ON $\tau$-$\oplus$-SUPPLEMENTED MODULES

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Abstract. Let $\tau$ be any preradical and $M$ any module. In [2], Al-Takhman, Lomp and Wisbauer defined $\tau$-supplemented module. In this paper we introduce the (completely) $\tau$-$\oplus$-supplemented modules. It is shown that (1) Any finite direct sum of $\tau$-$\oplus$-supplemented modules is $\tau$-$\oplus$-supplemented. (2) If $M$ is $\tau$-$\oplus$-supplemented module and $(D_3)$ then $M$ is completely $\tau$-$\oplus$-supplemented.

1. Introduction

Throughout this paper $R$ will denote an arbitrary associative ring with identity and all modules will be unitary right $R$-modules. A functor $\tau$ from the category of the right $R$-modules to itself is called a preradical if it satisfies the following properties:

(1) $\tau(M)$ is a submodule of an $R$-module $M$,
(2) If $f : M' \rightarrow M$ is an $R$-module homomorphism, then $f(\tau(M')) \subseteq \tau(M)$ and $\tau(f)$ is the restriction of $f$ to $\tau(M')$.

A preradical $\tau$ is called a right exact preradical if for any submodule $K$ of $M$, $\tau(K) = \tau(M) \cap K$. But it is well known if $K$ is a direct summand of $M$, then $\tau(K) = \tau(M) \cap K$ for a preradical.

Let $M$ be an $R$-module and $\tau$ denote a preradical. Like in [2], a submodule $K \leq M$ is called $\tau$-supplement (weak $\tau$-supplement) provided there exists some $U \leq M$ such that $M = U + K$ and $U \cap K \subseteq \tau(K) \ (U \cap K \subseteq \tau(M))$.

$M$ is called $\tau$-supplemented (weakly $\tau$-supplemented) if each of its submodules has a $\tau$-supplement (weak $\tau$-supplement) in $M$. $M$ is called amply $\tau$-supplemented, if for all submodules $K$ and $L$ of $M$ with $K + L = M$, $K$ contains a $\tau$-supplement of $L$ in $M$. Kosan and Harmanci [9] studied supplemented modules relative to torsion theories. Motivated by their work, we study $\oplus$-supplemented modules with respect to a preradical. Also another work has been done on $C_1$ modules (see [12]).

A module $M$ is called $\tau$-lifting if for every submodule $K$ of $M$, there is a decomposition $K = A \oplus B$, such that $A$ is a direct summand of $M$ and $B \subseteq \tau(M)$.

In this paper we introduce the (completely) $\tau$-$\oplus$-supplemented modules and investigate some properties of them.

Our paper is organized as follows.

In Section 2, we define the concept of $\tau$-$\oplus$-supplemented module. We call a module $M$ $\tau$-$\oplus$-supplemented if every submodule of $M$ has a $\tau$-supplement that is a direct summand of $M$. Then we show any finite direct sum of $\tau$-$\oplus$-supplemented modules is $\tau$-$\oplus$-supplemented. We also investigate when a direct summand of a $\tau$-$\oplus$-supplemented module is $\tau$-$\oplus$-supplemented.

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In Section 3, we call a module $M$ completely $\tau$-$\oplus$-supplemented if every direct summand of $M$ is $\tau$-$\oplus$-supplemented and prove that $M$ is $\tau$-$\oplus$-supplemented module and $(D_3)$, then $M$ is completely $\tau$-$\oplus$-supplemented.

The notation $N \leq_d M$ denotes that $N$ is a direct summand of $M$.

**Definition 1.1.** For any preradical $\tau$, we call a module $M$, $\tau$-$\oplus$-supplemented if every submodule of $M$ has a $\tau$-supplement that is a direct summand of $M$.

**Theorem 1.2.** For any preradical $\tau$, any finite direct sum of $\tau$-$\oplus$-supplemented modules is $\tau$-$\oplus$-supplemented.

