ALBANIAN JOURNAL
OF MATHEMATICS
Volume 4, Number 4, Pages 195–201
ISSN 1930-1235: (2010)

RELATIONS BETWEEN HEUN EQUATIONS AND PAINLEVE EQUATIONS

S.YU. SLAVYANOV, A.YA. KAZAKOV, F. R. VUKAJLOVIĆ

Special functions play significant role in Computer Algebra packages. Here we can mention all-purpose packages as Mathematika or Maple as well as specialized packages as SFTools. Further development would without doubt be focused on Heun functions and closely related Painleve transcendents. Partly the relationship between Heun equations and Painleve equations is presented in the package SFTools. However new studies induce revision of presentation of these relations. The items of these revisions are the following.

1. Relations between equations belonging to Heun class, the corresponding deformed equations with added apparent singularity and the corresponding 2×2 systems. It is needed to stress that two different 2×2 systems correspond to one deformed equation.
2. Relations between integral transforms linking different equations belonging to Heun class and Okamoto-type transforms linking Painleve equations. These give rise to symmetries in the class of corresponding functions.
3. Relations between known physical models which are solved in terms of Heun functions like two-Coulomb centers problem, Stark effect etc. and the corresponding problems in classical dynamics.

In the publications [1, 2] and later in the the book [3] the author formulated the statement that every equation belonging to Heun class induces the corresponding equation belonging to Painleve class. This statement has been implemented in the package SFTools which supplied different information on special functions [4]. The mentioned induction called later as "antiquantization" is realised by substitution instead of quantum variables: coordinate and momentum – in the hamiltonian for Heun equations – the classical variables in the corresponding classical Lagrangian. Newtonian equations of motion appear to be Painleve equations. However several aspects of the theory were missing at that stage. These there:

1. What transforms of Painleve equations are induced by s-homotopic transformations of Heun equations?
2. What are the deformed Heun equations generated from Heun equation by adding an apparent singularity?
3. What are the relations to 2×2 first order linear systems which often are assumed as basic in handling with Painleve equations?
4. Are there other linear systems related to Painleve equations?
5. What transforms of Painleve equations are induced by integral transforms of Heun equations?
6. What classical physical problems are related to the well-known quantum problems exposed in terms of particular equations belonging to Heun class?

In view of a large number of Heun equations the detailed answer to posed questions is rather complicated and needs to be coded in a software package. Here is given a general approach to basic Heun equation. The presentation is referring to previous publications of the author with collaborators. The canonical form of Heun equation is chosen as

$$w'' + \left[\frac{1-\theta_1}{z} + \frac{1-\theta_2}{z-1} + \frac{1-\theta_3}{z-t}\right] w' + \left[\frac{\alpha\beta}{z(z-1)} - \frac{t(t-1)H}{z(z-1)(z-t)}\right] w = 0. \tag{1}$$

Here θ_j are characteristic exponents for the solutions with singularities at the points z_j, $z_1 = 0$, $z_2 = 1$, $z_3 = t$.

Parameters α, β – are characteristic exponents at infinity. According to Fuchs theorem it holds

$$\sum_{j=1}^{3} \theta_j + \alpha + \beta = 1. \tag{2}$$

Parameter H is assumed to be the energy. It is normalized in such a way that the residue of of the corresponding term at $z = t$ is unity. A more general Heun equation can be obtained by applying linear transformation of independent variable and s-homotopic transformations [3] of dependent variable

$$y := (z - z_k)^{\gamma_k} w.$$

It is as following

$$\sigma(z)y''(z) + \sum_{j=1}^{3}(1-b_j)\sigma_j(z)y'(z) + \left[\sum_{j=1}^{3} \frac{a_j\sigma_j(z)}{(z-z_j)} + \delta(z-z_3) - \left(\frac{\lambda\sigma_3(z_3)}{(z_2-z_1)} + \frac{1}{2}\sum_{j=1}^{2}(1-b_3)(1-b_j)\frac{\sigma_3(z_3)}{z_3-z_j}\right)\right] y(z) = 0. \tag{3}$$

