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COMMON FIXED POINT THEOREMS IN FUZZY METRIC SPACES VIA PROPERTIES

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ABSTRACT. We prove common fixed point theorems for weakly compatible mappings via an implicit relation in fuzzy metric spaces using property (E.A) and a common property (E.A). Our theorems extend theorems of [1, 3, 4, 6, 15, 16] and a corollary of [2].

1. Introduction and Preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [29] in 1965. To use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and applications. George and Veeramani [12] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [19] and defined the Hausdorff topology of fuzzy metric spaces which have very important applications in quantum particle physics particularly in connections with both string and $E-$infinity theory which were given and studied by El Naschie and Tanaka [8, 9, 10, 11, 28]. They showed also that every metric induces a fuzzy metric. The authors [13, 14, 21] proved fixed point theorems in fuzzy (probabilistic) metric spaces and the authors [2, 4, 5, 7, 23, 27, 30] proved fixed and common fixed point theorems using contractive conditions of integral type and generalized contractive conditions.

Motivated by a work due to Popa [22], we have observed that proving fixed point theorems using an implicit relation is a good idea since it covers several contractive conditions rather than one contractive condition.

It is our purpose in this paper to prove common fixed point theorems in fuzzy metric spaces via an implicit relation for weakly compatible mappings satisfying the property (E.A) introduced by [1] and a common property (E.A) introduced by Liu et al [20]. Our Theorems generalize Theorems of [1, 3, 4, 6, 15, 16] and a corollary of [2].

Definition 1.1 ([24])**.** A binary operation $* : [0,1]^2 \longrightarrow [0,1]$ is called a continuous t-norm if $([0,1]\,,*)$ is an abelian topological monoid; i.e.,
(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0,1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

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Two typical examples of a continuous $t-$norm are $a*b = ab$ and $a*b = \min\{a, b\}$.

Definition 1.2 ([12])**.** The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

(FM-1) $M(x, y, t) > 0$,

(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(FM-5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable.

Example 1.3. Let $X = \mathbb{R}$. Denote $a * b = a.b$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

for all $x, y \in X$.

Definition 1.4 ([12])**.** Let $(X, M, *)$ be a fuzzy metric space.

1) A sequence $\{x_n\}$ in X converges to x if and only if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $M(x_n, x, t) > 1 - \epsilon$; i.e., $M(x_n, x, t) \to 1$ as $n \to \infty$ for all $t > 0$.

2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $M(x_n, x_m, t) > 1-\epsilon$; i.e., $M(x_n, x_m, t) \to 1$ as $n, m \to \infty$ for all $t > 0$.

3) A fuzzy metric space (X, M, t) in which every Cauchy sequence is convergent is said to be complete.

Lemma 1.5 ([13])**.** *For all $x, y \in X$, $M(x, y, .)$ is a non-decreasing function.*

Definition 1.6. Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t)$$

whenever $\{(x_n, y_n, t_n)\}$ is a sequence in $X^2 \times (0, \infty)$ which converges to a point $(x, y, t) \in X^2 \times (0, \infty)$; i.e.,

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t).$$

Lemma 1.7 ([13])**.** *M is continuous function on $X^2 \times (0, \infty)$.*

Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself.

Definition 1.8. A and S are said to be

1) compatible [17, 25] if

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1 \text{ for all } t > 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X.$$

2) A and S are said to be weakly compatible [18] if they commute at their coincidence points; i.e., $Ax = Sx$ for some $x \in X$ implies that $ASx = SAx$.

Remark 1.9. If A and S of are compatible, then they are weakly compatible and the converse is not true in general, see [26]

Definition 1.10. The pair (A, S) satisfies the property (E.A) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} M(Ax_n, u, t) = \lim_{n\to\infty} M(Sx_n, u, t) = 1$$

for some $u \in X$ and all $t > 0$.

Clearly, a pair of noncompatible mappings satisfies the property (E.A).

Example 1.11. Let $X = \mathbb{R}$ and $M(x, y, t) = \dfrac{t}{t + |x - y|}$ for every $x, y \in X$ and $t > 0$. Define A and S by

$$Ax = 2x + 1, \quad Sx = x + 2.$$

Define the sequence $\{x_n\}$ by $x_n = 1 + \dfrac{1}{n}$, $n = 1, 2, ...$ We have

$$\lim_{n\to\infty} M(Ax_n, 3, t) = \lim_{n\to\infty} M(Sx_n, 3, t) = 1$$

for every $t > 0$. Then, the pair (A, S) satisfies the property (E.A). However, A and S are not weakly compatible.