**Proof.** Let $M = M_1 \oplus M_2$ where $M_1$ and $M_2$ are two $\tau$-$\oplus$-supplemented modules. Let $P$ be any submodule of $M$. We have $P + M_2 = M_2 \oplus [(P + M_2) \cap M_1]$ and $(P + M_2) \cap M_1$ is a submodule of $M_1$. Since $M_1$ is $\tau$-$\oplus$-supplemented, there exists a direct summand $K_1$ of $M_1$ such that $[(P + M_2) \cap M_1] + K_1 = M_1$ and $(P + M_2) \cap K_1 \subseteq \tau(K_1)$. We have $(P + K_1) \cap M_2$ is a submodule of $M_2$, so there exists a direct summand $K_2$ of $M_2$ such that $[(P + K_1) \cap M_2] + K_2 = M_2$ and $(P + K_1) \cap K_2 \subseteq \tau(K_2)$. Let $K = K_1 \oplus K_2$, $K$ is a direct summand of $M$. Moreover $M_1 \leq P + M_2 + K_1$ and $M_2 \leq P + K_1 + K_2$. Hence $M = P + K_1 + K_2 = P + K$. Since $P \cap (K_1 + K_2) \leq [(P + K_1) \cap K_2] + [(P + K_2) \cap K_1]$, thus $P \cap (K_1 + K_2) \leq [(P + K_1) \cap K_2] + [(P + M_2) \cap K_1]$. As $(P + M_2) \cap K_1 \subseteq \tau(K_1) + \tau(K_2)$, we have $(P \cap K) \subseteq \tau(K)$. Thus $M$ is $\tau$-$\oplus$-supplemented.

A nonzero module $M$ is called completely torsion if for every proper submodule $K$ of $M$, $K \subseteq \tau(M)$.

**Corollary 1.3.** For any preradical $\tau$, any finite direct sum of completely torsion modules is $\tau$-$\oplus$-supplemented.

**Theorem 1.4.** Let $M_i (1 \leq i \leq n)$ be any finite collection of relatively projective modules. Then for any preradical $\tau$, the module $M = \bigoplus_{i=1}^{n} M_i$ is $\tau$-$\oplus$-supplemented if and only if $M_i$ is $\tau$-$\oplus$-supplemented for each $1 \leq i \leq n$.

**Proof.** The sufficiency is proved in Theorem 1.2 Conversely, we only prove $M_i$ to be $\tau$-$\oplus$-supplemented. Let $A \leq M_1$. Then there exists $B \leq M$ such that $M = A + B$, $B$ is a direct summand of $M$ and $A \cap B \subseteq \tau(B)$. Since $M = A + B = M_1 + B$, by [10] Lemma 4.47, there exists $B_1 \leq B$ such that $M = M_1 \oplus B_1$. Thus $B = B_1 \oplus (M_1 \cap B)$. Note that $M_1 = A + (M_1 \cap B)$ and $M_1 \cap B$ is a direct summand of $M_1$. Therefore $A \cap B = A \cap (M_1 \cap B) \subseteq \tau(B) \cap (M_1 \cap B) = \tau(M_1 \cap B)$. Hence $M_1$ is $\tau$-$\oplus$-supplemented.

A factor module of a $\tau$-$\oplus$-supplemented module need not be $\tau$-$\oplus$-supplemented for $\tau = \text{Rad}$ (see Examples 2.2 and 2.3).

**Theorem 1.5.** Let $M$ be a $\tau$-$\oplus$-supplemented module for any preradical $\tau$ and $X \leq M$. If for every direct summand $K$ of $M$, $(X + K)/X$ is a direct summand of $M/X$, then $M/X$ is $\tau$-$\oplus$-supplemented.

**Proof.** Let $N/X \leq M/X$. Since $M$ is $\tau$-$\oplus$-supplemented, there exists a direct summand $K$ of $M$ such that $N + K = M$ and $N \cap K \subseteq \tau(K)$. Then $N/X + (K + X)/X = M/X$. By assumption, $(K + X)/X$ is a direct summand of $M/X$. It is easy to check that $(N/X) \cap ((K + X)/X) \subseteq \tau((K + X)/X)$.
Let $M$ be a module. Then $M$ is called **distributive** if its lattice of submodules is a distributive lattice, equivalently for submodules $K, L, N$ of $M$, $N + (K \cap L) = (N + K) \cap (N + L)$ or $N \cap (K + L) = (N \cap K) + (N \cap L)$.

Let $M$ be a module. A submodule $X$ of $M$ is called **fully invariant**, if for every $f \in \text{End}(M)$, $f(X) \subseteq X$. The module $M$ is called **duo module**, if every submodule of $M$ is fully invariant. The submodule $A$ of $M$ is called **projection invariant** in $M$ if $f(A) \subseteq A$, for any idempotent $f \in \text{End}(M)$.