Here

$$b_j = (\rho_{1j} + \rho_{2j}), \quad a_j = \rho_{1j}\rho_{2j},\ j = 1,2,3, \quad a_\infty = \kappa_1\kappa_2,$$

$$\sigma(z) = \prod_{j=1}^{3}(z - z_j), \quad \sigma_j(z) = \frac{\sigma(z)}{z - z_j},$$

$$\delta = a_\infty - \sum_{j=1}^{3} a_j,$$

where ρ_{mj} are characteristic exponents at finite singular points and κ_1, κ_2 are characteristic exponents at infinity. It can be shown that the quantity λ stays invariant under transforms mentioned above. The other invariants are squares of differences between characteristic exponents

$$\Delta_j = (\rho_{1j} - \rho_{2j})^2,\ j = 1,2,3 \quad \Delta_\infty = (\kappa_1 - \kappa_2)^2.$$

Applying the antiquantization procedure we arrive to the following equation [5]

$$\frac{2\sigma_3(t)}{\sqrt{\sigma(q)}}\frac{d}{dt}\frac{\dot{q}\sigma_3(t)}{\sqrt{\sigma(q)}}+\frac{\dot{q}\sigma_3^2(t)}{\sigma(q)(q-t)}+$$

$$\left[-\Delta_\infty+\sum_{j=1}^{2}\frac{(\Delta_j+1-2b_j)\sigma_j(z_j)}{(q-z_j)^2}+\frac{(\Delta_3-1)\sigma_3(t)}{(q-t)^2}\right]=0. \tag{4}$$

This is a general form of the Painlevé equation P^6 generated by general form Heun equation. Two important features of equation (4) should be emphasized.

1. The role of the singular point $z_3 = t$ in (1) is specific in (4) compared to the other points z_1, z_2.
2. The only influence of generalization due to s-homotopic transformation is a slight dependence on b_j $j = 1, 2$ in (4).

Deformed Heun equations. Here only the canonical form of Heun equation is studied. The deformed Heun equation termed as Heun1 arises by adding an apparent singularity into Heun equation thus increasing the number of Fuchsian singular points up to five. It can be written as following.

$$w''+\left[\frac{1-\theta_1}{z}+\frac{1-\theta_2}{z-1}+\frac{1-\theta_3}{z-t}-\frac{1}{z-q}\right]w'+$$

$$+\left[\frac{\alpha\beta}{z(z-1)}+\frac{q(q-1)p}{z(z-1)(z-q)}-\frac{t(t-1)H}{z(z-1)(z-t)}\right]w=0, \tag{5}$$

where θ_j, $j = 1, 2, 3$, are the characteristic exponents for solutions with singularities at the singular points z_j. The set of parameters θ_1, θ_2, α, β, t, q and p corresponds to this equation. We note that θ_3 is considered a dependent parameter because the Fuchs condition slightly different from (2)

$$\sum_{j=1}^{3}\theta_j+\alpha+\beta=0 \tag{6}$$

related to the characteristic exponents at singularities must be satisfied (the choice of the one dependent parameter θ_3 among θ_1, θ_2, and θ_3 is arbitrary). The parameter H is not an independent parameter of (5) either; it is determined from the condition that the point $z = q$ is an apparent singularity of the equation. This condition leads to an explicit expression for H in terms of the parameters θ_1, θ_2, α, β, q, p, and t.

$$H=\frac{1}{\sigma_3(t)}\left[\sigma(q)p^2+p\sum_{j=1}^{3}\sigma_j(q)(1-\theta_j)+\alpha\beta(q-t)\right]. \tag{7}$$

These considerations can be inverted. Namely, if dependence on t is assumed for functions $p(t)$ and $q(t)$ then the property of the apparent singularity to stay an apparent singularity along the path $p(t)$, $q(t)$ in the phase space if $p(t)$, $q(t)$ obey the Hamilton system of equations generated by the hamiltonian H. This latter system is equivalent to P^6 derived above.

First order 2×2 linear system. Historically Painlevé equations are more often related to first order 2×2 systems. However the explicit derivation of P^6 from such systems is to the authors experience extremely boring. Moreover, several additional conditions on the system should be posed and it is not clear to what extent they are necessary. A thorough explanation of this general situation is presented in the recent article by M.V. Babich ([6]). Here we present a more particular approach to this problem referring to ([7]). What are the demands to the system if it is assumed to generate (5)?

