FIXED POINT PROPERTY FOR THE HYPERSPACES OF NON-METRIC CHAINABLE CONTINUA

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Abstract. The main purpose of this paper is to prove that some hyperspaces of a non-metric chainable continuum have the fixed point property.

1. Introduction

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space \( X \) is denoted by \( w(X) \).

A generalized arc is a Hausdorff continuum with exactly two non-separating points (end points) \( x, y \). Each separable arc is homeomorphic to the closed interval \( I = [0,1] \).

We say that a space \( X \) is arcwise connected if for every pair \( x, y \) of points of \( X \) there exists a generalized arc \( L \) with end points \( x, y \).

An inverse system [3, pp. 135-142] is denoted by \( X = \{ X_a, p_{ab}, A \} \). Suppose that we have two inverse systems \( X = \{ X_a, p_{ab}, A \} \) and \( Y = \{ Y_b, q_{bc}, B \} \). A morphism of the system \( X \) into the system \( Y \) [1, p. 15] is a family \( \{ \varphi, \{ f_b : b \in B \} \} \) consisting of a nondecreasing function \( \varphi : B \to A \) such that \( \varphi(B) \) is cofinal in \( A \), and of maps \( f_b : X_{\varphi(b)} \to Y_b \) defined for all \( b \in B \) such that the following diagram commutes:

\[
\begin{array}{ccc}
X_{\varphi(b)} & \xrightarrow{p_{\varphi(b)}q_{\varphi(c)}} & X_{\varphi(c)} \\
\downarrow f_b & & \downarrow f_c \\
Y_b & \xleftarrow{q_{bc}} & Y_c
\end{array}
\]

Any morphism \( \{ \varphi, \{ f_b : b \in B \} \} : X \to Y \) induces a map, called the limit map of the morphism

\[
\lim(\{ \varphi, \{ f_b : b \in B \} \} : X \to Y)
\]

In the present paper we deal with the inverse systems defined on the same indexing set \( A \). In this case, the map \( \varphi : A \to A \) is taken to be the identity and we use the following notation \( \{ f_a : X_a \to Y_a ; a \in A \} : X \to Y \).

The following result is well-known.

Theorem 1.1. [3, Exercise 2.5.D(b), p. 143]. If for every \( s \in S \) an inverse system \( X(s) = \{ X_a(s), p_{ab}(s), A \} \) is given, then the family \( \Pi\{ X(s) : s \in S \} = \{ \Pi\{ X_a(s) : s \in S \}, \Pi\{ p_{ab}(s) : s \in S \}, A \} \) is an inverse system and \( \lim(\Pi\{ X(s) : s \in S \}) \) is homeomorphic to \( \Pi(\lim X(s) : s \in S) \).

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If $X = \{X_a, p_{ab}, A\}$ is an inverse system, then we have inverse system $X \times X = \{X_a \times X_a, p_{ab} \times p_{ab}, A\}$. Let $X = \lim X$. By Theorem 1.1 we infer that $X \times X$ is homeomorphic to the limit of inverse system $X \times X$.

We say that an inverse system $X = \{X_a, p_{ab}, A\}$ is factorizing [1, p. 17] if for each real-valued mapping $f : \lim X \to \mathbb{R}$ there exist an $a \in A$ and a mapping $f_a : X_a \to \mathbb{R}$ such that $f = f_a p_a$.

An inverse system $X = \{X_a, p_{ab}, A\}$ is said to be $\sigma$-directed if for each sequence $a_1, a_2, ..., a_k, ...$ of the members of $A$ there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

**Lemma 1.2.** [1, Corollary 1.3.2, p. 18]. If $X = \{X_a, p_{ab}, A\}$ is a $\sigma$-directed inverse system of compact spaces with surjective bonding mappings, then it is factorizing.

An inverse system $X = \{X_a, p_{ab}, A\}$ is said to be $\tau$-continuous [1, p. 19] if for each chain $B$ in $A$ with $\text{card}(B) < \tau$ and $\sup B = b$, the diagonal product $\Delta \{p_{ab} : a \in B\}$ maps the space $X_b$ homeomorphically into the space $\lim X_a$.

An inverse system $X = \{X_a, p_{ab}, A\}$ is said to be $\tau$-system [1, p. 19] if:

a) $\tau \geq \omega(X_a)$ for every $a \in A$,

b) The system $X = \{X_a, p_{ab}, A\}$ is $\tau$-continuous,

c) The indexing set $A$ is $\tau$-complete.

If $\tau = \aleph_0$, then $\tau$-system is called a $\sigma$-system. The following theorem is called the Spectral Theorem [1, p. 19].