**Corollary 1.6.** Let $M$ be a $\tau \oplus$-supplemented module for any preradical $\tau$.

1. Let $N \leq M$ such that for each decomposition $M = M_1 \oplus M_2$ we have $N = (N \cap M_1) \oplus (N \cap M_2)$. Then $M/N$ is $\tau \oplus$-supplemented. (In particular, this is true for any distributive module). If moreover $N \leq_d M$, then $N$ is $\tau \oplus$-supplemented.

2. Let $X$ be a projection invariant submodule of $M$. Then $M/X$ is $\tau \oplus$-supplemented. In particular, for every fully invariant submodule $A$ of $M$, $M/A$ is $\tau \oplus$-supplemented.

**Proof.**

1. Let $L/N \leq M/N$. Since $M$ is $\tau \oplus$-supplemented, there exists a direct summand $D$ of $M$ such that $M = L + D$ and $L \cap D \subseteq \tau(D)$. Then $M/N = L/N + (D + N)/N$ and $L/N \cap (D + N)/N = (L \cap (D + N))/N \subseteq \tau((D + N)/N)$.

Let $M = D \oplus D'$. By assumption, $N = (N \cap D) \oplus (N \cap D') = (D + N) \cap (D' + N)$. So, $(D + N)/N \oplus (D' + N)/N = M/N$. It follows that $M/N$ is $\tau \oplus$-supplemented.

Now let $N \leq_d M$ and $V \leq N$. Then there exist submodules $K$ and $K'$ of such that $M = K \oplus K' = V + K$ and $V \cap K \subseteq \tau(K)$. Thus $N = V + N \cap K$. By assumption $N \cap K \leq_d N$. Moreover, $V \cap (N \cap K) \subseteq \tau(K)$. Then $V \cap (N \cap K) \subseteq \tau(N \cap K)$. Therefore, $N$ is $\tau \oplus$-supplemented.

2. Clear by (1). $\square$

Let $M$ be an $R$-module. By $P_\tau(M)$ we denote the sum of all submodules $N$ of $M$ with $\tau(N) = N$. Since $P_\tau(M)$ is a sum of some submodules of $M$, itself is a submodule of $M$.

**Corollary 1.7.** Let $M$ be a $\tau \oplus$-supplemented module for any preradical $\tau$. Then $M/P_\tau(M)$ is $\tau \oplus$-supplemented. If moreover $P_\tau(M) \leq_d M$, then $P_\tau(M)$ is $\tau \oplus$-supplemented.

**Proof.** By Corollary 1.6(1), it suffices to prove that $P_\tau(M)$ is a fully invariant submodule of $M$. Let $N \leq M$ such that $N = \tau(N)$ and $f \in \text{End}(M)$ and $g$ its restriction to $N$. But $\tau(N) = N$ and $f(N) = g(N)$, hence $f(N) \subseteq \tau(f(N))$. Thus, $\tau(f(N)) = f(N)$. This implies that $f(N) \subseteq P_\tau(M)$. This completes the proof. $\square$

We recall that a module $M$ is called **semi-Artinian** if every nonzero quotient module of $M$ has nonzero socle. For a module $M$, we define $Sa(M) = \sum\{U \leq M \mid U\text{-semi-Artinian}\}$.

**Corollary 1.8.** Let $M$ be a $\tau \oplus$-supplemented module for any preradical $\tau$. Then $M/Sa(M)$ is $\tau \oplus$-supplemented. If, moreover, $Sa(M)$ is a direct summand of $M$, then $Sa(M)$ is also $\tau \oplus$-supplemented.

**Proof.** Let $f \in \text{End}(M)$ and $U$ a semi-Artinian submodule. Let $g$ be restriction of $f$ to $U$. Thus $U/\text{Ker}(g) \cong g(U)$. Hence $f(U) \cong U/\text{Ker}(g)$. But it is easy to check that $U/\text{Ker}(g)$ is a semi-Artinian module. Therefore, $f(U)$ is semi-Artinian. This implies that $f(Sa(M)) \subseteq Sa(M)$. Thus $Sa(M)$ is a fully invariant submodule of $M$. The result follows from Corollary 1.6(1). $\square$
Remark 1.9. If $M$ is a $\tau$-$\oplus$-supplemented module for any preradical $\tau$, then $M/\tau(M)$ is semisimple and hence $\tau$-$\oplus$-supplemented.

Example 1.10. Let $M$ be the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. By [8] Example 10], $M$ is not lifting and it is not $\tau$-lifting. By [3] Theorem 1.4], $M$ is $\oplus$-supplemented and hence $\tau$-$\oplus$-supplemented for $\tau = \operatorname{Rad}$.