1. Firstly, regular singularities of this system must be $z_1 = 0$, $z_2 = 1$, $z_3 = t$, $z_4 = \infty$.
2. Secondly characteristic exponents at infinity must be α, β.
3. Transform from the system to a second order equation must lead to only one apparent singularity

The system for a vector function W is assumed to be

$$MW' = NW \tag{8}$$

with the following values of the matrix coefficients for matrices M and N

$$\begin{pmatrix} z^2 - z & \rho(z-1) \\ z & z - t - \rho \end{pmatrix} W' = \begin{pmatrix} -\alpha z + e_1 & e_2 \\ e_3 & -\beta \end{pmatrix} W. \tag{9}$$

Demands 1. and 2. can be easily checked. System (8) can be brought to the form

$$W' = TW, \qquad T = M^{-1}N = \big(\sigma(z)\big)^{-1} S, \tag{10}$$

where

$$\sigma(z) = \det M = \prod_{j=1}^{3} (z - z_j).$$

Solving system (10) for $w_1(z)$, we obtain the second-order equation

$$w_1''(z) + f^{(1)}(z) w_1'(z) + g^{(1)}(z) w_1(z) = 0, \tag{11}$$

where

$$f^{(1)}(z) = -T_{12}' T_{12}^{-1} - \mathrm{tr} T, \qquad g^{(1)}(z) = T_{12}' T_{12}^{-1} T_{11} - T_{11}' + \det T.$$

Next, solving system(10) for $w_2(z)$, we obtain the second-order equation

$$w_2''(z) + f^{(2)}(z) w_2'(z) + g^{(2)}(z) w_2(z) = 0, \tag{12}$$

where

$$f^{(2)}(z) = -T_{21}' T_{21}^{-1} - \mathrm{tr} T, \qquad g^{(2)}(z) = T_{21}' T_{21}^{-1} T_{22} - T_{22}' + \det T.$$

The matrix $S(z)$ is evaluated in accordance with (10) and is given by

$$\begin{pmatrix} -\alpha z^2 + z\big(e_1 - \rho g_2 + \alpha t\big) - t e_1 + \rho f_2 & g_1 z - f_1 \\ g_2 z^2 - z f_1 & -\beta z^2 + z(\beta - e_2) \end{pmatrix} \tag{13}$$

with

$$f_1 = \rho\beta + t - \rho e_2, \quad f_2 = e_3 + e_1.$$

$$g_1 = \rho\beta + e_2, \quad g_2 = e_3 + \alpha.$$

This implies that in addition to the regular singularities coincident with the regular singularities of system (10), Eqs. (11) and (12) each have only one apparent singularity,

$$q^{(1)} = \frac{\rho\beta + (t-\rho)e_2}{\rho\beta + e_2} \qquad \text{and} \qquad q^{(2)} = \frac{e_3 + e_1}{e_3 + \alpha}. \tag{14}$$

Therefore, these equation are Heun1 equations. We evaluate tr$T$ and $\det T$, which are the same for Eqs.(11) and (12):

$$\begin{aligned} -\operatorname{tr}T &= \frac{e_1}{z} + \frac{\alpha - e_1}{z-1} + \frac{\beta}{z-t} + \frac{\rho f_2}{z(z-t)} - \frac{1}{z-1}\left(\frac{e_2 - \rho(e_1-\alpha)}{t-1}\right) - \\ &\quad + \frac{1}{z-t}\left(\frac{e_2 - \rho(e_1-\alpha)}{t-1}\right), \\ \det T &= \frac{\alpha\beta}{z(z-1)} + \frac{t\alpha\beta - \beta e_1 - e_2 e_3}{\sigma}. \end{aligned} \tag{15}$$

