

## ON REGULAR SEMI GENERALIZED CLOSED SETS

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ABSTRACT. In this paper we introduce the concept of rsg-closed sets and investigate some of its properties in topological spaces. We also define an rsg-regular space and give some of its fundamental properties.

### 1. INTRODUCTION

In 1970, Levine [12] introduced the notion of generalized closed sets in topological spaces. In 1987, Battacharyya and Lahiri [2] used semi-open sets [11] to define the notion of semi-generalized closed sets. In 1990, Arya and Nour [1] introduced the concept of generalized semi-closed sets. The notion of  $s^*g$ -closed sets was introduced by Rao and Joseph [16]. In this paper, we investigate many properties of rsg-closed sets which are situated between  $s^*g$ -closed sets and rg-closed sets. We also show that arbitrary intersection of rsg-closed sets in a locally indiscrete space is rsg-closed. Moreover rsg-regular space is defined and some of its basic properties are investigated.

### 2. PRELIMINARY

Throughout this paper,  $(X, \tau)$  (or simply  $X$ ) will always represent a topological space on which no separation axioms are assumed, unless otherwise mentioned. When  $A$  is a subset of  $X$ ,  $cl(A)$  and  $Int(A)$  denote the closure and interior of a set  $A$ , respectively. A subset  $A$  of a space  $X$  is said to be semi-open [11] if there exists an open set  $U$  such that  $U \subset A \subset cl(U)$ . The complement of a semi-open set is said to be semi-closed. A subset  $A$  of a topological space  $X$  is said to be semi-regular [6] if it is both semi-open and semi-closed. In [6], it is pointed out that a set is semi-regular if and only if there exists a regular open set  $U$  such that  $U \subset A \subset cl(U)$ . Cameron [4] called semi regular sets regular semi-open.

**Definition 2.1.** A subset  $A$  of a space  $X$  is said to be

- (1): *generalized closed* [12] (*briefly, g-closed*) if  $cl(A) \subset U$  whenever  $A \subset U$  and  $U$  is open in  $X$ . The complement of a  $g$ -closed set is said to be  $g$ -open;
- (2):  *$s^*g$ -closed* [16] if  $cl(A) \subset G$  whenever  $A \subset G$  and  $G$  is semi-open in  $X$ . The complement of an  $s^*g$ -closed set is said to be  $s^*g$ -open;
- (3): *regular generalized closed* [15] (*briefly, rg-closed*) if  $cl(A) \subset U$  whenever  $A \subset U$  and  $U$  is regular-open in  $X$ . The complement of an  $rg$ -closed set is said to be  $rg$ -open;
- (4): *semi-generalized closed* [3] (*briefly, sg-closed*) if  $scl(A) \subset U$  whenever  $A \subset U$  and  $U$  is semi-open in  $X$ .

## 3. RSG-CLOSED SETS

**Definition 3.1.** A subset  $A$  of a space  $X$  is said to be

- (1): regular semi generalized closed (briefly, rsg-closed) if  $cl(A) \subset G$  whenever  $G \subset A$  for every semi-regular set  $G$  in  $X$ ;
- (2): regular semi generalized open (briefly, rsg-open) if  $X - A$  is rsg-closed.

**Theorem 3.2.** A subset  $A$  of a space  $(X, \tau)$  is rsg-open if and only if  $G \subset Int(A)$  whenever  $G \subset A$  for every semi-regular set  $G$  in  $X$ .

*Proof.* Let  $A$  be an rsg-open set and  $G$  a semi-regular set such that  $G \subset A$ . Then  $X - A$  is rsg-closed and  $X - A \subset X - G$ . Since  $X - G$  is semi-regular in  $X$ ,  $cl(X - A) \subset X - G$  and hence  $X - Int(A) \subset X - G$ . Therefore,  $G \subset Int(A)$ .

Conversely, let  $G \subset Int(A)$  whenever  $G \subset A$  and  $G$  is semi-regular in  $X$ . This implies that  $X - Int(A) = cl(X - A) \subset X - G$  whenever  $X - A \subset X - G$  and  $X - G$  is semi-regular in  $X$ . This proves that  $X - A$  is rsg-closed in  $X$  and hence  $A$  is rsg-open in  $X$ .