A $\tau$-lifting module is $\tau$-$\oplus$-supplemented. But the converse does not hold. The following proposition shows that under some assumption it can be true.

Proposition 1.11. Assume $M$ is $\tau$-$\oplus$-supplemented for any preradical $\tau$ such that whenever $M = M_1 \oplus M_2$ then $M_1$ and $M_2$ are relatively projective. Then $M$ is $\tau$-lifting.

Proof. Let $N \leq M$. Since $M$ is $\tau$-$\oplus$-supplemented, there exists a decomposition $M = M_1 \oplus M_2$ such that $M = N + M_2$ and $N \cap M_2 \subseteq \tau(M_2)$ for submodules $M_1, M_2$ of $M$. By hypothesis, $M_1$ is $M_2$-projective. By [10] Lemma 4.47, we obtain $M = A \oplus M_2$ for some submodule $A$ of $M$ such that $A \leq N$. Then $N = A \oplus (M_2 \cap N)$. So $M$ is $\tau$-lifting by [2], 2.8.

Corollary 1.12. Let $M$ be a $\tau$-$\oplus$-supplemented module for any preradical $\tau$. If $M$ is projective then $M$ is $\tau$-lifting.

Now we give a characterization of $\tau$-$\oplus$-supplemented rings.

Theorem 1.13. Let $\tau$ be any preradical. Then the following are equivalent:

1. $R$ is $\tau$-$\oplus$-supplemented;
2. Every finitely generated free $R$-module is $\tau$-$\oplus$-supplemented;
3. If $F$ is a finitely generated free $R$-module and $N$ a fully invariant submodule, then $F/N$ is $\tau$-$\oplus$-supplemented.

Proof. (1) $\Rightarrow$ (2) Let $M$ be a finitely generated free $R$-module. Then $M \cong \bigoplus_{i=1}^{n} R$. Since any finite direct sum of $\tau$-$\oplus$-supplemented modules is $\tau$-$\oplus$-supplemented, the result follows.

(2) $\Rightarrow$ (3) By (2), $F$ is $\tau$-$\oplus$-supplemented. The result follows from Corollary 1.6 (2).

(3) $\Rightarrow$ (1) is clear.

Lemma 1.14. Let $M = M_1 \oplus M_2$. Then for any preradical $\tau$, $M_2$ is $\tau$-$\oplus$-supplemented if and only if for every submodule $N/M_1$ of $M/M_1$, there exists a direct summand $K$ of $M$ such that $K \leq M_2$, $M = K + N$ and $N \cap K \subseteq \tau(M)$.

Proof. Suppose that $M_2$ is $\tau$-$\oplus$-supplemented. Let $N/M_1 \leq M/M_1$. As $M_2$ is $\tau$-$\oplus$-supplemented, there exists a decomposition $M_2 = K \oplus K'$ such that $M_2 = (N \cap M_2) + K$ and $N \cap K \subseteq \tau(K)$. Note that $M = (N \cap M_2) + K + M_1$ gives $M = N + K$.

Conversely, suppose that $M/M_1$ has the stated property. Let $H$ be a submodule of $M_2$. Consider the submodule $(H \oplus M_1)/M_1 \leq M/M_1$. By hypothesis, there exists a direct summand $L$ of $M$ such that $L \leq M_2$, $M = (L + H) + M_1$ and $L \cap (H + M_1) \subseteq \tau(M)$. By modularity, $M_2 = L + H$. Then $L \cap H \subseteq \tau(L)$. Thus, $L$ is a $\tau$-supplement of $H$ in $M_2$ and it is a direct summand of $M_2$. Therefore, $M_2$ is $\tau$-$\oplus$-supplemented.
Theorem 1.15. Let \( \tau \) be any preradical and \( M \) a direct summand of a \( \tau \)-supplemented module \( M \) such that for every direct summand \( K \) of \( M \) with \( M = K + M \), \( K \cap M \) is a direct summand of \( M \). Then \( M \) is \( \tau \)-supplemented.

Proof. Suppose that \( M = M_1 \oplus M_2 \) and let \( N/M_1 \leq M/M_1 \). Consider the submodule \( N \cap M_2 \) of \( M \). Since \( M \) is \( \tau \)-supplemented, there exists a direct summand \( K \) of \( M \) such that \( M = (N \cap M_2) + K \) and \( N \cap M_2 \cap K \subseteq \tau(K) \). Note that \( M = N + M_2 \).