**Theorem 1.3.** [1, Theorem 1.3.4, p. 19]. If a $\tau$-system $X = \{X_a, p_{ab}, A\}$ with surjective limit projections is factorizing, then each map of its limit space into the limit space of another $\tau$-system $Y = \{Y_a, q_{ab}, A\}$ is induced by a morphism of cofinal and $\tau$-closed subsystems. If two factorizing $\tau$-systems with surjective limit projections and the same indexing set have homeomorphic limit spaces, then they contain isomorphic cofinal and $\tau$-closed subsystems.

Let us remark that the requirement of surjectivity of limit projections of systems in Theorem 1.3 is essential [1, p. 21].

In the sequel we will need the following theorem.

**Theorem 1.4.** [7, Theorem 1.6, p. 402]. If $X$ is the Cartesian product $X = \prod \{X_s : s \in S\}$, where $\text{card}(S) > \aleph_0$ and each $X_s$ is compact, then there exists a $\sigma$-directed inverse system $X = \{Y_a, P_{ab}, A\}$ of the countable products $Y_a = \prod \{X_\mu : \mu \in a\}$, $\text{card}(a) = \aleph_0$, such that $X$ is homeomorphic to $\lim X$.

2. Fixed point property for non-metric compact spaces

A fixed point of a function $f : X \to X$ is a point $p \in X$ such that $f(p) = p$. A space $X$ is said to have the fixed point property provided that every surjective mapping $f : X \to X$ has a fixed point.

First Step in the proving fixed point property for hyperspaces of non-metric chainable continua is the following general Theorem for fixed point property for non-metric continua.

**Theorem 2.1.** Let $X = \{X_a, p_{ab}, A\}$ be a $\sigma$-system of compact spaces with the limit $X$ and onto projections $p_a : X \to X_a$. Let $\{f_a : X_a \to X_a\} : X \to X$ be a morphism. Then the induced mapping $f = \lim f_a$ has a fixed point if and only if each mapping $f_a : X_a \to X_a$, $a \in A$, has a fixed point.
Proof. The if part. Let $F_a, a \in A$, be a set of fixed points of the mapping $f_a$.

Claim 1. Every set $F_a$ is closed. This is a consequence of the following theorem [3, Theorem 1.5.4., p. 59]. For any pair $f,g$ of mappings of a space $X$ into a Hausdorff space $Y$, the set

$$\{x \in X : f(x) = g(x)\}$$

is closed in $X$.

It suffices to set $g(x) = x$ and $Y = X$.

Claim 2. If $b \geq a$, then $p_{ab}(F_b) \subseteq F_a$. Let $x_b$ be any point of $F_b$. From the commutativity of the diagram (1.1) it follows $p_{ab}(f_b(x)) = f_a(p_{ab}(x))$. We have $p_{ab}(x) = f_a(p_{ab}(x))$ since $f_b(x) = x$. This means that for the point $y = p_{ab}(x) \in X_a$ we have $y = f_a(y)$, i.e., $y \in F_a$. We infer that $p_{ab}(x) \in F_a$ and $p_{ab}(F_b) \subseteq F_a$.

Claim 3. $F = \{F_a, p_{ab}|F_b, A\}$ is an inverse system of compact spaces with the non-empty limit $F$.

Claim 4. The set $F \subseteq X$ is the set of fixed points of the mapping $f$. Let $x \in F$ and let $x_a = p_a(x), a \in A$. Now, $f_a(x_a) = x_a$ since $x_a \in F_a$. We infer that $f(x) = x$ since the morphism $\{f_a : a \in A\}$ induces $f$. The proof of the “if” part is complete.

The only if part. Suppose that the induced mapping $f$ has a fixed point $x$. Let us prove that every mapping $f_a, a \in A$, has a fixed point. Now we have $f_a p_a(x) = p_a f(x)$. From $f(x) = (x)$ it follows $f_a p_a(x) = p_a(x)$. We infer that $p_a(x)$ is a fixed point for $f_a$.

As an immediate consequence of this theorem and the Spectral theorem 1.3 we have the following result.

Theorem 2.2. Let a non-metric continuum $X$ be the inverse limit of an inverse $\sigma$-system $X = \{X_a, p_{ab}, A\}$ such that each $X_a$ has the fixed point property and each bonding mapping $p_{ab}$ is onto. Then $X$ has the fixed point property.

The following result is an application of Theorem 2.2.

Theorem 2.3. Let $S$ be an infinite set and $Q = \Pi\{X_s : s \in S\}$ Cartesian product of compact spaces. If each product $X_{s_1} \times X_{s_2} \times \ldots \times X_{s_n}$ of finitely many spaces $X_s$ has the fixed point property, then $Q$ has the fixed point property.

Proof. We shall consider the following cases.