- Remark 3.3.**
- (1): Every closed set is rsg-closed;
  - (2): Every open set is rsg-open;
  - (3): Semi open sets and rsg-open sets are independent of each other.

**Example 3.4.** Let  $X = \{a, b, c, d\}$  and let

- (1):  $\tau = \{\phi, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, c, d\}, X\}$ . Then  $\{a, b, c\}$  is semi open but not rsg-open, similarly let
- (2):  $\tau = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$ . Then  $\{b\}$  is rsg-open but not semi open.

**Example 3.5.** The union of two rsg-open sets is generally not rsg-open. To see this in Example 3.4(1),  $\{a\}$  and  $\{b\}$  are rsg-open sets in  $X$  but  $\{a, b\}$  is not rsg-open. Therefore, the intersection of two rsg-closed sets is generally not rsg-closed.

**Theorem 3.6.** If  $A$  and  $B$  are rsg-open, then  $A \cap B$  is rsg-open.

*Proof.* If  $G \subset A \cap B$  and  $G$  is semi-regular, then  $G \subset Int(A)$  and  $G \subset Int(B)$  and hence  $G \subset Int(A) \cap Int(B) = Int(A \cap B)$ . By Theorem 3.2,  $A \cap B$  is rsg-open.

**Theorem 3.7.** The union of two rsg-closed sets is rsg-closed.

*Proof.* This is an immediate consequence of Theorem 3.6.

Diagram

$$\begin{array}{ccccc} \text{closed} & \longrightarrow & \text{s}^*\text{g-closed} & \longrightarrow & \text{g-closed} \\ & & \searrow & & \searrow \\ & & \text{rsg-closed} & \longrightarrow & \text{rg-closed} \end{array}$$

**Remark 3.8.** In Example 3.4(1),  $\{a, c, d\}$  is rsg-closed but it is neither g-closed nor sg-closed.  $\{c, d\}$  is sg-closed but not rsg-closed. Let  $X = \{a, b, c, d\}$  and let  $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ , then  $\{c\}$  is g-closed but not rsg-closed.

**Remark 3.9.** By Remark 3.8, we have

- (1): rsg-closedness and g-closedness are independent of each other.
- (2): rsg-closedness and sg-closedness are also independent of each other.

**Theorem 3.10.** If a set  $A$  is rsg-closed, then  $cl(A) - A$  contains no non empty semi-regular set.

*Proof.* Let  $F$  be a semi-regular subset of  $cl(A) - A$ . Then  $A \subset X - F$  and since  $A$  is rsg-closed and  $X - F$  is semi-regular, we have  $cl(A) \subset X - F$  or  $F \subset X - cl(A)$ . Thus  $F \subset cl(A) \cap (X - cl(A)) = \phi$ . Therefore  $F$  is empty.

**Theorem 3.11.** *If  $A$  is an rsg-closed subset of  $X$ , then  $cl(A) - A$  is rsg-open.*

*Proof.* Let  $A$  be an rsg-closed subset of  $X$  and  $G$  be a semi-regular subset of  $X$  such that  $G \subset cl(A) - A$ . By Theorem 3.10,  $G = \phi$  and thus  $G \subset Int[cl(A) - A]$ . By Theorem 3.2,  $cl(A) - A$  is an rsg-open set.

**Definition 3.12.** A subset  $A$  of a space  $X$  is said to be preopen [14] if  $A \subset Int(cl(A))$ .

**Lemma 3.13.** (Dorsett [8]). *Let  $A$  be a preopen set in a space  $(X, \tau)$ , then  $SR(A, \tau_A) = SR(X, \tau) \cap A$ , where  $SR(X, \tau)$  denotes the family of all semi-regular sets of  $(X, \tau)$ .*

**Definition 3.14.** A subset  $B$  of a space  $X$  is said to be rsg-closed relative to  $A$  if  $cl_A(B) \subset G$  whenever  $B \subset G$  for every semi-regular set  $G$  in  $A$ .