By [2] Lemma 1.2, \( M = (K \cap M_2) + N \). Since \( M = K + M_2 \), \( K \cap M_2 \) is a direct summand of \( M \) by hypothesis. By Lemma 1.14, \( M \) is \( \tau \)-supplemented.

\[ \square \]

Corollary 1.16. Let \( M \) be a \( \tau \)-supplemented module for any preradical \( \tau \) and \( K \) a direct summand of \( M \) such that \( M/K \) is \( K \)-projective. Then \( K \) is \( \tau \)-supplemented.

Proof. Let \( L \) be a direct summand of \( M \) with \( M = L + K \). Since \( K \) is a direct summand of \( M \), \( M = K \oplus K_0 \) for some submodule \( K_0 \) of \( M \). Therefore, \( K_0 \) is \( K \)-projective. Then by [16] 41.14, there exists a submodule \( L_0 \) of \( L \) such that \( M = L_0 \oplus K \). Now \( L = L' \oplus (L \cap K) \) implies that \( L \cap K \) is a direct summand of \( M \). By Theorem 1.15, \( K \) is \( \tau \)-supplemented.

\[ \square \]

Corollary 1.17. Let \( M \) be a \( \tau \)-supplemented module for any preradical \( \tau \) and \( N \leq_d M \) such that \( M/N \) is projective. Then \( N \) is \( \tau \)-supplemented.

A submodule \( N \) of \( M \) is called small in \( M \) (notation \( N \leq M \)) if \( \forall L \leq M, L+N \neq M \). A module \( M \) is called hollow if every proper submodule of \( M \) is small in \( M \).

Let \( M \) be a module and \( S \) denote the class of all small modules. Talebi and Vana [13] defined \( Z(M) \) as follows:

\[ Z(M) = \bigcap \{ \ker g \mid g \in \text{Hom}(M, L), L \in S \}. \]

The module \( M \) is called cosingular (non-cosingular) if \( Z(M) = \{0\} (Z(M) = M) \). Clearly every non-cosingular module is \( Z \)-supplemented. Also if \( R \) is a non-cosingular ring, then every \( R \)-module is \( Z \)-supplemented by [13] Proposition 2.4.

In [11] for any preradical \( \tau \), the authors call a module \( M \), \( \tau \)-semiprfect if it satisfies one of the following conditions (see [11] Proposition 2.1):

1. For every submodule \( K \) of \( M \) there exists a decomposition \( K = A \oplus B \) such that \( A \) is a projective direct summand of \( M \) and \( B \subseteq \tau(M) \);
2. For every submodule \( K \) of \( N \), there exists a decomposition \( M = A \oplus B \) such that \( A \) is a projective direct summand of \( M \), \( A \leq K \) and \( K \cap B \subseteq \tau(M) \).

By this definition every \( \tau \)-semiprfect module is \( \tau \)-lifting and hence \( \tau \)-supplemented.

Also if \( M \) is projective we have the following:

\[ \tau \text{-semiprfect} \iff \tau \text{-lifting} \iff \tau \text{-supplemented}. \]

A \( \tau \)-supplemented module need not be \( \oplus \)-supplemented and the converse also hold.

Example 1.18. Let \( K \) be a field and let \( R = \prod_{n \geq 1} K_n \) with \( K_n = K \). By [14] Example 4.1(1)] \( R \) is not semiprfect. Since \( R \) is projective, \( R \) is not \( \oplus \)-supplemented by [13] Lemma 1.2]. Again by [14] Example 4.1(1)], the module \( R \) is \( Z \)-semiprfect and so it is \( Z \)-supplemented.

If \( R \) is a DVR (Discrete Valuation Ring ), then by [14] Example 4.1(1)] the \( R \)-module \( R_R \) is semiprfect and hence \( \oplus \)-supplemented but it is not \( Z \)-semiprfect and so it is not \( Z \)-supplemented.

Now we give an equivalent condition for a module to be \( Z \)-supplemented under some assumptions.
Proposition 1.19. Let $R$ be a commutative ring and $P$ a projective module with $\text{Rad}(P) \ll P$ and $P$ has finite hollow dimension. Then the following are equivalent:

1. $P$ is $\mathbb{Z}\oplus$-supplemented;
2. $P = P_1 \oplus P_2 \oplus P_3$ with $P_1$ is $\oplus$-supplemented and $\text{Rad}(P_1) = \mathbb{Z}(P_1)$, $P_2$ is semisimple and $\mathbb{Z}(P_3) = P_3$.