Case 1. $\text{card}(S) = N_0$. We may assume that $S = N$. The proof is a straightforward modification of the proof of [9, Corollary 3.5.3, pp. 106-107]. Let $f : Q \rightarrow Q$ be continuous. For every $n \in N$ define

$$K_n = \{x \in Q : (x_1, \ldots, x_n) = (f(x)_1, \ldots, f(x)_n)\}.$$ 

It is clear that for every $n$ the set $K_n$ is closed in $Q$ and that $K_{n+1} \subseteq K_n$. For every $n \in N$, let $o_n$ be a given point of $X_n$ and $p_n : Q \rightarrow X_1 \times \ldots \times X_n$ be the projection. Define continuous function $f_n : X_1 \times \ldots \times X_n \rightarrow X_1 \times \ldots \times X_n$ by

$$f_n(x_1, \ldots, x_n) = (p_n f)(x_1, \ldots, x_n, o_{n+1}, o_{n+2}, \ldots).$$

By assumption of Theorem $f_n$ has the fixed point property, say $(x_1, \ldots, x_n)$. It follows that

$$(x_1, \ldots, x_n, o_{n+1}, o_{n+2}, \ldots) \in K_n.$$ 

We conclude that $\{K_n : n \in N\}$ is a decreasing collection of nonempty closed subsets of $Q$. By compactness of $Q$ we have that

$$K = \cap \{K_n : n \in N\}$$
is nonempty. It is clear that every point in \( K \) is a fixed point of \( f \).

**Case 2.** \( \text{card}(A) \geq \aleph_1 \). By Theorem 1.4 there exists a \( \sigma \)-directed inverse system \( X = \{ Y_a, q_{ab}, A \} \) of the countable products \( Y_a = \prod \{ X_\mu : \mu \in a \} \), \( \text{card}(a) = \aleph_0 \), such that \( Q \) is homeomorphic to \( \lim X \). By Case 1 each \( Y_a \) has the fixed point property. Finally, by Theorem 2.2 we infer that \( Q \) has the fixed point property. \( \square \)

3. **Fixed point property for the hyperspaces of non-metric chainable continua**

In this Section we shall study the fixed point property of the hyperspaces of chainable continua.

A chain \( \{ U_1, ..., U_n \} \) is a finite collection of sets \( U_i \) such that \( U_i \cap U_j \neq \emptyset \) if and only if \( |i - j| \leq 1 \). A continuum \( X \) is said to be chainable or arc-like if each open covering of \( X \) can be refined by an open covering \( u = \{ U_1, ..., U_n \} \) such that \( \{ U_1, ..., U_n \} \) is a chain.

Second Step in the proving fixed point property for hyperspaces of non-metric chainable continua is the following general expanding Theorem for non-metric chainable continua into inverse \( \sigma \)-system.

**Theorem 3.1.** If \( X \) is a chainable continuum, then there exists a \( \sigma \)-system \( Q_{\sigma} = \{ Q_\Delta, p_\Delta, A_\sigma \} \) such that each \( Q_\Delta \) is a metric chainable continuum, \( p_\Delta \) are surjections and \( X \) is homeomorphic with the inverse limit \( \lim Q_{\sigma} \).

**Proof.** The proof is broken into several steps.

**Step 1.** If \( X \) is a chainable continuum, then there exists a system \( Q = \{ Q_a, q_{ab}, A \} \) such that each \( Q_a \) is a metric chainable continuum and \( X \) is homeomorphic with the inverse limit \( \lim Q \). By [8, Theorem 2] every chainable continuum \( X \) is homeomorphic with the inverse limit of an inverse system \( \{ Q_a, q_{ab}, A \} \) of metric chainable continua \( Q_a \). One can assume that \( q_{ab} \) are onto mappings since a closed connected subset \( C \) of chainable continuum is chainable.

**Step 2.** There exists a \( \sigma \)-system of chainable continua such that \( X \) is homeomorphic with its inverse limit. The inverse system \( \{ Q_a, q_{ab}, A \} \) is not a \( \sigma \)-system. Now we shall prove that such inverse system exists. For each subset \( \Delta_0 \) of \( (A, \leq) \) we define sets \( \Delta_n, n = 0, 1, ..., \) by the inductive rule \( \Delta_{n+1} = \Delta_n \cup \{ m(x, y) : x, y \in \Delta_n \} \), where \( m(x, y) \) is a member of \( A \) such that \( x, y \leq m(x, y) \). Let \( \Delta = \bigcup \{ \Delta_n : n \in \mathbb{N} \} \).