From (15), we obtain the residue of tr$T$ at infinity:

$$\lim_{z\to\infty} z \operatorname{tr}T = -\alpha - \beta. \tag{16}$$

Using (13), we next evaluate the following expressions, which determine the coefficients of Eqs. (11) and (12):

$$\begin{aligned} -T'_{12}T_{12}^{-1} &= -\frac{1}{z-q^{(1)}} + \sum_{j=1}^{3}\frac{1}{z-z_j}, \quad -T'_{21}T_{21}^{-1} = -\frac{1}{z-q^{(2)}} + \sum_{j=2}^{3}\frac{1}{z-z_j}, \\ T'_{12}T_{12}^{-1}T_{11} - T'_{11} &= \frac{\alpha}{z(z-1)} + \left[\alpha t - e_1 + \rho\frac{f_2 - q^{(1)}g_2}{t-q^{(1)}}\right]\frac{1}{\sigma} + \\ &\quad + \left[-\alpha q^{(1)} + e_1 - \rho\frac{f_2 - q^{(1)}g_2}{t-q^{(1)}}\right]\frac{1}{\sigma^{(1)}}, \\ T'_{21}T_{21}^{-1}T_{22} - T'_{22} &= \frac{e_2 + \beta(q^{(2)}-1)}{t-q^{(2)}}\left(\frac{q^{(2)}}{\sigma^{(2)}} - \frac{t}{\sigma}\right), \end{aligned} \tag{17}$$

where

$$\sigma^{(k)}(z) = z(z-1)\big(z - q^{(k)}\big), \quad k = 1, 2.$$

This preliminary computations enable to find explicit expressions for the coefficients of Eqs. (11) and (12) and as a result explicit formulas for ρ, e_j, $j = 1, 2, 3$ in terms of θ_j, $j = 1, 2$, and p, q. The calculations are troublesome and can be simplified by Computer Algebra systems. Here are given final results only for equation (12) omitting index $^{(2)}$.

$$\begin{aligned} e_1 &= -\sigma_3(q)p - \frac{1}{q-t}(t(q-1)(\theta_1 - 1) + q(t-1)(\theta_2 - \alpha) + \beta\sigma_3(q)), \\ e_2 &= -\sigma_1(q)p - \beta(q-1), \\ e_3 &= -qp - \frac{1}{q-t}\left(t(\theta_1 - 1) + \frac{q}{q-1}(t-1)\theta_2 + q(\alpha+\beta)\right), \\ \rho &= t\frac{q-1}{q}\frac{e_1 + \theta_1 - 1}{e_1 - \alpha} \end{aligned} \tag{18}$$

Of course, inverse formulas can also be obtained.

Integral transform for 2×2 systems. We have studied the Fuchsian system of equations

$$(z^2 A + zB + C)\frac{dW}{dz} = (-\alpha z A + E)W, \tag{19}$$

where A, B, C, and E are $2{\times}2$ matrices independent on z

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} -1 & \rho \\ 1 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & -\rho \\ 0 & \rho - t \end{pmatrix}, E = \begin{pmatrix} e_1 & e_2 \\ e_3 & -\beta \end{pmatrix}.$$

If the solution of system (19) is represented as

$$W(z) = \int_L (z-\xi)^{-\alpha}\Phi(\xi)\, dt, \tag{20}$$

where $\Phi(t)$ is a two-vector function and the integration contour L in the complex plane is specified properly then $\Phi(\xi)$ should be a solution of the similar system but with modified matrix coefficients

$$(\xi^2 A + \xi B + C)\frac{d\Phi(\xi)}{d\xi} = \big((\alpha-2)\xi A + E + (\alpha-1)B\big)\Phi(\xi) = 0, \tag{21}$$

Therefore we arrive to the following chain: Heun1 $\to$ Fuchsian system $\to$ modified Fuchsian system $\to$ modified Heun1. If at the first stage the Painlevé equation is generated then at the end the transformed Painleve equation is obtained. This transformation of Painlevé equations belongs to the Okamoto-type transforms. The other way of derivation the Okamoto transforms was proposed in [8].