Proof. (1) $\Rightarrow$ (2) By the proof of [14] Corollary 4.3 and since every semiperfect is $\oplus$-supplemented.

(2) $\Rightarrow$ (1) By [14] Corollary 4.3 all $P_1$, $P_2$ and $P_3$ are $\mathbb{Z}$-semiperfect and hence $\mathbb{Z}\oplus$-supplemented. Since any finite direct sum of $\mathbb{Z}\oplus$-supplemented modules is $\mathbb{Z}\oplus$-supplemented, $P$ is $\mathbb{Z}\oplus$-supplemented.

Let $e = e^2 \in R$. Then $e$ is called a left (right) semicentral idempotent if $xe = exe$ (ex = exe), for all $x \in R$. The set of all left (right) semicentral idempotents is denoted by $S_l(R)$ ($S_r(R)$). A ring $R$ is called Abelian if every idempotent is central.

Let $M$ be a module. We consider the following condition.

$(D_3)$ If $M_1$ and $M_2$ are direct summands of $M$ with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of $M$.

By [14] Lemma 4.6 and Proposition 4.38, every quasi-projective module is $(D_3)$.

Proposition 1.20. Let $M$ be an $R$-module such that $\text{End}(M)$ is Abelian and $X \subseteq M$ implies $X = \sum_{i \in I} h_i(M)$ where $h_i \in \text{End}(M)$. Then for any preradical $\tau$, $M$ is $\tau$-$\oplus$-supplemented if and only if $M$ is $\tau$-lifting and has $(D_3)$-condition.

Proof. The sufficiency is obvious. Conversely, let $X \subseteq M$, $X = \sum_{i \in I} h_i(M)$ with $h_i(M) \in \text{End}(M)$. Since $M$ is $\tau\oplus$-supplemented, there exists a direct summand $eM$ such that $X + eM = M$ and $(X \cap eM) \subseteq \tau(eM)$ for some $e^2 = e \in \text{End}(M)$.

Since $\text{End}(M)$ is Abelian, $(1-e)X = (1-e)M = (1-e)\sum_{i \in I} h_i(M) = \sum_{i \in I} h_i(1-e)(M) \subseteq X$. Therefore $X = (1-e)M \oplus (X \cap eM)$. Hence $M$ is $\tau$-lifting. If $eM + fM = M$ for $e^2 = e$, $f^2 = f \in \text{End}(M)$, then $eM \cap fM = efM$ with $(ef)^2 = ef$. So $M$ has $(D_3)$-condition.

Recall that an $R$-module $M$ is said to be a multiplication module if for each $X \subseteq M$ there exists $A_M \subseteq RR$ such that $X = MA$.

Corollary 1.21. If $M$ satisfies one of the following conditions, then $M$ is $\tau$-lifting if and only if $M$ is $\tau\oplus$-supplemented for any preradical $\tau$.

1. $M$ is cyclic and $R$ is commutative.
2. $M$ is a multiplication module and $R$ is commutative.

Proof. (1) Assume that $M$ is cyclic and $R$ is commutative. There exists $B_R \subseteq RR$ such that $M \cong R/B$. Let $Y/B \subseteq R/B$, $Y/B = \sum_{i \in I} (y_i + B) = \sum_{i \in I} y_i R + B$ where each $y_i \in Y$. Define $h_i : R/B \to R/B$ by $h_i(r + B) = y_i r + B$, $i \in I$. Then it is easy to check that $h_i \in \text{End}_R(R/B)$. Hence $Y/B = \sum_{i \in I} h_i(R/B)$. Since $R$ is commutative, $\text{End}_R(R/B)$ is also commutative. By Proposition 1.13, $M$ is $\tau$-lifting.

(2) Assume $M$ is a multiplication module. Let $X \subseteq M$. Then $X = MA$ for some $A_M \subseteq RR$. For each $a \in A$, define $h_a : M \to M$ by $h_a(m) = ma$ for all $m \in M$. Then $h_a$ is an $R$-homomorphism and $X = MA = \sum_{a \in A} h_a(M)$. Since every multiplication module is a duo module, thus if $e^2 = e \in S = \text{End}(M)$, then $e$,
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$1 - e \in S_i(S)$. Therefore $e$ is central. So $End(M)$ is Abelian. Again by Proposition 1.20 $M$ is $\tau$-lifting.