It is clear that \( \text{card}(\Delta) = \text{card}(\Delta_0) \). Moreover, \( \Delta \) is directed by \( \leq \). For each directed set \( (A, \leq) \) we define

\[ A_\sigma = \{ \Delta : \emptyset \neq \Delta \subset A, \text{card}(\Delta) \leq \aleph_0 \text{ and } \Delta \text{ is directed by } \leq \}. \]

Let us prove that \( A_\sigma \) is \( \sigma \)-directed and \( \sigma \)-complete. Let \( \{ \Delta^1, \Delta^2, ..., \Delta^n, ... \} \) be a countable subset of \( A_\sigma \). Then \( \Delta_0 = \bigcup \{ \Delta^1, \Delta^2, ..., \Delta^n, ... \} \) is a countable subset of \( A_\sigma \). Define sets \( \Delta_n, n = 0, 1, ..., \) by the inductive rule \( \Delta_{n+1} = \Delta_n \cup \{ m(x, y) : x, y \in \Delta_n \} \), where \( m(x, y) \) is a member of \( A \) such that \( x, y \leq m(x, y) \). Let \( \Delta = \bigcup \{ \Delta_n : n \in \mathbb{N} \} \).

It is clear that \( \text{card}(\Delta) = \text{card}(\Delta_0) \). This means that \( \Delta \) is countable. Moreover \( \Delta \supseteq \Delta^i, i \in \mathbb{N} \). Hence \( A_\sigma \) is \( \sigma \)-complete. Let \( \Delta^1 \subset \Delta^2 \subset ... \subset \Delta^n \subset ... \) be a countable chain in \( A_\sigma \). Then \( \Delta = \bigcup \{ \Delta^i : i \in \mathbb{N} \} \) is countable and directed subset of \( A \), i.e., \( \Delta \in A_\sigma \). It is clear that \( \Delta \supseteq \Delta^i, i \in \mathbb{N} \). Moreover, for each \( \Gamma \in A_\sigma \) with property \( \Gamma \supseteq \Delta^i, i \in \mathbb{N} \), we have \( \Gamma \supseteq \Delta \). Hence \( \Delta = \sup \{ \Delta^i : i \in \mathbb{N} \} \). This means that \( A_\sigma \) is \( \sigma \)-complete.
For any finitely many subsets denoted by projections \( q^b : Q^b \rightarrow Q^\delta \), let \( p_{\Delta} : Q^\delta \rightarrow Q_{\Delta} \) denote the map induced by the projections \( q^b : Q^b \rightarrow Q^\delta \), \( \delta \in \Delta \), of the inverse system \( Q^\delta \).

Now we shall prove that if \( Q = \{Q_a, q_{ab}, A\} \) is an inverse system, then \( Q_\sigma = \{Q_\Delta, p_{\Delta}, A\} \) is a \( \sigma \)-directed and \( \sigma \)-complete inverse system such that \( \lim Q \) and \( \lim Q_\sigma \) are homeomorphic. Each thread \( x = (x_a : a \in A) \) induces the thread \( (x_a : a \in A) \) for each \( \Delta \in A_\sigma \), i.e., the point \( q_\Delta \in Q_\Delta \). This means that we have a mapping \( H : \lim Q \rightarrow \lim Q_\sigma \) such that \( H(x) = (q_\Delta : \Delta \in A_\sigma) \). It is obvious that \( H \) is continuous and 1-1. The mapping \( H \) is onto since the collections of the threads \( \{q_\Delta : \Delta \in A_\sigma\} \) induces the thread in \( Q \). We infer that \( H \) is a homeomorphism since \( \lim Q \) is compact.

Finally, let us prove that every \( Q_{\Delta} \) is chainable. We may assume that \( Q^\Delta = \{Q_b, q_{bb}, \Delta\} \) is an inverse sequence since \( \Delta \) is countable and \( Q_{\Delta} = \lim Q^\Delta \). Let \( u = \{U_1, ..., U_n\} \) be an open covering of \( Q_{\Delta} \). There exists a \( b \in \Delta \) and an open covering \( u_b = \{U^b_1, ..., U^b_m\} \) of \( Q_b \) such that \( \{q^{-1}_b(U^b_1), ..., q^{-1}_b(U^b_m)\} \) refines the covering \( u = \{U_1, ..., U_n\} \). There is a chain \( \{V^b_1, ..., V^b_p\} \) which refines \( u_b \) since \( Q_b \) is chainable. It is clear that \( \{q^{-1}_b(V^b_1), ..., q^{-1}_b(V^b_p)\} \) is a chain which refines the covering \( u \). Hence, \( Q_{\Delta} \) is chainable.

**Step 3.** One can assume that \( p_{\Delta} \) and \( p_{\Delta} : \lim Q_\sigma \rightarrow Q_{\Delta} \) are onto mappings. If \( p_{\Delta} \) and \( p_{\Delta} : \lim Q_\sigma \rightarrow Q_{\Delta} \) are not onto mappings, then we shall use the inverse system \( Q_\sigma^b = \{Q_{\Delta}(\lim Q_\sigma), p_{\Delta_{\Gamma}}|p_{\Delta}(\lim Q_\sigma), A_\sigma\} \). Each \( p_{\Delta} \) is chainable since a closed connected subset of chainable continuum is chainable.

