such that the following

$$\begin{array}{ccc} X_{\varphi(b)} & \overset{p_{\varphi(b)\varphi(c)}}{\longleftarrow} & X_{\varphi(c)} \\ \downarrow f_b & & \downarrow f_c \\ Y_b & \overset{q_{bc}}{\longleftarrow} & Y_c \end{array} \tag{1.1}$$

diagram commutes. Any morphism $\{\varphi, \{f_b : b \in B\}\} : \mathbf{X} \to \mathbf{Y}$ induces a map, called the *limit map of the morphism*

$$\lim\{\varphi, \{f_b : b \in B\}\} : \lim \mathbf{X} \to \lim \mathbf{Y}$$

In the present paper we deal with the inverse systems defined on the same indexing set A. In this case, the map $\varphi : A \to A$ is taken to be the identity and we use the following notation $\{f_a : X_a \to Y_a; a \in A\} : \mathbf{X} \to \mathbf{Y}$.

The following result is well-known.

Theorem 1.1. [3, Exercise 2.5.D(b), p. 143]. *If for every $s \in S$ an inverse system $\mathbf{X}(s) = \{X_a(s), p_{ab}(s), A\}$ is given, then the family $\Pi\{\mathbf{X}(s) : s \in S\} = \{\Pi\{X_a(s) : s \in S\}, \Pi\{ p_{ab}(s) : s \in S\}, A\}$ is an inverse system and $\lim(\Pi\{\mathbf{X}(s) : s \in S\})$ is homeomorphic to $\Pi\{\lim \mathbf{X}(s) : s \in S\}$.*

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If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an inverse system, then we have inverse system $\mathbf{X} \times \mathbf{X} = \{X_a \times X_a, p_{ab} \times p_{ab}, A\}$. Let $X = \lim \mathbf{X}$. By Theorem 1.1 we infer that $X \times X$ is homeomorphic to the limit of inverse system $\mathbf{X} \times \mathbf{X}$.

We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *factorizing* [1, p. 17] if for each real-valued mapping $f : \lim \mathbf{X} \to \mathbb{R}$ there exist an $a \in A$ and a mapping $f_a : X_a \to \mathbb{R}$ such that $f = f_a p_a$.

An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be *σ-directed* if for each sequence $a_1, a_2, ..., a_k, ...$ of the members of A there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

Lemma 1.2. [1, Corollary 1.3.2, p. 18]. *If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a σ-directed inverse system of compact spaces with surjective bonding mappings, then it is factorizing.*

An inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is said to be *τ-continuous* [1, p. 19] if for each chain B in A with $\mathrm{card}(B) < \tau$ and $\sup B = b$, the diagonal product $\Delta \{p_{ab} : a \in B\}$ maps the space X_b homeomorphically into the space $\lim\{X_a, p_{ab}, B\}$.

An inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is said to be *τ-system* [1, p. 19] if:

a) $\tau \geq \mathrm{w}(X_a)$ for every $a \in A$,

b) The system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is τ-continuous,

c) The indexing set A is τ-complete.

If $\tau = \aleph_0$, then τ-system is called a σ-system. The following theorem is called the *Spectral Theorem* [1, p. 19].

Theorem 1.3. [1, Theorem 1.3.4, p. 19]. *If a τ-system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ with surjective limit projections is factorizing, then each map of its limit space into the limit space of another τ-system $\mathbf{Y} = \{Y_a,\ q_{ab},\ A\}$ is induced by a morphism of cofinal and τ-closed subsystems. If two factorizing τ-systems with surjective limit projections and the same indexing set have homeomorphic limit spaces, then they contain isomorphic cofinal and τ-closed subsystems.*

Let us remark that the requirement of surjectivity of limit projections of systems in Theorem 1.3 is essential [1, p. 21].

In the sequel we will need the following theorem.

Theorem 1.4. [7, Theorem 1.6, p. 402]. *If X is the Cartesian product $X = \prod\{X_s : s \in S\}$, where $\mathrm{card}(S) > \aleph_0$ and each X_s is compact, then there exists a σ-directed inverse system $\mathbf{X} = \{Y_a, P_{ab}, A\}$ of the countable products $Y_a = \prod\{X_\mu : \mu \in a\}$, $\mathrm{card}(a) = \aleph_0$, such that X is homeomorphic to $\lim \mathbf{X}$.*

2. Fixed point property for non-metric compact spaces

A *fixed point of a function* $f : X \to X$ is a point $p \in X$ such that $f(p) = p$. A space X is said to have the *fixed point property* provided that every surjective mapping $f : X \to X$ has a fixed point.