The following example shows that there are some pairs of mappings which do not satisfy the property (E.A).

Example 1.12. Let $X = \mathbb{R}$ and $M(x, y, t) = \dfrac{t}{t + |x - y|}$ for every $x, y \in X$ and $t > 0$. Define A and S by $Ax = x + 1$ and $Sx = x + 2$. Assume that there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} M(Ax_n, u, t) = \lim_{n\to\infty} M(Sx_n, u, t) = 1$$

for some $u \in X$ and all $t > 0$. Therefore

$$\lim_{n\to\infty} M(x_n + 1, u, t) = \lim_{n\to\infty} M(x_n + 2, u, t) = 1.$$

We conclude that $x_n \to u - 1$ and $x_n \to u - 2$ which is a contradiction. Hence, the pair (A, S) do not satisfy the property (E.A).

Definition 1.13. The pairs (A, S) and (B, T) of a fuzzy metric space $(X, M, *)$ satisfy a common property (E.A) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that for some $u \in X$ and for all $t > 0$

(1.1)

$$\lim_{n\to\infty} M(Ax_n, u, t) = \lim_{n\to\infty} M(Sx_n, u, t) = \lim_{n\to\infty} M(By_n, u, t) = \lim_{n\to\infty} M(Ty_n, u, t) = 1.$$

If $B = A$ and $T = S$ in (1.1), we obtain the definition of the property (E.A).

Example 1.14. Let $X = [1, \infty)$ and $M(x, y, t) = \dfrac{t}{t + |x - y|}$ for every $x, y \in X$ and $t > 0$. Define A, B, S, T by

$$Ax = 2 + \frac{x}{3}, Bx = 2 + \frac{x}{2}, Sx = 1 + \frac{2}{3}x, Tx = 1 + x.$$

Define sequences $\{x_n\}$ and $\{y_n\}$ by $x_n = 3 + \dfrac{1}{n}$, $y_n = 2 + \dfrac{1}{n}$, $n = 1, 2, ...$ Since for all $t > 0$

$$\lim_{n\to\infty} M(Ax_n, 3, t) = \lim_{n\to\infty} M(By_n, 3, t) = \lim_{n\to\infty} M(Sx_n, 3, t) = \lim_{n\to\infty} M(Ty_n, 3, t) = 1$$

Therefore, the pairs (A, S) and (B, T) satisfy a common property (E.A)

Let Φ be the set of all continuous functions $\phi : [0, \infty[\longrightarrow [0, \infty[$ such that $\phi(t) < t$ for all $t > 0$.

2. Implicit relations

Let F_6 be the set of all continuous functions $F(t_1, t_2, t_3, t_4, t_5, t_6) : [0, 1]^6 \to \mathbb{R}$ satisfying the following conditions:

$(F_1) : F(u, 1, u, 1, 1, u) < 0$ for all $u \in (0, 1)$.
$(F_2) : F(u, 1, 1, u, u, 1) < 0$ for all $u \in (0, 1)$.
$(F_3) : F(u, u, 1, 1, u, u) < 0$ for all $u \in (0, 1)$.

The aim of this section is to give several examples of the function F.

Example 2.1. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \phi(\min\{t_2, t_3, t_4, t_5, t_6\})$, where $\phi : [0, 1] \longrightarrow [0, 1]$ is increasing and continuous function such that $\phi(t) > t$ for all $t \in (0, 1)$.

$(F_1) : F(u, 1, u, 1, 1, u) = u - \phi(u) < 0$ for all $u \in (0, 1)$.
$(F_2) :$ Similarly, $F(u, 1, 1, u, u, 1) < 0$ for all $u \in (0, 1)$.
$(F_3) : F(u, u, 1, 1, u, u) = u - \phi(u) < 0$ for all $u \in (0, 1)$.

Example 2.2. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - c_1 \min\{t_2^2, t_3^2, t_4^2\} - c_2 \min\{t_3 t_6, t_4 t_5\} - c_3 t_5 t_6$

$c_1, c_2, c_3 > 0$, $c_1 + c_2 \geq 1, c_1 + c_3 \geq 1$.
$(F_1) : F(u, 1, u, 1, 1, u) = u^2(1 - c_1 - c_2) - c_3 u < 0$ for all $u \in (0, 1)$.
$(F_2) :$ Similarly, $F(u, 1, 1, u, u, 1) < 0$ for all $u \in (0, 1)$.
$(F_3) : F(u, u, 1, 1, u, u) = u^2(1 - c_1 - c_3) - c_2 u < 0$ for all $u \in (0, 1)$.