The proof is completed since \( X \) is representable as the inverse limit of \( \sigma \)-system \( Q_{\sigma} = \{Q_{\Delta}, p_{\Delta}, A_\sigma\} \) of metric chainable continua \( Q_{\Delta} \).

Finally, we represent the various hyperspaces of a non-metric chainable continuum \( X \) as the inverse limits of hyperspaces of metric chainable continua.

Let \( X \) be a space. We define its hyperspaces as the following sets:

\[
X = \{F \subseteq X : F \text{ is closed and nonempty}\},
\]

\[
C(X) = \{F \subseteq X : F \text{ is connected}\},
\]

\[
F_n(X) = \{A \subseteq X : A \text{ is nonempty and } A \text{ has at most } n \text{ points}\}.
\]

For any finitely many subsets \( S_1, ..., S_n \), let

\[
\langle S_1, ..., S_n \rangle = \left\{ F \subseteq X : F \subseteq \bigcup_{i=1}^n S_i, \text{ and } F \cap S_i \neq \emptyset, \text{ for each } i \right\}.
\]

The topology on \( 2^X \) is the Vietoris topology, i.e., the topology with a base \( \{< U_1, ..., U_n > : U_i \text{ is an open subset of } X \text{ for each } i \text{ and each } n < \infty \} \), and \( C(X), X(n) \) are subspaces of \( 2^X \). Moreover, \( X(1) \) is homeomorphic to \( X \).

The topology on \( 2^X \) is the Vietoris topology and \( C(X) \) and \( F_n(X) \) is a subspace of \( 2^X \).

Let \( X \) be the spaces and let \( f : X \rightarrow Y \) be a mapping. Define \( 2^f : 2^X \rightarrow 2^Y \) by \( 2^f(F) = f(F) \) for \( F \subseteq 2^X \). It is known that \( 2^f \) is continuous and \( 2^f(C(X)) \subseteq C(Y) \). Moreover, \( 2^f(C_n(X)) \subseteq F_n(Y) \). The restriction \( 2^f|C(X) \) is denoted by \( C(f) \). Similarly, the restriction \( 2^f|F_n(X) \) is denoted by \( F_n(f) \).

Let \( X = \{X_a, p_{ab}, A\} \) be an inverse system of compact spaces with the natural projections \( p_a : \lim X \rightarrow X_a, a \in A \). Then \( 2^X = \{2^{X_a}, p_{ab}, A\} \), \( C(X) = \{C(X_a), C(p_{ab}), A\} \) and \( F_n(X) = \{F_n(X_a), F_n(p_{ab}), A\} \) form inverse systems.
Lemma 3.2. Let $X = \lim X$. Then $2^X = \lim 2^X$, $C(X) = \lim C(X)$ and $F_n(X) = \lim F_n(X)$.

In [6, Corollary 5, p. 616] it is proved the following result.

**Theorem 3.3.** The third symmetric product $F_3(X)$ of a metric chainable continuum $X$ has the fixed point property.

The proof given there is purely metric. This means that it is reasonable to give the proof for non-metric chainable continua.

**Theorem 3.4.** The third symmetric product $F_3(X)$ of a chainable continuum $X$ (metric or non-metric) has the fixed point property.

**Proof.** If $X$ is a metric chainable continuum, then apply Theorem 3.3. In order to complete the proof, we shall assume that $X$ is non-metric chainable continuum. By Theorem 3.1 there exists a $\sigma$-system $X = \{X_a, p_{ab}, A\}$ such that each $X_a$ is a metric chainable continuum, $p_{ab}$ are surjections and $X$ is homeomorphic with the inverse limit $\lim X$. Now we have a $\sigma$-system $F_3(X) = \{F_3(X_a), F_3(p_{ab}), A\}$ whose limit is homeomorphic to $F_3(X)$. In order to apply Theorem 2.2 it suffices to prove that $F_3(p_{ab})$ are surjections for every $a \leq b$. Let $\{x_1, x_2, x_3\} \in F_3(X_a)$. The sets $p_{ab}^{-1}(x_1), p_{ab}^{-1}(x_2), p_{ab}^{-1}(x_3)$ are non-empty since $p_{ab}$ is onto. If $y_1 \in p_{ab}^{-1}(x_1), y_2 \in p_{ab}^{-1}(x_2)$ and $y_3 \in p_{ab}^{-1}(x_3)$, then $\{y_1, y_2, y_3\} \in F_3(X_b)$ and $F_3(p_{ab})(\{y_1, y_2, y_3\}) = \{x_1, x_2, x_3\} \in F_3(X_a)$. Hence, $F_3(p_{ab})$ is a surjection. By Theorem 3.3 each $X_a$ has the fixed point property. Finally, by Theorem 2.2 we infer that $F_3(X)$ has the fixed point property. \[
\square
\]

From the proof of the above theorem it is clear that is true the following theorem.