First Step in the proving fixed point property for hyperspaces of non-metric chainable continua is the following general Theorem for fixed point property for non-metric continua.

Theorem 2.1. *Let $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ be a σ-system of compact spaces with the limit X and onto projections $p_a : X \to X_a$. Let $\{f_a : X_a \to X_a\} : \mathbf{X} \to \mathbf{X}$ be a morphism. Then the induced mapping $f = \lim\{f_a\} : X \to X$ has a fixed point if and only if each mapping $f_a : X_a \to X_a$, $a \in A$, has a fixed point.*

Proof. The if part. Let $F_a, a \in A$, be a set of fixed points of the mapping f_a.

Claim 1. *Every set F_a is closed.* This is a consequence of the following theorem [3, Theorem 1.5.4., p. 59]. *For any pair f,g of mappings of a space X into a Hausdorff space Y, the set*

$$\{x \in X : f(x) = g(x)\}$$

is closed in X.

It suffices to set $g(x) = x$ and $Y = X$.

Claim 2. *If $b \geq a$, then $p_{ab}(F_b) \subset F_a$.* Let x_b be any point of F_b. From the commutativity of the diagram (1.1) it follows $p_{ab}(f_b(x)) = f_a(p_{ab}(x))$. We have $p_{ab}(x) = f_a(p_{ab}(x))$ since $f_b(x) = x$. This mens that for the point $y = p_{ab}(x) \in X_a$ we have $y = f_a(y)$, i.e., $y \in F_a$. We infer that $p_{ab}(x) \in F_a$ and $p_{ab}(F_b) \subset F_a$.

Claim 3. $\mathbf{F} = \{F_a, p_{ab}|F_b, A\}$ *is an inverse system of compact spaces with the non-empty limit F.*

Claim 4. *The set $F \subset X$ is the set of fixed points of the mapping f.* Let $x \in F$ and let $x_a = p_a(x)$, $a \in A$. Now, $f_a(x_a) = x_a$ since $x_a \in F_a$. We infer that $f(x) = x$ since the morphism $\{f_a : a \in A\}$ induces f. The proof of the "if" part is complete.

The only if part. Suppose that the induced mapping f has a fixed point x. Let us prove that every mapping $f_a, a \in A$, has a fixed point. Now we have $f_a p_a(x) = p_a f(x)$. From $f(x) = (x)$ it follows $f_a p_a(x) = p_a(x)$. We infer that $p_a(x)$ is a fixed point for f_a. □

As an immediate consequence of this theorem and the Spectral theorem 1.3 we have the following result.

Theorem 2.2. *Let a non-metric continuum X be the inverse limit of an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a has the fixed point property and each bonding mapping p_{ab} is onto. Then X has the fixed point property.*

The following result is an application of Theorem 2.2.

Theorem 2.3. *Let S be an infinite set and $Q = \Pi\{X_s : s \in S\}$ Cartesian product of compact spaces. If each product $X_{s_1} \times X_{s_2} \times ... \times X_{s_n}$ of finitely many spaces X_s has the fixed point property, then Q has the fixed point property.*

Proof. We shall consider the following cases.

Case 1. $\text{card}(S) = \aleph_0$. We may assume that $S = \mathbb{N}$. The proof is a straightforward modification of the proof of [9, Corollary 3.5.3, pp. 106-107]. Let $f : Q \to Q$ be continuous. For every $n \in \mathbb{N}$ define

$$K_n = \{x \in Q : (x_1, ..., x_n) = (f(x)_1, ..., f(x)_n)\}.$$

It is clear that for every n the set K_n is closed in Q and that $K_{n+1} \subset K_n$. For every $n \in \mathbb{N}$, let o_n be a given point of X_n and $p_n : Q \to X_1 \times ...\times X_n$ be the projection. Define continuous function $f_n : X_1 \times ... \times X_n \to X_1 \times ... \times X_n$ by

$$f_n(x_1, ..., x_n) = (p_n f)(x_1, ..., x_n, o_{n+1}, o_{n+2}, ...).$$

By assumption of Theorem f_n has the fixed point property, say $(x_1, ..., x_n)$. It follows that

$$(x_1, ..., x_n, o_{n+1}, o_{n+2}, ...) \in K_n.$$

We conclude that $\{K_n : n \in \mathbb{N}\}$ is a decreasing collection of nonempty closed subsets of Q. By compactness of Q we have that

$$K = \cap\{K_n : n \in \mathbb{N}\}$$

is nonempty. It is clear that every point in K is a fixed point of f.