Example 2.3. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^3 - a \min\{t_1^2 t_2, t_1 t_3 t_4, t_5^2 t_6, t_5 t_6^2\}$, $a > 1$.

$(F_1) : F(u, 1, u, 1, 1, u) = u^3 - a \min\{u^2, u\} < 0$ for all $u \in (0, 1)$.
$(F_2) :$ Similarly, $F(u, 1, 1, u, u, 1) < 0$ for all $u \in (0, 1)$.
$(F_3) : F(u, u, 1, 1, u, u) = u^3 - a \min\{u^3, u, u^3\} < 0$ for all $u \in (0, 1)$.

Example 2.4. $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^3 - a \dfrac{t_3^2 t_4^2 + t_5^2 t_6^2}{t_2 + t_3 + t_4}$, $a \geq 2$.

$(F_1) : F(u, 1, u, 1, 1, u) = u^3 - \dfrac{2au^2}{u + 2} < 0$ for all $u \in (0, 1)$.
$(F_2) :$ Similarly, $F(u, 1, 1, u, u, 1) < 0$ for all $u \in (0, 1)$.
$(F_3) : F(u, u, 1, 1, u, u) = u^3 - a \dfrac{u^4 + 1}{u + 2} < 0$ for all $u \in (0, 1)$.

Example 2.5. $F(t_1, t_2, t_3, t_4, t_5, t_6) = (1 + pt_2)t_1 - p \min\{t_3 t_4, t_5 t_6\} - \phi(\min\{t_2, t_3, t_4, t_5, t_6\})$, where $p \geq 0$ and $\phi : [0, 1] \longrightarrow [0, 1]$ is increasing and continuous function such that $\phi(t) > t$ for all $t \in (0, 1)$.

$(F_1), (F_2)$ and (F_3) as in Example 2.1.

Example 2.6. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - a\dfrac{t_2^2+t_3^2+t_4^2}{t_5+t_6}$, $a \geq 1$.

(F_1) : $F(u,1,u,1,1,u) = u^2 - a\dfrac{u^2+2}{u+1} = \dfrac{u^3+(1-a)u^2-2a}{u+1} < 0$ for all $u \in (0,1)$.

(F_2) Similarly, $F(u,1,1,u,u,1) < 0$ for all $u \in (0,1)$.

(F_3) : $F(u,u,1,1,u,u) = u^2 - a\dfrac{u^2+2}{2u} < 0$ for all $u \in (0,1)$.

Example 2.7. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^3 - a\dfrac{t_3^2t_4^2}{t_2+t_5+t_6}$, $a > 3$.

(F_1) : $F(u,1,u,1,1,u) = u^3 - a\dfrac{u^2}{u+2} < 0$ for all $u \in (0,1)$.

(F_2) Similarly, $F(u,1,1,u,u,1) < 0$ for all $u \in (0,1)$.

(F_3) : $F(u,u,1,1,u,u) = u^3 - \dfrac{a}{3u} < 0$ for all $u \in (0,1)$.

Example 2.8. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - a\min\{t_2^2,t_3^2,t_4^2\} - b\dfrac{t_5}{t_5+t_6}$, $a \geq 1$ and $b > 0$.

(F_1) : $F(u,1,u,1,1,u) = (1-a)u^2 - b\dfrac{u}{u+1} < 0$ for all $u \in (0,1)$.

(F_2) Similarly, $F(u,1,1,u,u,1) < 0$ for all $u \in (0,1)$.

(F_3) : $F(u,u,1,1,u,u) = (1-a)u^2 - \dfrac{b}{2} < 0$ for all $u \in (0,1)$.

Example 2.9. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - a\min\{t_2^2,t_5^2,t_6^2\} - b\dfrac{t_3}{t_3+t_4}$, $a > 1$ and $b > 0$.

(F_1) , (F_2) and (F_3) as in Example 2.8.

Example 2.10. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a_1t_2 - a_2t_3 - a_3t_4 - a_4t_5 - a_5t_6$,
$a_1,a_2,a_3,a_4,a_5 > 0$, $a_2+a_5 \geq 1$, $a_3+a_4 \geq 1$ and $a_1+a_4+a_5 \geq 1$.

(F_1) : $F(u,1,u,1,1,u) = u - a_1 - a_2u - a_3 - a_4 - a_5u < 0$ for all $u \in (0,1)$.