**Theorem 3.15.** *Let  $B \subset A \subset X$  and  $X$  be a space. If  $B$  is an rsg-closed set relative to  $A$  and  $A$  is open and  $s^*g$ -closed in  $X$ , then  $B$  is rsg-closed relative to  $X$ .*

*Proof.* Let  $B \subset G$  and suppose that  $G$  is semi-regular in  $X$ . Then  $B \subset A \cap G$ . Therefore  $cl_A(B) \subset A \cap G$  since by Lemma 3.13,  $A \cap G$  is semi-regular in  $A$ . It follows that  $A \cap cl_X(B) \subset A \cap G$  or  $A \subset G \cup (X - cl_X(B))$ . Since  $A$  is  $s^*g$ -closed,  $cl_X(A) \subset G \cup (X - cl_X(B))$  or  $cl_X(B) \subset G$ . This proves that  $B$  is rsg-closed relative to  $X$ .

**Corollary 3.16.** *Let  $A$  be an open and  $s^*g$ -closed subset of the space  $X$  and  $F$  be a closed subset of  $X$ . Then  $A \cap F$  is an rsg-closed set.*

*Proof.*  $A \cap F$  is closed in  $A$  and hence rsg-closed in  $A$ . By Theorem 3.15,  $A \cap F$  is rsg-closed relative to  $X$ .

**Theorem 3.17.** *Let  $B \subset A \subset X$  and suppose that  $B$  is rsg-closed in  $X$  and  $A$  is pre-open in  $X$ . Then  $B$  is rsg-closed relative to  $A$ .*

*Proof.* Let  $B \subset A \cap G$  and suppose that  $G$  is semi-regular in  $X$  then by Lemma 3.13,  $A \cap G$  is semi-regular in  $A$ . Now  $B \subset G$  implies that  $cl_A(B) \subset G$ . It follows that  $A \cap cl_X(B) \subset A \cap G$ . This gives  $cl_A(B) \subset A \cap G$ . This proves that  $B$  is rsg-closed relative to  $A$ .

**Corollary 3.18.** *Let  $B \subset A \subset X$  where  $A$  is open and  $s^*g$ -closed. Then  $B$  is rsg-closed relative to  $A$  if and only if  $B$  is rsg-closed in  $X$ .*

*Proof.* This is an immediate consequence of Theorems 3.15 and 3.17.

**Theorem 3.19.** *If  $B$  is a subset of a space  $X$  such that  $A \subset B \subset cl(A)$  and  $A$  is an rsg-closed set in  $X$ , then  $B$  is also rsg-closed in  $X$ .*

*Proof.* Let  $G$  be a semi-regular set containing  $B$ , then  $A \subset G$ . Since  $A$  is rsg-closed, therefore  $cl(A) \subset G$ . This gives  $cl(B) \subset G$ . Hence  $B$  is rsg-closed in  $X$ .

**Corollary 3.20.** *If  $B$  is a subset of a space  $X$  such that  $Int(A) \subset B \subset A$ , where  $A$  is an rsg-open set in the space  $X$ , then  $B$  is also rsg-open in  $X$ .*

*Proof.* Let  $F$  be any semi-regular set contained in  $B$ . Then  $F \subset A$ . Since  $A$  is rsg-open, therefore  $F \subset Int(A)$ . This gives  $F \subset Int(B)$ . Hence  $B$  is rsg-open.

**Definition 3.21.** A space  $X$  is said to be locally indiscrete [7] if every open set in it is closed.

**Theorem 3.22.** *In a locally indiscrete space  $X$ , a subset  $A$  is rsg-open in  $X$  if and only if  $G = X$  whenever  $G$  is semi-regular and  $\text{Int}(A) \cup (X - A) \subset G$ .*

*Proof. Necessity.* Suppose that  $G$  is semi-regular and that  $\text{Int}(A) \cup (X - A) \subset G$ . Now  $(X - G) \subset \text{cl}(X - A) \cap A = \text{cl}(X - A) - (X - A)$ . Since  $(X - G)$  is semi-regular and  $(X - A)$  is rsg-closed, by Theorem 3.10 it follows that  $(X - G) = \phi$  or  $X = G$ .

*Sufficiency.* Suppose that  $F$  is a semi-regular set and  $F \subset A$ . It suffices to show that  $F \subset \text{Int}(A)$ . Now  $\text{Int}(A) \cup (X - A) \subset \text{Int}(A) \cup (X - F)$  and hence  $\text{Int}(A) \cup (X - F) = X$ . It follows that  $F \subset \text{Int}(A)$ .