$\square$

2. Completely $\tau$-\$\oplus$-Supplemented Modules

**Definition 2.1.** For any preradical $\tau$, we call a module $M$ **completely $\tau$-\$\oplus$-supplemented** for any preradical $\tau$ if every direct summand of $M$ is a $\tau$-\$\oplus$-supplemented.

**Theorem 2.2.** Let $M$ be a module with $(D_3)$ and $\tau$ a preradical. Then $M$ is $\tau$-\$\oplus$-supplemented if and only if $M$ is completely $\tau$-\$\oplus$-supplemented.

**Proof.** Sufficiency is clear. Conversely, assume that $M$ is $\tau$-\$\oplus$-supplemented and $K$ a direct summand of $M$ and $A$ a submodule of $K$. We show $A$ has a $\tau$-supplement in $K$ that is a direct summand of $K$. Since $M$ is $\tau$-\$\oplus$-supplemented, there exists a direct summand $B$ of $M$ such that $M = A + B$ and $A \cap B \subseteq \tau(B)$. Then $K = A + (K \cap B)$. Furthermore $K \cap B$ is a direct summand of $M$ because $M$ has $(D_3)$. Then $A \cap (K \cap B) = (A \cap B) \cap (K \cap B) \subseteq \tau(B) \cap (K \cap B) = \tau(K \cap B)$. $\square$

A submodule $K$ of $M$ is called **essential in $M$** (notation $K \leq_e M$) if $K \cap A \neq 0$ for any nonzero submodule $A$ of $M$.

**Proposition 2.3.** Let $M$ be a $\tau$-supplemented module for any preradical $\tau$. Then $M = M_1 \oplus M_2$, where $M_1$ is semisimple module and $M_2$ is a module with $\tau(M_2)$ essential in $M_2$.

**Proof.** See [2, 2.2]. $\square$

Recall that a module $M$ has the **Summand Sum Property** (SSP) if the sum of any two direct summands of $M$ is again a direct summand.

**Theorem 2.4.** (1) Every $\tau$-lifting module is completely $\tau$-\$\oplus$-supplemented for any preradical $\tau$.

(2) Let $M$ be a $\tau$-\$\oplus$-supplemented module for any preradical $\tau$. If $M$ has the (SSP), then $M$ is completely $\tau$-\$\oplus$-supplemented.

**Proof.** (1) By [2, 2.10] every direct summand of a $\tau$-lifting module is $\tau$-lifting. The rest is clear.

(2) Assume that $M$ is $\tau$-\$\oplus$-supplemented and $M$ has the (SSP). Let $N$ be a direct summand of $M$. We will show that $N$ is $\tau$-\$\oplus$-supplemented. Let $M = N \oplus N'$ for some submodule $N'$ of $M$. Suppose that $A$ is a direct summand of $M$. Since $M$ has the (SSP), $A + N'$ is a direct summand of $M$. Let $M = (A + N') \oplus B$ for some $B \leq M$. Then $M/N' = (A + N')/N' \oplus (B + N')/N'$. Hence by Theorem 1.5 $M/N'$ is $\tau$-\$\oplus$-supplemented and so $N$ is $\tau$-\$\oplus$-supplemented. $\square$

We give a decomposition of any $\tau$-\$\oplus$-supplemented $(D_3)$-module by the second singular submodule $Z_2(M)$ of $M$. We will show that if $M$ is $\tau$-\$\oplus$-supplemented and $N \leq M$ with $M/N$ projective, then $N$ is $\tau$-\$\oplus$-supplemented.

Recall that the **singular submodule $Z(M)$** of a module $M$ is defined by $Z(M) = \{ m \in M \mid mE = 0, E \leq_e R \}$.

The **Goldie torsion submodule (or second singular submodule)** $Z_2(M)$ of $M$ is a submodule of $M$ containing $Z(M)$ such that $Z_2(M)/Z(M)$ is the singular submodule of $M/Z(M)$.
Proposition 2.5. Let $M$ be a module with $(D_3)$. Suppose that $Z_2(M)$ is $\tau$-co-closed in $M$. Then for any preradical $\tau$, $M$ is $\tau$-$\oplus$-supplemented if and only if $M = Z_2(M) \oplus K$ for some submodule $K$ of $M$ and, $Z_2(M)$ and $K$ are $\tau$-$\oplus$-supplemented.