(F_2) Similarly, $F(u,1,1,u,u,1) < 0$ for all $u \in (0,1)$.

(F_3) : $F(u,u,1,1,u,u) = u - a_1u - a_2 - a_3 - a_4u - a_5u < 0$ for all $u \in (0,1)$.

Example 2.11. $F(t_1,t_2,t_3,t_4,t_5,t_6) = \int\limits_0^{t_1}\varphi(t)dt - \phi(\int\limits_0^{\min\{t_2,t_3,t_4,t_5,t_6\}}\varphi(t)dt)$, where $\phi : [0,1] \longrightarrow [0,1]$ is increasing and continuous function such that $\phi(t) > t$ for all $t \in (0,1)$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integrable mapping which is summable and satisfies

$$0 < \int_0^{\epsilon}\varphi(s)ds < 1 \quad \text{for all } 0 < \epsilon < 1 \quad \text{and} \quad \int_0^1 \varphi(s)ds = 1. \tag{2.1}$$

(F_1) : $F(u,1,u,1,1,u) = \int\limits_0^u\varphi(t)dt - \phi(\int\limits_0^u\varphi(t)dt) < 0$ for all $u \in (0,1)$.

(F_2) : Similarly, $F(u,1,1,u,u,1) < 0$ for all $u \in (0,1)$.

(F_3) : $F(u,u,1,1,u,u) = \int\limits_0^u\varphi(t)dt - \phi(\int\limits_0^u\varphi(t)dt) < 0$ for all $u \in (0,1)$.

Example 2.12. $F(t_1,t_2,t_3,t_4,t_5,t_6) = (\int\limits_0^{t_1}\varphi(s)ds)^p - a(\int\limits_0^{t_2}\varphi(s)ds)^p -$

$$b\min\{\int_0^{t_3}\varphi(s)ds, \int_0^{t_4}\varphi(s)ds, (\int_0^{t_3}\varphi(s)ds)^{\frac{1}{2}}\cdot(\int_0^{t_5}\varphi(s)ds)^{\frac{1}{2}},$$
$$(\int_0^{t_5}\varphi(s)ds)^{\frac{1}{2}}(\int_0^{t_6}\varphi(s)ds)^{\frac{1}{2}}\}^p,$$

where $a > 0$, $0 < b \leq 1$, $a + b > 1$, $p > 0$, $\phi : [0,1] \longrightarrow [0,1]$ is increasing and continuous function such that $\phi(t) > t$ for all $t \in (0,1)$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integrable mapping which is summable and satisfies 2.1.

$$(F_1) : F(u,1,u,1,1,u) = (\int_0^u\varphi(s)ds)^p - a(\int_0^1\varphi(s)ds)^p$$
$$- b\min\{\int_0^u\varphi(s)ds, \int_0^1\varphi(s)ds, (\int_0^u\varphi(s)ds)^{\frac{1}{2}}\cdot(\int_0^1\varphi(s)ds)^{\frac{1}{2}},$$
$$(\int_0^1\varphi(s)ds)^{\frac{1}{2}}(\int_0^u\varphi(s)ds)^{\frac{1}{2}}\}^p$$
$$= (\int_0^u\varphi(s)ds)^p - a - b(\int_0^u\varphi(s)ds)^p < 0 \text{ for all } u \in (0,1).$$

(F_2) : Similarly, $F(u,1,1,u,u,1) < 0$ for all $u \in (0,1)$.

(F_3) : $F(u,u,1,1,u,u) = (1-a-b)\phi(\int_0^u\varphi(t)dt) < 0$ for all $u \in (0,1)$.

Define $G : R_A^+ \to \mathbb{R}$ satisfying:

Example 2.13. (i) $G(0) = 0$ and $G(t) > 0$ for each $t \in (0,A)$, $A \in (0,\infty]$, $R_A^+ = [0,A)$,
(ii) G is increasing on R_A^+,
(iii) G is continuous.

Example 2.14. $F(t_1,t_2,t_3,t_4,t_5,t_6) = G(t_1) - \phi(G(\min\{t_2,t_3,t_4,t_5,t_6\})$, where $\phi : [0,1] \longrightarrow [0,1]$ is increasing and continuous function such that $\phi(t) > t$ for all $t \in (0,1)$

(F_1) : $F(u,1,u,1,1,u) = G(u) - \phi(G(u)) < 0$ for all $u \in (0,1)$.
(F_2) : Similarly, $F(u,1,1,u,u,1) < 0$ for all $u \in (0,1)$.
(F_3) : $F(u,u,1,1,u,u) = G(u) - \phi(G(u)) < 0$ for all $u \in (0,1)$.