Fuchsian system 3×3. A particular Fuchsian system of 3×3 first order equations with three Fuchsian singularities at finite points z_j can also be regarded in respect to Heun equation

$$A(z)\vec{w}'(z) = B\vec{w}(z), \quad \vec{w}(z) = \begin{pmatrix} w_1(z) \\ w_2(z) \\ w_3(z) \end{pmatrix}. \tag{22}$$

The matrices $A(z)$ and B is supposed to be of the form

$$A(z) = \begin{pmatrix} z - z_1 & 0 & 0 \\ 0 & z - z_2 & 0 \\ 0 & 0 & z - z_3 \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}. \tag{23}$$

The particularity of (22) is determined by the specific values of Frobenius exponents at singularities

$$\rho_{mj} = 0, 1, b_{jj}$$

With already introduced notation for σ and σ_j and parameters k and q introduced by

$$k = b_{13}B_{21} + b_{12}B_{31}$$

$$q = \frac{b_{13}B_{21}z_2 + b_{12}B_{31}z_3}{b_{13}B_{21} + b_{12}B_{31}}$$

the following third order Fuchsian equation can be derived [9]

$$\sigma(z)w_1'''(z) - \left(\sum_{j=1}^{3}\sigma_j(z)(b_{jj}-1) - \sigma_1(z) + \frac{\sigma}{z-q}\right)w_1''(z) +$$
$$\left(\sum_{j=1}^{3}B_{jj}(z-z_j) - \frac{(1-b_{11})\sigma_1}{z-q} + \frac{B_{11}b_{13}b_{12}(z_2-z_3)z}{k(z-q)}\right)w_1' -$$
$$\frac{(z_2-z_3)b_{13}b_{12}\det B}{k(z-q)}w_1 = 0. \tag{24}$$

It has Fuchsian singularities at $z = z_j$ and one additional apparent singularity at $z = q$.

Along with (24) a particular Fuchsian third-order equation with singularities located at the points $z_1 = 0$, $z_2 = 1$, $z_3 = t$ can be considered

$$\sigma y(z)''' + \sum_{j=1}^{3} b_j\sigma_j y(z)'' + ((\Delta_2+\Delta_1+1)(z-z_3)+\lambda)y(z)' + \Delta_3 y(z) = 0. \tag{25}$$

Here Δ_1, Δ_2, Δ_3 are standard symmetric functions of three parameters a, b, c

$$\Delta_1 = a+b+c, \quad \Delta_2 = ab+bc+ac, \quad \Delta_3 = abc.$$

Parameters a, b, c, b_j, $j = 1, 2, 3$ determine local behaviour of solutions at singularities z_j and ∞. Parameter λ is an accessory parameter. The Riemann scheme for this equation

$$\begin{pmatrix} z_1 & z_2 & z_3 & \infty & z \\ 0 & 0 & 0 & a & \lambda \\ 1 & 1 & 1 & b & \\ 2-b_1 & 2-b_2 & 2-b_3 & c & \end{pmatrix} \tag{26}$$

shows the Frobenius characteristic exponents. It means that at each finite singularity there is one holomorphic solution depending on two initial data and one solution which in general is not holomorphic.

Comparing (24) and (25) one sees that in principle they only differ in existence of an additional apparent singularity in (24). Equation (25) is obtained from (24) by specification of parameters and additional s-homotopic transform. Assuming, for example, $a = 0$ we arrive to one equation with the solution equal to a sum of a constant and general solution of Heun equation. The other possibility to obtain this result is the use of an appropriate Euler transform [10].

References

[1] S. Slavyanov, J.Phys A., 29, 7329-7335, (1996).
[2] S. Slavyanov, Theor. Mat. Phys., **123**,744-753, (2000).
[3] S. Slavyanov, W. Lay, Special functions: a unified theory based on singularities. OUP, 2000.
[4] S. Slavyanov, W. Lay, A. Akopyan, A. Pirozhnikov, V. Dmitriev, A. Yatzik, V. Zhegunov, SIGSAM Bulletin, **33**, 21-27, (1999).
[5] S. Slavyanov, Operator theory: Advances and Applications, **132**, 395-402, (2002).
[6] M.V. Babich, Russian Math. Surv. **64**, 45-127, (2009).
[7] A.Ya. Kazakov, S.Yu. Slavyanov, Theor. Math. Phys, **155**, 721-732, (2008).
[8] D.P. Novikov, Theor. Math. Phys., **146**, 355-364,(2006).
[9] S.Yu. Slavyanov, F.R. Vukajlović,Theor. Math. Phys., **150**, 123-131, (2007).
[10] A.Ya. Kazakov,Theor. Math. Phys, **116**, 323-329, (1998).