Case 2 .card$(A) \geq \aleph_1$. By Theorem 1.4 there exists a σ-directed inverse system $\mathbf{X} = \{Y_a, P_{ab}, A\}$ of the countable products $Y_a = \prod\{X_\mu : \mu \in a\}$, card$(a) = \aleph_0$, such that Q is homeomorphic to $\lim \mathbf{X}$. By Case 1 each Y_a has the fixed point property. Finally, by Theorem 2.2 we infer that Q has the fixed point property. $\square$

3. Fixed point property for the hyperspaces of non-metric chainable continua

In this Section we shall study the fixed point property of the hyperspaces of chainable continua.

A *chain* $\{U_1, ..., U_n\}$ is a finite collection of sets U_i such that $U_i \bigcap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$. A continuum X is said to be *chainable* or *arc-like* if each open covering of X can be refined by an open covering $u = \{U_1, ..., U_n\}$ such that $\{U_1, ..., U_n\}$ is a chain.

Second Step in the proving fixed point property for hyperspaces of non-metric chainable continua is the following general expanding Theorem for non-metric chainable continua into inverse σ-system.

Theorem 3.1. *If X is a chainable continuum, then there exists a σ-system $\mathbf{Q}_\sigma = \{Q_\Delta, p_{\Delta\Gamma}, A_\sigma\}$ such that each Q_Δ is a metric chainable continuum, $p_{\Delta\Gamma}$ are surjections and X is homeomorphic with the inverse limit* $\lim \mathbf{Q}_\sigma$.

Proof. The proof is broken into several steps.

Step 1. *If X is a chainable continuum, then there exists a system $\mathbf{Q} = \{Q_a, q_{ab}, A\}$ such that each Q_a is a metric chainable continuum and X is homeomorphic with the inverse limit* $\lim \mathbf{Q}$. By [8, Theorem 2*] every chainable continuum X is homeomorphic with the inverse limit of an inverse system $\{Q_a, q_{ab}, A\}$ of metric chainable continua Q_a. One can assume that q_{ab} are onto mappings since a closed connected subset C of chainable continuum is chainable.

Step 2. *There exists a σ-system of chainable continua such that X is homeomorphic with its inverse limit.* The inverse system $\{Q_a, q_{ab}, A\}$ is not a σ-system. Now we shall prove that such inverse system exists. For each subset Δ_0 of $(A, \leq)$ we define sets Δ_n, $n = 0, 1, ...$, by the inductive rule $\Delta_{n+1} = \Delta_n \bigcup \{m(x, y) : x, y \in \Delta_n\}$, where $m(x, y)$ is a member of A such that $x, y \leq m(x, y)$. Let $\Delta = \bigcup\{\Delta_n : n \in \mathbb{N}\}$. It is clear that card$(\Delta) =$ card(Δ_0). Moreover, Δ is directed by $\leq$. For each directed set $(A, \leq)$ we define

$$A_\sigma = \{\Delta : \emptyset \neq \Delta \subset A, \text{card}(\Delta) \leq \aleph_0 \quad \text{and } \Delta \text{ is directed by } \leq\}.$$

Let us prove that A_σ is σ-directed and σ-complete. Let $\{\Delta^1, \Delta^2, ..., \Delta^n, ...\}$ be a countable subset of $A_\sigma = \{\Delta : \emptyset \neq \Delta \subset A, \text{card}(\Delta) \leq \aleph_0 \quad \text{and } \Delta \text{ is directed by } \leq\}$. Then $\Delta_0 = \cup\{\Delta^1, \Delta^2, ..., \Delta^n, ...\}$ is a countable subset of A_σ. Define sets Δ_n, $n = 0, 1, ...$, by the inductive rule $\Delta_{n+1} = \Delta_n \bigcup \{m(x, y) : x, y \in \Delta_n\}$, where $m(x, y)$ is a member of A such that $x, y \leq m(x, y)$. Let $\Delta = \bigcup\{\Delta_n : n \in \mathbb{N}\}$. It is clear that card$(\Delta) =$ card(Δ_0). This means that Δ is countable. Moreover $\Delta \supseteq \Delta^i$, $i \in \mathbb{N}$. Hence A_σ is σ-directed. Let us prove that A_σ is σ-complete. Let $\Delta^1 \subset \Delta^2 \subset ... \subset \Delta^n \subset ...$ be a countable chain in A_σ. Then $\Delta = \cup\{\Delta^i : i \in \mathbb{N}\}$ is countable and directed subset of A, i.e., $\Delta \in A_\sigma$. It is clear that $\Delta \supseteq \Delta^i$, $i \in \mathbb{N}$. Moreover, for each $\Gamma \in A_\sigma$ with property $\Gamma \supseteq \Delta^i$, $i \in \mathbb{N}$, we have $\Gamma \supseteq \Delta$. Hence $\Delta = \sup\{\Delta^i : i \in \mathbb{N}\}$. This means that A_σ is σ-complete.