Example 2.15. $F(t_1,t_2,t_3,t_4,t_5,t_6) = (G(t_1))^p - \phi[a(G(t_2))^p +$
$b\min\{G(t_3), G(t_4), (G(t_3))^{\frac{1}{2}}\cdot(G(t_5))^{\frac{1}{2}}, (G(t_5))^{\frac{1}{2}}\cdot(G(t_6))^{\frac{1}{2}}\}^p]$,
where $a > 0$, $0 < b \leq 1$, $a + b > 1$, $p > 0$ and $\phi : [0,1] \longrightarrow [0,1]$ is increasing and continuous function such that $\phi(t) > t$ for all $t \in (0,1)$.

Define $\Phi[0,A) = \{G : G \text{ satisfies (i)–(iii)}\}$.
The following examples were given by [30].
1) Let $G(t) = t$, then $G \in \Phi[0,A)$ for each $A \in (0,+\infty]$.
2) Suppose that φ is nonnegative, Lebesgue integrable on $[0,A)$ and satisfies

$$\int_0^\epsilon \varphi(t)dt > 0 \text{ for each } \epsilon \in (0,A).$$

Let $G(t) = \int_0^t\varphi(s)ds$, then $G \in \Phi[0,A)$.

3) Suppose that ψ is nonnegative, Lebesgue integrable on $[0, A)$ and satisfies

$$\int_0^{\epsilon} \psi(t)dt > 0 \text{ for each } \epsilon \in (0, A)$$

and φ is nonnegative, Lebesgue integrable on $[0, \int_0^A \psi(s)ds)$ and satisfies

$$\int_0^{\epsilon} \varphi(t)dt > 0 \text{ for each } \epsilon \in (0, \int_0^A \psi(s)ds).$$

Let $G(t) = \int_0^{\int_0^t \psi(s)ds} \varphi(u)du$, then $G \in \Phi[0, A)$.

4) If $H \in \Phi[0, A)$ and $G \in \Phi[0, H(A-0))$, then a composition mapping $G \circ H \in \Phi[0, A)$. For instance, let $L(t) = \int_0^{G(t)} \varphi(s)ds$, then $L \in \Phi[0, A)$ whenever $G \in \Phi[0, A)$ and φ is nonnegative, Lebesgue integrable on $\Phi[0, G(A-0))$ and satisfies

$$\int_0^{\epsilon} \varphi(t)dt > 0 \text{ for each } \epsilon \in (0, G(A-0)).$$

Lemma 2.16 ([30])**.** *Let $A \in (0, +\infty]$ and $G \in \Phi[0, A)$. If $\lim_{n\to\infty} G(\epsilon_n) = 0$ for $\epsilon_n \in R_A^+$, then $\lim_{n\to\infty} \epsilon_n = 0$.*

3. **Main Results**

The purpose of this section is to give our main results Theorem 3.1 and Theorem 3.3..

Theorem 3.1. *Let A, B, S and T be self-mappings of a fuzzy metric space $(X, M, *)$ satisfying the following conditions*

$$A(X) \subset T(X) \text{ and } B(X) \subset S(X), \tag{3.1}$$

$$\begin{aligned} & F(M(Ax, By, t), M(Sx, Ty, t), M(Ax, Sx, t), \\ & M(By, Ty, t), M(Sx, By, t), M(Ax, Ty, t)) \\ \geq\ & 0 \end{aligned} \tag{3.2}$$

for all x, y in X and $F \in F_6$. Suppose that the pair (A, S) or (B, T) satisfies the property (E.A) and (A, S) and (B, T) are weakly compatible. If the range of one A, B, S and T is a closed subset of X, then A, B, S and T have a unique common fixed point in X.

Proof. Suppose that the pair (B, T) satisfies the property (E.A). Then, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Therefore, we have $\lim_{n\to\infty} M(Bx_n, Tx_n, t) = 1$. Since $B(X) \subset S(X)$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$.