**Theorem 3.5.** The $n$th-symmetric product $F_n(X)$ of a chainable non-metric continuum $X$ has the fixed point property if the $n$th-symmetric product of every chainable metric continuum has the fixed point property.

From this theorem we shall give the following result.

**Theorem 3.6.** If $X$ is a non-metric chainable continuum, then $X$ has the fixed point property.

**Proof.** Now $X$ is homeomorphic to $F_1(X)$ which is homeomorphic to $\lim F_1(X)$. From Theorem 3.5 it follows that $\lim F_1(X)$ has the fixed point property since each metric chainable continuum $F_1(Y)$ (homeomorphic to $Y$) has the fixed point property [4]. \[
\square
\]

Another hyperspace of a continuum is the hyperspace $C(X) = \{F \in 2^X : F$ is connected\}. The following result is known.

**Theorem 3.7.** [11]. If $Y$ is a metric chainable continuum, then $C(Y)$ has the fixed point property.

For non-metric chainable continua we have the following result.

**Theorem 3.8.** If $X$ is a non-metric chainable continuum, then $C(X)$ has the fixed point property.
Proof. If $X$ is a metric chainable continuum, then apply Theorem 3.7. In order to complete the proof, we shall assume that $X$ is non-metric chainable continuum. By Theorem 3.1 there exists a $σ$-system $X = \{X_α, p_{ab}, A\}$ such that each $X_α$ is a metric chainable continuum, $p_{ab}$ are surjections and $X$ is homeomorphic with the inverse limit $\lim lim X$. Now we have a $σ$-system $C(X) = \{C(X_α), C(p_{ab}), A\}$ whose limit is homeomorphic to $C(X)$. In order to apply Theorem 2.2 it suffices to prove that $C(p_{ab})$ are surjections for every $a ≤ b$. Let $C \in C(X_α)$. The set $p_a^{-1}(C)$ contains a continuum $D$ in $Y$ such that $p_a(D) = C ([10, Theorem 12.46, p. 262])$. Hence, $C(p_{ab})$ is a surjection. By Theorem 3.7 each $X_α$ has the fixed point property. Finally, by Theorem 2.2 we infer that $C(X)$ has the fixed point property.

4. Fixed point property of the product of chainable continua

Dyer [2, Theorem 1, p. 663] showed the following result.

**Theorem 4.1.** Suppose that $M$ is the Cartesian product of $n$ compact chainable metric continua $X_1, X_2, ..., X_n$ and $f$ is a continuous mapping of $M$ into itself. Then there is a point $x ∈ M$ such that $x = f(x)$.

For $n = 2$ we have the following result.

**Theorem 4.2.** [5, p. 199, Exercise 22.26]. If $X$ and $Y$ are metric chainable continua, then $X × Y$ has the fixed point property.

Dyer [2, Corollary, p. 665] showed the following general result.

**Theorem 4.3.** Cartesian product of the elements of any collection of chainable metric continua has the fixed point property.

We will show that last Theorem 4.3 is true for non-metrizable chainable continua.

**Theorem 4.4.** Cartesian product of the elements of any collection of chainable continua of the same weight has the fixed point property.

**Proof.** If for every $s ∈ S$ we have a chainable non-metrizable continuum $X(s)$, then, for every $s ∈ S$, there exists an inverse system $X(s) = \{X_α(s), p_{ab}s, A(s)\}$ such that $X(s)$ is homeomorphic to $lim X(s)$ and every $X_α(s)$ is a metric chainable continuum (Theorem 3.1). If $w(X(s_1)) = w(X(s_2)), s_1, s_2 ∈ S$, then $A(s_1) = A(s_2)$ and we may suppose that $A(s) = A$ for every $s ∈ S$. By Theorem 1.1 the family $Π\{X(s) : s ∈ S\} = \{Π\{X_α(s) : s ∈ S\}, Π\{p_{ab}(s) : s ∈ S\}, A\}$ is an inverse system and $lim(Π\{X(s) : s ∈ S\})$ is homeomorphic to $Π\{lim X(s) : s ∈ S\}$. From Theorem 4.3 it follows that each $Π\{X_α(s) : s ∈ S\}$ has the fixed point property. Finally, from Theorem 2.2 it follows that $Π\{X(s) : s ∈ S\}$ has the fixed point property.