If $\Delta \in A_\sigma$, let $\mathbf{Q}^\Delta = \{Q_b,\, q_{bb'},\, \Delta\}$ and $Q_\Delta = \lim \mathbf{Q}^\Delta$. If $\Delta, \Gamma \in A_\sigma$ and $\Delta \subseteq \Gamma$, let $p_{\Delta\Gamma}$: $Q_\Gamma \to Q_\Delta$denote the map induced by the projections $q_\delta^\Gamma : Q_\Gamma \to Q_\delta$, $\delta \in \Delta$, of the inverse system $\mathbf{Q}^\Gamma$.

Now we shall prove that if $\mathbf{Q} = \{Q_a,\, q_{ab},\, A\}$ is an inverse system, then $\mathbf{Q}_\sigma = \{Q_\Delta,\, p_{\Delta\Gamma},\, A_\sigma\}$ is a σ-directed and σ-complete inverse system such that $\lim \mathbf{Q}$ and $\lim \mathbf{Q}_\sigma$ are homeomorphic. Each thread $x = (x_a : a \in A)$ induces the thread $(x_a : a \in \Delta)$ for each $\Delta \in A_\sigma$, i.e., the point $q_\Delta \in Q_\Delta$. This means that we have a mapping $H : \lim \mathbf{Q} \to \lim \mathbf{Q}_\sigma$ such that $H(x) = (q_\Delta : \Delta \in A_\sigma)$. It is obvious that H is continuous and 1-1. The mapping H is onto since the collections of the threads $\{q_\Delta : \Delta \in A_\sigma\}$ induces the thread in $\mathbf{Q}$. We infer that H is a homeomorphism since $\lim \mathbf{Q}$ is compact.

Finally, let us prove that every Q_Δ is chainable. We may assume that $\mathbf{Q}^\Delta = \{Q_b, q_{bb'},\, \Delta\}$ is an inverse sequence since Δ is countable and $Q_\Delta = \lim \mathbf{Q}^\Delta$. Let $u = \{U_1, ..., U_n\}$ be an open covering of Q_Δ. There exists a $b \in \Delta$ and an open covering $u_b = \{U_1^b, ..., U_m^b\}$ of Q_b such that $\{q_b^{-1}(U_1^b), ..., q_b^{-1}(U_m^b)\}$ refines the covering $u = \{U_1, ..., U_n\}$. There is a chain $\{V_1^b, ..., V_p^b\}$ which refines u_b since Q_b is chainable. It is clear that $\{q_b^{-1}(V_1^b), ..., q_b^{-1}(V_p^b)\}$ is a chain which refines the covering u. Hence, Q_Δ is chainable.

Step 3. *One can assume that* $p_{\Delta\Gamma}$ *and* $p_\Delta : \lim Q_\sigma \to Q_\Delta$ *are onto mappings.* If $p_{\Delta\Gamma}$ and $p_\Delta : \lim Q_\sigma \to Q_\Delta$ are not onto mappings, then we shall use the inverse system $\mathbf{Q}_\sigma^p = \{p_\Delta(\lim Q_\sigma),\, p_{\Delta\Gamma}|p_\Delta(\lim Q_\sigma),\, A_\sigma\}$. Each $p_{\Delta\Gamma}|p_\Delta(\lim Q_\sigma)$ is chainable since a closed connected subset of chainable continuum is chainable.

The proof is completed since X is representable as the inverse limit of σ-system $\mathbf{Q}_\sigma = \{Q_\Delta,\, p_{\Delta\Gamma},\, A_\sigma\}$ of metric chainable continua Q_Δ. □

Finally, we represent the various hyperspaces of a non-metric chainable continuum X as the inverse limits of hyperspaces of metric chainable continua.

Let X be a space. We define its hyperspaces as the following sets:

$2^X = \{F \subseteq X : F \text{ is closed and nonempty}\}$,
$\mathcal{C}(X) = \{F \in 2^X : F \text{ is connected}\}$,
$\mathcal{F}_n(X) = \{A \subset X : A \text{ is nonempty and } A \text{ has at most } n \text{ points}\}$.