Hence, $\lim_{n\to\infty} Sy_n = z$. Let us show that $\lim_{n\to\infty} Ay_n = z$. Using (3.2) we have

$$\begin{aligned} & F(M(Ay_n, Bx_n, t), M(Sy_n, Tx_n, t), M(Ay_n, Sy_n, t), \\ & M(Bx_n, Tx_n, t), M(Sy_n, Bx_n, t), M(Ay_n, Tx_n, t)) \\ = \ & F(M(Ay_n, Bx_n, t), M(Bx_n, Tx_n, t), M(Ay_n, Bx_n, t), \\ & M(Bx_n, Tx_n, t), 1, M(Ay_n, Tx_n, t)) \\ \geq \ & 0. \end{aligned}$$

Assume that $\limsup_{n\to\infty} M(Ay_n, Bx_n, t) = l < 1$. Taking the limit as $n \to \infty$ we get

$$F(l, 1, l, 1, 1, l) \geq 0$$

which is a contradiction of (F_1) and so $l = 1$; i.e., $\lim_{n\to\infty} Ay_n = z$.

Suppose that $S(X)$ is a closed subspace of X. Then, $z = Su$ for some $u \in X$.

If $z \neq Au$, applying (3.2) we obtain

$$\begin{aligned} & F(M(Au, Bx_n, t), M(Su, Tx_n, t), M(Au, Su, t), \\ & F(M(Au, Bx_n, t), M(Su, Tx_n, t), M(Au, Su, t)) \\ \geq \ & 0 \end{aligned}$$

Letting $n \to \infty$ we have

$$F(M(Au, z, t), 1, M(Au, z, t), 1, 1, M(Au, z, t)) \geq 0$$

which is a contradiction of (F_1). Hence, $z = Au = Su$.

Since $A(X) \subset T(X)$, there exists $v \in X$ such that $z = Au = Tv$.

If $z \neq Bv$, using (3.2) we have

$$\begin{aligned} & F(M(Au, Bv, t), M(Su, Tv, t), M(Au, Su, t), \\ & M(Bv, Tv, t), M(Su, Bv, t), M(Au, Tv, t)) \\ = \ & F(M(z, Bv, t), 1, 1, M(z, Bv, t), M(z, Bv, t), 1) \geq 0 \end{aligned}$$

which is a contradiction of (F_2) and therefore $Au = Su = z = Bv = Tv$.

Since the pairs (A, S) and (B, T) are weakly compatible, we have $ASu = SAu$ and $BTv = TBv$; i.e., $Az = Sz$ and $Bz = Tz$. If $Az \neq z$, using (3.2) we get

$$\begin{aligned} & F(M(Az, Bv, t), M(Sz, Tv, t), M(Az, Sz, t), \\ & M(Bv, Tv, t), M(Sz, Bv, t), M(Az, Tv, t)) \\ = \ & F(M(Az, z, t), M(Az, z, t), 1, 1, \\ & M(Az, z, t), M(Az, z, t)) \\ \geq \ & 0 \end{aligned}$$

which is a contradiction of (F_3). Then, $Az = Sz = z$.

Similarly, we can prove that $Bz = Tz = z$. Hence, $z = Bz = Tz = Az = Sz$ and z is a common fixed point of A, B, S and T. The uniqueness of z follows from (3.2) and (F_3). □

If $B = A$ and $T = S$ in Theorem 3.1, we get the following Corollary.

Corollary 3.2. *Let A and S be self-mappings of a fuzzy metric space $(X, M, *)$ satisfying*

$$A(X) \subset S(X),$$

$$\begin{aligned} & F(M(Ax,Ay,t),M(Sx,Sy,t),M(Ax,Sx,t), \\ & M(Ay,Sy,t),M(Sx,Ay,t),M(Ax,Sy,t)) \\ \geq\ & 0. \end{aligned}$$

Suppose that the pair (A,S) satisfies the property (E.A) and (A,S) is weakly compatible. If the range of one A and S is a closed subset of X, then A and S have a unique common fixed point in X.

Theorem 3.3. *Let A,B,S and T be self-mappings of a fuzzy metric space $(X,M,*)$ satisfying (3.2). Suppose that the pairs (A,S) and (B,T) satisfy a common property (E.A) and (A,S) and (B,T) are weakly compatible. If $S(X)$ and $T(X)$ are closed subsets of X, then A,B,S and T have a unique common fixed point in X.*

Proof. Suppose that the pairs (A,S) and (B,T) satisfy a common property (E.A). Then, there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z \in X.$$

Since $S(X)$ and $T(X)$ are closed subsets of X, we obtain $z = Su = Tv$ for some $u,v \in X$.