**QUESTION.** Is it true that the assumption ”of the same weight” in Theorem 4.4 can be omitted?

As an immediate application of Theorem 4.4 we give the following generalization of Brouwer Fixed-Point Theorem. Let $L$ be a non-metric arc. The space $X$ is said to be a generalized $n$-cell if it is homeomorphic to $L^n = L × L × ... × L$ ($n$ factors).

**Theorem 4.5.** Every mapping $f : L^n → L^n$ has a fixed point, i.e., $L^n$ has the fixed point property.

Theorem 4.5 implies the following result.
Theorem 4.6. If $L_1, ..., L_n$ are arcs (metric or non-metric), then $L_1 \times L_2 \times ... \times L_n$ has the fixed point property.

Proof. Step 1. If $M$ is a subarc of the arc $L$, then there exists a retraction $r : L \to M$. Let $a, b, c, d$ be end points of $L$ and $M$ such that $a \leq c < d \leq b$. We define $r : L \to M$ as follows:

$$r(x) = \begin{cases} c & \text{if } a \leq x \leq c, \\ x & \text{if } c \leq x \leq d, \\ d & \text{if } d \leq x \leq b. \end{cases}$$

Step 2. If $L_1, L_2, ..., L_n$ is a finite collection of arcs, then there are arcs $L$ such that $L_1, L_2, ..., L_n$ are subarcs of $L$. For each $i \in \{1, 2, ..., n\}$ let $a_i, b_i$ be a pair of end points of $L_i$ such that $a_i < b_i$. If we identify the pair of points $\{b_1, a_2\}, \{b_2, a_3\} ,..., \{b_{n-1}, a_n\}$ we obtain an arc $L$ such that $L_i \subset L$ for each $i \in \{1, 2, ..., n\}$.

Step 3. $L_1 \times L_2 \times ... \times L_n$ is a retract of $L^n$. Let $L$ and $L_1, L_2, ..., L_n$ be as in Step 2. Let $r_i : L \to L_i, i \in \{1, 2, ..., n\}$ be a retraction defined in Step 1. Let us prove that $r = r_1 \times r_2 \times ... \times r_n$ is a retraction of $L^n$ onto $L_1, L_2, ..., L_n$. If $(y_1, y_2, ..., y_n) \in L^n$, then we have: $r_1 \times r_2 \times ... \times r_n(y_1, y_2, ..., y_n) = (r_1(y_1), r_2(y_2), ..., r_n(y_n)) \in L_1 \times L_2 \times ... \times L_n$ since $r_i(y_i) \in L_i$. If $(x_1, x_2, ..., x_n) \in L_1 \times L_2 \times ... \times L_n$, then $r_1 \times r_2 \times ... \times r_n(x_1, x_2, ..., x_n) = (r_1(x_1), r_2(x_2), ..., r_n(x_n)) = (x_1, x_2, ..., x_n) \in L_1 \times L_2 \times ... \times L_n$ since $r_i(x_i) \in x_i$.

Step 4. The product $L_1 \times L_2 \times ... \times L_n$ has the fixed point property since it is retract of the product $L^n$ which has the fixed point property (Theorem 4.5). The proof is completed.

□

Theorem 4.7. If $L = \Pi \{L_s : s \in S\}$ is a Cartesian product of arcs $L_s$, then $L$ has the fixed point property.

Proof. Apply Theorems 4.6 and 2.3.

□

For Cartesian product of two chainable continua the assumption concerning the weight in Theorem 4.4 can be omitted.

Theorem 4.8. If $X$ and $Y$ are non-metrizable chainable continua, then $X \times Y$ has the fixed point property.

Proof. First we shall prove that if $X$ is any chainable continuum and if $Y$ is a metric chainable continuum, then $X \times Y$ has the fixed point property. By Theorem 3.1 there exists a $\sigma$-directed inverse system $X = \{X_a, p_{ab}, A\}$ such that each $X_a$ is a metric chainable continuum and $X$ is homeomorphic to $\lim X$. It is clear that $X \times Y = \{X_a \times Y, p_{ab} \times id, A\}$ is a $\sigma$-directed inverse system whose limit is homeomorphic to $X \times Y$. Every $X_a \times Y$ has the fixed point property since it is the product of metric chainable continua (Theorem 4.2). Applying Theorem 2.2 we infer that $X \times Y$ has the fixed point property.

Suppose now that $X$ and $Y$ are non-metric chainable continua. Using again Theorem 3.1 we obtain a $\sigma$-directed inverse system $X = \{X_a, p_{ab}, A\}$ such that each $X_a$ is a metric chainable continuum and $X$ is homeomorphic to $\lim X$. It is clear that $X \times Y = \{X_a \times Y, p_{ab} \times id, A\}$ is a $\sigma$-representation of $X \times Y$. From the first part of this proof it follows that every $X_a \times Y$ has the fixed point property since it is the product of metric chainable continuum $X_a$ and a chainable continuum $Y$. Applying Theorem 2.2 we infer that $X \times Y$ has the fixed point property. □
We close this section with the fixed point property for multifunctions on chainable continua.