For any finitely many subsets S_1, ..., S_n, let

$$\langle S_1, ..., S_n\rangle = \left\{F \in 2^X : F \subset \bigcup_{i=1}^n S_i, \text{ and } F \cap S_i \neq \emptyset, \text{ for each } i\right\}.$$

The topology on 2^X is the Vietoris topology, i.e., the topology with a base $\{< U_1, ..., U_n >: U_i$ is an open subset of X for each i and each $n < \infty$ $\}$, and $C(X), X(n)$ are subspaces of 2^X. Moreover, $X(1)$ is homeomorphic to X.

The topology on 2^X is the Vietoris topology and $\mathcal{C}(X)$ and $\mathcal{F}_n(X)$ is a subspaces of 2^X.

Let X and Y be the spaces and let $f : X \to Y$ be a mapping. Define $2^f : 2^X \to 2^Y$ by $2^f(F) = f(F)$ for $F \in 2^X$. It is known that 2^f is continuous and $2^f(\mathcal{C}(X)) \subset \mathcal{C}(Y)$. Moreover, $2^f(\mathcal{F}_n(X)) \subset \mathcal{F}_n(Y)$. The restriction $2^f|\mathcal{C}(X)$ is denoted by $\mathcal{C}(f)$. Similarly, the restriction $2^f|\mathcal{F}_n(X)$ is denoted by $\mathcal{F}_n(f)$.

Let $\mathbf{X} = \{X_a,\, p_{ab},\, A\}$ be an inverse system of compact spaces with the natural projections $p_a : \lim X \to X_a$, $a \in A$. Then $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$, $\mathcal{C}(\mathbf{X}) = \{\mathcal{C}(X_a), \mathcal{C}(p_{ab}), A\}$ and $\mathcal{F}_n(\mathbf{X}) = \{\mathcal{F}_n(X_a), \mathcal{F}_n(p_{ab}), A\}$ form inverse systems.

Lemma 3.2. *Let* $X = \lim \mathbf{X}$*. Then* $2^X = \lim 2^{\mathbf{X}}$*,* $\mathcal{C}(X) = \lim \mathcal{C}(\mathbf{X})$ *and* $\mathcal{F}_n(X) = \lim \mathcal{F}_n(\mathbf{X})$*.*

In [6, Corollary 5, p. 616] it is proved the following result.

Theorem 3.3. *The third symmetric product* $\mathcal{F}_3(X)$ *of a metric chainable continuum* X *has the fixed point property.*

The proof given there is purely metric. This means that it is reasonable to give the proof for non-metric chainable continua.

Theorem 3.4. *The third symmetric product* $\mathcal{F}_3(X)$ *of a chainable continuum* X (*metric or non-metric*) *has the fixed point property.*

Proof. If X is a metric chainable continuum, then apply Theorem 3.3. In order to complete the proof, we shall assume that X is non-metric chainable continuum. By Theorem 3.1 there exists a σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metric chainable continuum, p_{ab} are surjections *and* X is homeomorphic with the inverse limit $\lim \mathbf{X}$. Now we have a σ-system $\mathcal{F}_3(\mathbf{X}) = \{\mathcal{F}_3(X_a), \mathcal{F}_3(p_{ab}), A\}$ whose limit is homeomorphic to $\mathcal{F}_3(X)$. In order to apply Theorem 2.2 it suffices to prove that $\mathcal{F}_3(p_{ab})$ are surjections for every $a \leq b$. Let $\{x_1, x_2, x_3\} \in \mathcal{F}_3(X_a)$. The sets $p_{ab}^{-1}(x_1), p_{ab}^{-1}(x_2)$, $p_{ab}^{-1}(x_3)$ are non-empty since p_{ab} is onto. If $y_1 \in p_{ab}^{-1}(x_1), y_2 \in p_{ab}^{-1}(x_2)$ and $y_3 \in p_{ab}^{-1}(x_3)$, then $\{y_1, y_2, y_3\} \in \mathcal{F}_3(X_b)$ and $\mathcal{F}_3(p_{ab})(\{y_1, y_2, y_3\}) = \{x_1, x_2, x_3\} \in \mathcal{F}_3(X_a)$. Hence, $\mathcal{F}_3(p_{ab})$ is a surjection. By Theorem 3.3 each X_a has the fixed point property. Finally, by Theorem 2.2 we infer that $\mathcal{F}_3(X)$ has the fixed point property. $\square$

From the proof of the above theorem it is clear that is true the following theorem.