If $z \neq Au$, using (3.2) we obtain

$$\begin{aligned} & F(M(Au,By_n,t),M(Su,Ty_n,t),M(Au,Su,t), \\ & M(By_n,Ty_n,t),M(Su,By_n,t),M(Au,Ty_n,t)) \\ \geq\ & 0. \end{aligned}$$

Letting $n \to \infty$ we have

$$F(M(Au,z,t),1,M(Au,z,t),1,1,M(Au,z,t)) \geq 0$$

which is a contradiction of (F_1) and so $z = Au = Su = Tv$. The rest of the proof follows as in Theorem 3.1. □

If we take examples 2.1, 2.11 and 2.14, we get Corollaries which generalize Theorems of [1, 4, 6, 7] and a corollary of [2]

If we take examples 2.2-2.10, we get several Corollaries.

Theorems 3.1 and 3.3 extend theorems of [3, 15, 16].

Example 3.4. Let $(X,M,*)$ be a fuzzy metric space, where $X = [0,2)$ with a t-norm defined by $a*b = \min\{a,b\}$ for all $a,b \in [0,2)$ and $M(x,y,t) = \dfrac{t}{t+|x-y|}$ for all $x,y \in X$ and $t > 0$. Define A,B,S and T by:

$$\begin{aligned} Ax &= Bx = 1, \\ Sx &= \begin{cases} 1 & \text{if } x \text{ is rational,} \\ \frac{2}{3} & \text{if } x \text{ is irrational} \end{cases}, \quad Tx = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ \frac{1}{3} & \text{if } x \text{ is irrational} \end{cases}, \end{aligned}$$

$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \phi(\min\{t_2,t_3,t_4,t_5,t_6\})$, $\phi(s) = \sqrt{s}$ for all $s \in [0,1]$.

It is easy to see that for all $x,y \in X$ and $t > 0$

$$\begin{aligned} M(Ax,By,t) \geq\ & \phi(\min\{M(Sx,Ty,t),M(Ax,Sx,t), \\ & M(By,Ty,t),M(Sx,By,t),M(Ax,Ty,t)\}) \end{aligned}$$

and the other conditions of Theorem 3.1 are satisfied, consequently, 1 is the unique common fixed point of A,B,S and T.

Example 3.5. Let $(X, M, *)$ and F as in example 3.4. Define A, B, S and T by:

$$Ax = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ \frac{3}{4} & \text{if } x \text{ is irrational} \end{cases}, \; Bx = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ \frac{1}{2} & \text{if } x \text{ is irrational} \end{cases},$$

$$Sx = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ \frac{2}{3} & \text{if } x \text{ is irrational} \end{cases}, \; Tx = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ \frac{1}{3} & \text{if } x \text{ is irrational} \end{cases},$$

It is easy to see that for all $x, y \in X$ and $t > 0$

$$M(Ax, By, t) \geq \phi(\min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Sx, By, t), M(Ax, Ty, t)\})$$

and the other conditions of Theorem 3.3 are satisfied, consequently, 1 is the unique common fixed point of A, B, S and T.

Note that Theorem 3.1 is not applicable in example 3.5 since (3.1) is not verified.

Example 3.6. Let $(X, M, *)$, A, B, S and T as in example 3.4,

$F(t_1, t_2, t_3, t_4, t_5, t_6) = G(t_1) - \phi(G(\min\{t_2, t_3, t_4, t_5, t_6\}))$, $\phi(s) = \sqrt{s}$ for all $s \in [0, 1]$ and $G(s) = s^{\frac{1}{s}}$ for $s > 0$, $G(0) = 0$.

It is easy to see that $D = diam(X) = 2$, $G \in \digamma[0, A)$, where $A = e > D = 2$ and for all $x, y \in X$ and $t > 0$

$$G(M(Ax, By, t)) \geq \phi(G(L(x, y, t))),$$

where

$$L(x, y, t) = \min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Sx, By, t), M(Ax, Ty, t)\}.$$

and the other conditions of Theorem 3.1 are satisfied, consequently, 1 is the unique common fixed point of A, B, S and T.

Example 3.7. Let $(X, M, *)$, A, B, S and T as in example 3.5 and F as in example 3.6.

It is easy to see that for all $x, y \in X$ and $t > 0$

$$G(M(Ax, By, t)) \geq \phi(G(L(x, y, t))),$$

where

$$L(x, y, t) = \min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Sx, By, t), M(Ax, Ty, t)\}.$$

and the other conditions of Theorem 3.3 are satisfied, consequently, 1 is the unique common fixed point of A, B, S and T.

Note that Theorem 3.1 is not applicable in example 3.7 since (3.1) is not verified.