A multifunction, \( F : X \to Y \), from a space \( X \) to a space \( Y \) is a point-to-set correspondence such that, for each \( x \in X \), \( F(x) \) is a subset of \( Y \). For any \( y \in Y \), we write \( F^{-1}(y) = \{ x \in X : y \in F(x) \} \). If \( A \subset X \) and \( B \subset Y \), then \( F(A) = \cup \{ F(x) : x \in A \} \) and \( F^{-1}(B) = \cup \{ F^{-1}(y) : y \in B \} \).

A multifunction \( F : X \to Y \) is said to be continuous if and only if: (i) \( F(x) \) is closed for each \( x \in X \), (ii) \( F^{-1}(B) \) is closed for each closed set \( B \) in \( Y \), (iii) \( F^{-1}(V) \) is open for each open set \( V \) in \( Y \).

A topological space \( X \) is said to have F.p.p (fixed point property for multi-valued functions) if for every multi-valued continuous function \( F : X \to X \) there exists a point \( x \in X \) such that \( x \in F(x) \). It follows that \( X \) has F.p.p if for every single-valued continuous function \( F : X \to 2^X \) there exists a point \( x \in X \) such that \( x \in F(x) \).

**Theorem 4.9.** [12]. If \( X \) is any metric chainable continuum, then \( X \) has the F.p.p.

Now we shall prove the following result.

**Theorem 4.10.** Each chainable continuum \( X \) has the F.p.p.

**Proof.** If an chainable continuum is metrizable, then it has F.p.p (Theorem 4.9). Suppose that chainable continuum \( X \) is non-metrizable. By virtue of Theorem 3.1 there exists a \( \sigma \)-system \( X_\sigma = \{ X_\Delta, P_\Delta, A_\sigma \} \) of metric chainable continua \( X_\Delta \) and onto mappings \( P_\Delta \) such that \( X \) is homeomorphic to \( \lim X_\sigma \). Moreover, we have the inverse system \( 2^X = \{ 2^{X_\delta}, 2^{P_\delta}, A \} \) whose limit is \( 2^X \). Let \( F : X \to 2^X \) be a continuous mapping. From Theorem 1.3 it follows that there exists a subset \( B \) cofinal in \( A \) such that for every \( b \in B \) there exists a continuous mapping \( F_b : X_b \to 2^{|b|} \) with the property that \( \{ F_b : b \in B \} \) is a morphism which induce \( F \). Theorem 4.9 implies that the set \( Y_b \subset X_b, b \in B \), of fixed points of \( F_b \) is non-empty. Let us prove that \( Y_b \) is a closed subset of \( X_b \). We shall prove that \( U_b = X_b \setminus Y_b \) is open. Let \( x_b \in U_b \). This means that \( x_b \) and \( F_b(x_b) \) are disjoint closed subset of \( X_b \). By the normality of \( X_b \) there exists a pair of open sets \( U, V \) such that \( x \in U \) and \( Y_b \subset V \). From the continuity of \( F_b \) it follows that there exists an open set \( W \subset U \) such that for every \( x \in W \) we have \( f(x) \subset Y \). Hence, \( U_b \) is open and, consequently, \( Y_b \) is closed. Now, we shall prove that the collection \( \{ Y_b, p_{bc}[Y_c, B] \} \) is an inverse system. To do this we have to prove that if \( c > b \), then \( p_{bc}(Y_c) \subset Y_b \). Let \( x_c \) be a point of \( Y_c \). This means that \( x_c \in f_c(x_c) \). Hence, \( p_{bc}(x_c) \in p_{bc}(F_c(x_c)) = F_b p_{bc}(x_c) \). We conclude that the point \( x_b = p_{bc}(x_c) \) has the property \( x_b \in f_b(x_b) \), i.e., \( x_b = p_{bc}(x_c) \in Y_b \). Finally, \( p_{bc}(Y_c) \subset Y_b \) and \( \{ Y_b, p_{bc}[Y_c, B] \} \) is an inverse system with non-empty limit. Let \( Y = \lim \{ Y_b, p_{bc}[Y_c, B] \} \). In order to complete the proof we shall prove that for every \( x \in Y \) we have \( x \in F(x) \). Now we have \( p_b(x) \in Y_b \), i.e., \( p_b(x) \in F_b(p_b(x)) = p_b F(x) \), for every \( b \in B \). It follows that \( x \in F(x) \) since \( x \notin F(x) \) implies that there is a \( b \in B \) such that \( p_b(x) \notin p_b F(x) \). We conclude that \( F \) has the fixed point property. \( \square \)
References


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