Theorem 3.5. *The nth-symmetric product* $\mathcal{F}_n(X)$ *of a chainable non-metric continuum* X *has the fixed point property if the nth-symmetric product of every chainable metric continuum has the fixed point property.*

From this theorem we shall give the following result.

Theorem 3.6. *If* X *is a non-metric chainable continuum, then* X *has the fixed point property.*

Proof. Now X is homeomorphic to $\mathcal{F}_1(X)$ which is homeomorphic to $\lim \mathcal{F}_1(\mathbf{X})$. From Theorem 3.5 it follows that $\lim \mathcal{F}_1(\mathbf{X})$ has the fixed point property since each metric chainable continuum $\mathcal{F}_1(Y)$ (homeomorphic to Y) has the fixed point property [4]. $\square$

Another hyperspace of a continuum is the hyperspace $\mathcal{C}(X) = \{F \in 2^X : F$ is connected$\}$. The following result is known.

Theorem 3.7. [11]. *If* Y *is a metric chainable continuum, then* $\mathcal{C}(Y)$ *has the fixed point property.*

For non-metric chainable continua we have the following result.

Theorem 3.8. *If* X *is a non-metric chainable continuum, then* $\mathcal{C}(X)$ *has the fixed point property.*

Proof. If X is a metric chainable continuum, then apply Theorem 3.7. In order to complete the proof, we shall assume that X is non-metric chainable continuum. By Theorem 3.1 there exists a σ-system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ such that each X_a is a metric chainable continuum, p_{ab} are surjections *and* X is homeomorphic with the inverse limit $\lim \mathbf{X}$. Now we have a σ-system $\mathcal{C}(\mathbf{X}) = \{\mathcal{C}(X_a), \mathcal{C}(p_{ab}), A\}$ whose limit is homeomorphic to $\mathcal{C}(X)$. In order to apply Theorem 2.2 it suffices to prove that $\mathcal{C}(p_{ab})$ are surjections for every $a \leq b$. Let $C \in \mathcal{C}(X_a)$. The set $p_{ab}^{-1}(C)$ contains a continuum D in Y such that $p_{ab}(D) = C$ ([10, Theorem 12.46, p. 262]). Hence, $\mathcal{C}(p_{ab})$ is a surjection. By Theorem 3.7 each X_a has the fixed point property. Finally, by Theorem 2.2 we infer that $\mathcal{C}(X)$ has the fixed point property. □

4. Fixed point property of the product of chainable continua

Dyer [2, Theorem 1, p. 663] showed the following result.

Theorem 4.1. *Suppose that M is the Cartesian product of n compact chainable metric continua $X_1, X_2, ..., X_n$ and f is a continuous mapping of M into itself. Then there is a point $x \in M$ such that $x = f(x)$.*

For $n = 2$ we have the following result.

Theorem 4.2. [5, p. 199, Exercise 22.26]. *If X and Y are metric chainable continua, then $X \times Y$ has the fixed point property.*

Dyer [2, Corollary, p.665] showed the following general result.

Theorem 4.3. *Cartesian product of the elements of any collection of chainable metric continua has the fixed point property.*

We will show that last Theorem 4.3 is true for non-metrizable chainable continua.

Theorem 4.4. *Cartesian product of the elements of any collection of chainable continua of the same weight has the fixed point property.*

Proof. If for every $s \in S$ we have a chainable non-metrizable continuum $X(s)$, then, for every $s \in S$, there exists an inverse system $\mathbf{X}(s) = \{X_a(s),\ p_{ab}(s),\ A(s)\}$ such that $X(s)$ is homeomorphic to $\lim \mathbf{X}(s)$ and every $X_a(s)$ is a metric chainable continuum (Theorem 3.1). If $\mathrm{w}(X(s_1)) = \mathrm{w}(X(s_2)), s_1, s_2 \in S$, then $A(s_1) = A(s_2)$ and we may suppose that $A(s) = A$ for every $s \in S$. By Theorem 1.1 the family $\Pi\{\mathbf{X}(s) : s \in S\} = \{\Pi\{X_a(s) : s \in S\}, \Pi\{\ p_{ab}(s) : s \in S\},\ A\}$ is an inverse system and $\lim(\Pi\{\mathbf{X}(s) : s \in S\})$ is homeomorphic to $\Pi\{\lim \mathbf{X}(s) : s \in S\}$. From Theorem 4.3 it follows that each $\Pi\{X_a(s) : s \in S\}$ has the fixed point property. Finally, from Theorem 2.2 it follows that $\Pi\{X(s) : s \in S\}$ has the fixed point property. □

QUESTION. Is it true that the assumption "*of the same weight*" in Theorem 4.4 can be omitted?