Proof. Sufficiency is clear by Theorem 1.2 Conversely, assume that $M$ is $\tau$-$\oplus$-supplemented. There exist submodules $K$ and $K'$ of $M$ such that $M = K \oplus K' = Z_2(M) + K$ and $Z_2(M) \cap K \subseteq \tau(K)$. Now $Z_2(M) = Z_2(K) \oplus Z_2(K')$. Thus, $M = K \oplus Z_2(K')$ and hence $Z_2(K') = K'$. Note that $Z_2(M) \cap K = Z_2(K) \subseteq \tau(K)$. So, we can obtain that $Z_2(M)/K' \subseteq \tau(M/K')$. Therefore, $Z_2(M) = K'$ because $Z_2(M)$ is $\tau$-co-closed in $M$. So, $M = K \oplus Z_2(M)$. Clearly $K$ and $Z_2(M)$ are $\tau$-$\oplus$-supplemented.

Proposition 2.6. Let $M$ be a $\tau$-supplemented module for any preradical $\tau$. Then $M = M_1 \oplus M_2$, where $M_1$ is semisimple module and $M_2$ is a module with $\tau(M_2)$ essential in $M_2$.

Proof. See [2, 2.2].

Corollary 2.7. Let $M$ be a $\tau$-$\oplus$-supplemented module for any preradical $\tau$. Then $M = M_1 \oplus M_2$ where $M_1$ is a semisimple module and $M_2$ is a module with $\tau(M_2)$ essential in $M_2$.

Proof. Since each $\tau$-$\oplus$-supplemented module is $\tau$-supplemented the result follows from Proposition 2.6.

Proposition 2.8. Let $M$ be a $\tau$-$\oplus$-supplemented module for a left exact preradical $\tau$. Then $M = M_1 \oplus M_2$ such that $\tau(M_2) = M_2$.

Proof. Suppose that $M$ is a $\tau$-$\oplus$-supplemented module. There exists a direct summand $M_1$ of $M$ such that $M = M_1 + \tau(M)$ and $M_1 \cap \tau(M) = \tau(M_1)$ since $\tau$ is a left exact preradical and $M = M_1 \oplus M_2$ for some submodule $M_2$ of $M$. Then $M = \tau(M_2) \oplus M_1$. Thus $M_2 = \tau(M_2)$.

Theorem 2.9. For module $M$ with $(D_3)$ and a left exact preradical $\tau$ the following statements are equivalent:

1. $M$ is completely $\tau$-$\oplus$-supplemented;
2. $M$ is $\tau$-$\oplus$-supplemented;
3. $M = M_1 \oplus M_2$, where $M_1$ is semisimple module and $M_2$ is a $\tau$-$\oplus$-supplemented module with $\tau(M_2)$ essential in $M_2$;
4. $M = M_1 \oplus M_2$ such that $M_1$ is a $\tau$-$\oplus$-supplemented module and $M_2$ is a $\tau$-$\oplus$-supplemented module with $\tau(M_2) = M_2$.

Proof. (1) $\Rightarrow$ (2) Clear from definition.

(2) $\Rightarrow$ (1) It follows from Theorem 2.2.

(1) $\Rightarrow$ (3) By Proposition 2.6, $M = M_1 \oplus M_2$, where $M_1$ is semisimple module and $M_2$ is module with $\tau(M_2)$ essential in $M_2$. By (1), $M_2$ is $\tau$-$\oplus$-supplemented.

(1) $\Rightarrow$ (4) By Proposition 2.8, $M = M_1 \oplus M_2$ such that $\tau(M_2) = M_2$ and $M_1, M_2$ are $\tau$-$\oplus$-supplemented by (1).

(3) $\Rightarrow$ (2), (4) $\Rightarrow$ (2) follows by Theorem 1.2.

Lemma 2.10. Let $M$ be an indecomposable module. Then for any preradical $\tau$, $M$ is completely torsion if and only if $M$ is completely $\tau$-$\oplus$-supplemented.

Proof. Clear.
Proposition 2.11. Let $M = M_1 \oplus M_2$ such that $M_1$ and $M_2$ have local endomorphism rings. Then for any preradical $\tau$, $M$ is completely $\tau$-$\oplus$-supplemented if and only if $M_1$ and $M_2$ are completely torsion modules.

Proof. The necessity is clear from Lemma 2.10. Conversely, let $K$ be a direct summand of $M$. If $K = M$ then by Corollary 2.3 $K$ is $\tau$-$\oplus$-supplemented. Assume $K \neq M$. Then either $K \cong M_1$ or $K \cong M_2$ by Corollary 12.7. In either case $K$ is $\tau$-$\oplus$-supplemented. Thus $M$ is completely $\tau$-$\oplus$-supplemented. $\square$

References


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