References

[1] M. Aamri and D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl., 270 (2002), 181-188.

[2] A. Aliouche, A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type, J. Math. Anal. Appl., 322 (2) (2006), 796-802.

[3] A. Aliouche, Common fixed point theorems via an implicit relation and new properties, Soochow J. Math, 33 (4) (2007), 593-601.

[4] A. Aliouche, Common fixed point theorems of Gregus type for weakly compatible mappings satisfying generalized contractive conditions, J. Math Anal. Appl., 341 (2008), 707–719.

[5] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Sci., 29 (9) (2002), 531-536.

[6] A. Djoudi and L. Nisse, Gregus type fixed points for weakly compatible mappings, Bull. Belg. Math. Soc., 10 (2003), 369-378.

[7] A. Djoudi and A. Aliouche, Common fixed point theorems of Gregus type for weakly compatible mappings satisfying contractive conditions of integral type, J. Math Anal. Appl., 329 (1) (2007), 31-45.

[8] M. S. El Naschie, On the uncertainty of Cantorian geometry and two-slit experiment. Chaos, Solitons and Fractals., 9 (1998), 517–29.

[9] M. S. El Naschie, A review of E-infinity theory and the mass spectrum of high energy particle physics. Chaos, Solitons and Fractals., 19 (2004), 209–36.

[10] M. S. El Naschie, On a fuzzy Kahler-like Manifold which is consistent with two-slit experiment. Int. J of Nonlinear Science and Numerical Simulation., 6 (2005), 95–98.

[11] M. S. El Naschie, The idealized quantum two-slit gedanken experiment revisited—-Criticism and reinterpretation. Chaos, Solitons and Fractals., 27 (2006), 9–13.

[12] A. George and P. Veeramani, On some result in fuzzy metric space. Fuzzy Sets and Systems., 64 (1994), 395–399.

[13] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets and System., 27 (1988), 385-389.

[14] V. Gregori and A. Sapena, On fixed-point theorem in fuzzy metric spaces. Fuzzy Sets and Systems., 125 (2002), 245–252.

[15] M. Imdad and J. Ali, A general fixed point theorem in fuzzy metric spaces via an implicit function, J. Appl. Math. & Informatics, 26 (3 - 4) (2008), 591-603.

[16] J. Ali and M. Imdad, An implicit function implies several contraction conditions, Sarajevo J. Math., 4 (2) (2008), 269-285.

[17] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. & Math. Sci., 9 (4) (1986), 771-779.

[18] G. Jungck, Common fixed points for non-continuous non-self maps on non metric spaces, Far East J. Math. Sci., 4 (2) (1996), 199-215.

[19] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces. Kybernetica., 11 (1975), 326–334.

[20] Y. Liu, J. Wu and Z. Li, Common fixed points of single-valued and multi-valued maps, Internat J. Math. Math. Sci., 19 (2005), 3045-3055.

[21] D. Mihet, Banach contraction theorem in fuzzy metric spaces, Fuzzy Sets Systems., 144 (2004), 431–439.

[22] V. Popa, Some fixed point theorems for compatible mappings satisfying an implicit relation, Demonstratio Math., 32 (1999),157-163.

[23] B. E. Rhoades, Two fixed-Point Theorems for mappings satisfying a general contractive condition of integral type. Inter. J. Math and Math. Sci., 63 (2003), 4007-4013.

[24] B. Schweizer and A. Sklar, Statistical metric spaces. Pacific J. Math., 10 (1960), 313-334.

[25] B. Singh and M. S. Chauhan, Common fixed point of compatible maps in fuzzy metric space, Fuzzy Sets and System 115 (2000), 471-475.

[26] B. Singh and S. Jain, Semi-compatibility, compatibility and fixed point Theorems in fuzzy metric space, J. Chungcheong Math. Soc., 18 (1) (2005), 1-23.

[27] P. Vijayaraju, B. E. Rhoades and R. Mohanraj, A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type, Internat J. Math. Math. Sci., 15 (2005), 2359-2364.

[28] Tanaka. Y, Mizno Y, Kado T. Chaotic dynamics in Friedmann equation. Chaos, Solitons and Fractals., 24 (2005), 407–422.

[29] L. A. Zadeh, Fuzzy sets. Inform and Control., 8 (1965), 338–353.

[30] X. Zhang, Common fixed point theorems for some new generalized contractive type mappings, J. Math. Anal. Appl., 333 (2) (2007), 780-786.

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