As an immediate application of Theorem 4.4 we give the following generalization of Brouwer Fixed-Point Theorem. Let L be a non-metric arc. The space X is said to be a *generalized n-cell* if it is homeomorphic to $L^n = L \times L \times ... \times L$ (n factors).

Theorem 4.5. *Every mapping $f : L^n \to L^n$ has a fixed point, i.e., L^n has the fixed point property.*

Theorem 4.5 implies the following result.

Theorem 4.6. *If $L_1, ..., L_n$ are arcs (metric or non-metric), then $L_1 \times L_2 \times ... \times L_n$ has the fixed point property.*

Proof. **Step 1**. *If M is a subarc of the arc L, then there exists a retraction $r : L \to M$.* Let a, b, c, d be end points of L and M such that $a \leq c < d \leq b$. We define $r : L \to M$ as follows:

$$r(x) = \begin{cases} c & \text{if} \quad a \leq x \leq c, \\ x & \text{if} \quad c \leq x \leq d, \\ d & \text{if} \quad d \leq x \leq b. \end{cases}$$

Step 2. *If $L_1, L_2, ..., L_n$ is a finite collection of arcs, then there an arc L such that $L_1, L_2, ..., L_n$ are subarc of L.* For each $i \in \{1, 2, ..., n\}$ let a_i, b_i be a pair of end points of L_i such that $a_i < b_i$. If we identify the pair of points $\{b_1, a_2\}, \{b_2, a_3\}, ..., \{b_{n-1}, a_n\}$ we obtain an arc L such that $L_i \subset L$ for each $i \in \{1, 2, ..., n\}$.

Step 3. *$L_1 \times L_2 \times ... \times L_n$ is a retract of L^n.* Let L and $L_1, L_2, ..., L_n$ be as in Step 2. Let $r_i : L \to L_i, i \in \{1, 2, ..., n\}$ be a retraction defined in Step 1. Let us prove that $r = r_1 \times r_2 \times ... \times r_n$ is a retraction of L^n onto $L_1, L_2, ..., L_n$. If $(y_1, y_2, ..., y_n) \in L^n$, then we have: $r_1 \times r_2 \times ... \times r_n(y_1, y_2, ..., y_n) = (r_1(y_1), r_2(y_2), ..., r_n(y_n)) \in L_1 \times L_2 \times ... \times L_n$ since $r_i(y_i) \in L_i$. If $(x_1, x_2, ..., x_n) \in L_1 \times L_2 \times ... \times L_n$, then $r_1 \times r_2 \times ... \times r_n(x_1, x_2, ..., x_n) = (r_1(x_1), r_2(x_2), ..., r_n(x_n)) = (x_1, x_2, ..., x_n) \in L_1 \times L_2 \times ... \times L_n$ since $r_i(x_i) \in x_i$.

Step 4. The product $L_1 \times L_2 \times ... \times L_n$ has the fixed point property since it is retract of the product L^n which has the fixed point property (Theorem 4.5). The proof is completed. □

Theorem 4.7. *If $L = \Pi\{L_s : s \in S\}$ is a Cartesian product of arcs L_s, then L has the fixed point property.*

Proof. Apply Theorems 4.6 and 2.3. □

For Cartesian product of two chainable continua the assumption concerning the weight in Theorem 4.4 can be omitted.

Theorem 4.8. *If X and Y are non-metrizable chainable continua, then $X \times Y$ has the fixed point property.*

Proof. First we shall prove that if X is any chainable continuum and if Y is a metric chainable continuum, then $X \times Y$ has the fixed point property. By Theorem 3.1 there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metric chainable continuum and X is homeomorphic to $\lim \mathbf{X}$. It is clear that $\mathbf{X} \times Y = \{X_a \times Y, p_{ab} \times id, A\}$ is a σ-directed inverse system whose limit is homeomorphic to $X \times Y$. Every $X_a \times Y$ has the fixed point property since it is the product of metric chainable continua (Theorem 4.2). Applying Theorem 2.2 we infer that $X \times Y$ has the fixed point property.

Suppose now that X and Y are non-metric chainable continua. Using again Theorem 3.1 we obtain a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metric chainable continuum and X is homeomorphic to $\lim \mathbf{X}$. It is clear that $\mathbf{X} \times Y = \{X_a \times Y, p_{ab} \times id, A\}$ is a σ-representation of $X \times Y$. From the first part of this proof it follows that every $X_a \times Y$ has the fixed point property since it is the product of metric chainable continuum X_a and an chainable continuum Y. Applying Theorem 2.2 we infer that $X \times Y$ has the fixed point property. □

We close this section with the fixed point property for multifunctions on chainable continua.

A *multifunction*, $F : X \to Y$, from a space X to a space Y is a point-to-set correspondence such that, for each $x \in X$, $F(x)$ is a subset of Y. For any $y \in Y$, we write $F^{-1}(y) = \{x \in X : y \in F(x)\}$. If $A \subset X$ and $B \subset Y$, then $F(A) = \cup\{F(x) : x \in A\}$ and $F^{-1}(B) = \cup\{F^{-1}(y) : y \in B\}$.

A multifunction $F : X \to Y$ is said to be *continuous* if and only if: (i) $F(x)$ is closed for each $x \in X$, (ii) $F^{-1}(B)$ is closed for each closed set B in Y, (iii) $F^{-1}(V)$ is open for each open set V in Y.

A topological space X is said to have F.p.p (fixed point property for multi-valued functions) if for every multi-valued continuous function $F : X \to X$ there exists a point $x \in X$ such that $x \in F(x)$. It follows that X has F.p.p if for every single-valued continuous function $F : X \to 2^X$ there exists a point $x \in X$ such that $x \in F(x)$.

Theorem 4.9. [12]. *If X is any metric chainable continuum, then X has the F.p.p.*

Now we shall prove the following result.

Theorem 4.10. *Each chainable continuum X has the F.p.p.*

Proof. If an chainable continuum is metrizable, then it has F.p.p (Theorem 4.9). Suppose that chainable continuum X is non-metrizable. By virtue of Theorem 3.1 there exists a σ-system $\mathbf{X}_\sigma = \{X_\Delta,\ P_{\Delta\Gamma},\ A_\sigma\}$ of metric chainable continua X_Δ and onto mappings $P_{\Delta\Gamma}$ such that X is homeomorphic to $\lim \mathbf{X}_\sigma$. Moreover, we have the inverse system $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$ whose limit is 2^X. Let $F : X \to 2^X$ be a continuous mapping. From Theorem 1.3 it follows that there exists a subset B cofinal in A such that for every $b \in B$ there exists a continuous mapping $F_b : X_b \to 2^{X_b}$ with the property that $\{F_b : b \in B\}$ is a morphism which induce F. Theorem 4.9 implies that the set $Y_b \subset X_b, b \in B$, of fixed points of F_b is non-empty. Let us prove that Y_b is a closed subset of X_b. We shall prove that $U_b = X_b \diagdown Y_b$ is open. Let $x_b \in U_b$. This means that x_b and $F_b(x_b)$ are disjoint closed subset of X_b. By the normality of X_b there exists a pair of open sets U, V such that $x \in U$ and $Y_b \subset V$. From the continuity of F_b it follows that there exists an open set $W \subset U$ such that for every $x \in W$ we have $f(x) \subset V$. Hence, U_b is open and, consequently, Y_b is closed. Now, we shall prove that the collection $\{Y_b, p_{bc}|Y_c, B\}$ is an inverse system. To do this we have to prove that if $c > b$, then $p_{bc}(Y_c) \subset Y_b$. Let x_c be a point of Y_c. This means that $x_c \in f_c(x_c)$. Hence, $p_{bc}(x_c) \in p_{bc}(F_c(x_c)) = F_b p_{bc}(x_c)$. We conclude that the point $x_b = p_{bc}(x_c)$ has the property $x_b \in f_b(x_b)$, i.e., $x_b = p_{bc}(x_c) \in Y_b$. Finally, $p_{bc}(Y_c) \subset Y_b$. and $\{Y_b, p_{bc}|Y_c, B\}$ is an inverse system with non-empty limit. Let $Y = \lim\{Y_b, p_{bc}|Y_c, B\}$. In order to complete the proof we shall prove that for every $x \in Y$ we have $x \in F(x)$. Now we have $p_b(x) \in Y_b$, i.e., $p_b(x) \in F_b(p_b(x)) = p_b F(x)$, for every $b \in B$. It follows that $x \in F(x)$ since $x \notin F(x)$ implies that there is a $b \in B$ such that $p_b(x) \notin p_b F(x)$. We conclude that F has the fixed point property. $\square$

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Faculty of Organizations and Informatics Varaždin, Croatia
E-mail address: iloncar@tele2.hr or ivan.loncar@foi.hr