**Theorem 3.23.** *If  $A \subset Y \subset X$  where  $A$  is rsg-open relative to  $Y$  and  $Y$  is open in  $X$ , then  $A$  is rsg-open relative to  $X$ .*

*Proof.* Let  $F$  be any semi-regular subset of  $X$  contained in  $A$ . Since  $Y$  is open, therefore by Lemma 3.13,  $F$  is semi-regular in  $Y$ . Since  $A$  is rsg-open relative to  $Y$ , therefore  $F \subset \text{Int}_Y(A)$ . Since  $Y$  is open in  $X$ ,  $F \subset \text{Int}_Y(A) = \text{Int}_X(A)$ . This proves that  $A$  is rsg-open in  $X$ .

**Theorem 3.24.** *For each  $x \in X$ , either  $\{x\}$  is semi-regular or  $X - \{x\}$  is rsg-closed.*

*Proof.* If  $\{x\}$  is not semi-regular, then the only semi-regular superset of  $X - \{x\}$  is  $X$  itself. Hence the closure of  $X - \{x\}$  is contained in each of its semi-regular neighbourhoods and  $X - \{x\}$  is rsg-closed.

**Theorem 3.25.** *Let  $A$  and  $B$  be subsets of spaces  $X$  and  $Y$ , respectively, then  $A$  and  $B$  are rsg-closed in  $X$  and  $Y$ , respectively, if  $A \times B$  is rsg-closed in  $X \times Y$ .*

*Proof.* Let  $G$  and  $H$  be semi-regular subsets of  $X$  and  $Y$ , respectively, such that  $A \subset G$  and  $B \subset H$ . This implies  $A \times B \subset G \times H$  where  $G \times H$  is semi-regular in  $X \times Y$ . Since  $A \times B$  is rsg-closed in  $X \times Y$ , therefore  $\text{cl}(A \times B) = \text{cl}(A) \times \text{cl}(B) \subset G \times H$  or  $\text{cl}(A) \subset G$  and  $\text{cl}(B) \subset H$ . This proves that  $A$  and  $B$  are rsg-closed in  $X$  and  $Y$ , respectively.

**Theorem 3.26.** *Let  $X$  and  $Y$  be two spaces and  $A$  be a subset of a space  $X$ ,*

- (1): *If  $A \times Y$  is rsg-open in  $X \times Y$ , then  $A$  is rsg-open in  $X$ ;*
- (2): *If  $A \times Y$  is rsg-closed in  $X \times Y$ , then  $A$  is rsg-closed in  $X$ .*

*Proof.* (1) Let  $G$  be a semi-regular set in  $X$  such that  $G \subset A$ . Since  $G \times Y$  is a semi-regular set in  $X \times Y$ , then by definition  $G \times Y \subset \text{Int}(A \times Y) = \text{Int}(A) \times \text{Int}(Y) = \text{Int}(A) \times Y$ . This gives that  $G \subset \text{Int}(A)$ . This proves that  $A$  is rsg-open in  $X$ .

(2) Let  $G$  be a semi-regular set in  $X$  such that  $A \subset G$ . Since  $G \times Y$  is semi-regular in  $X \times Y$  and  $A \times Y \subset G \times Y$ . By definition  $\text{cl}(A) \times Y = \text{cl}(A) \times \text{cl}(Y) = \text{cl}(A \times Y) \subset G \times Y$ . This gives that  $\text{cl}(A) \subset G$ . This proves that  $A$  is rsg-closed in  $X$ .

**Theorem 3.27.** *Let  $A$  be an open and rsg-closed set, then  $\text{cl}(A)$  is clopen in  $X$ .*

*Proof.* Since  $A$  is open,  $\text{Int}(A) = A \subset \text{Int}(\text{cl}(A))$ . Since  $\text{Int}(\text{cl}(A))$  is semi-regular and  $A$  is rsg-closed, we obtain  $\text{cl}(A) \subset \text{Int}(\text{cl}(A))$ . This proves that  $\text{cl}(A)$  is clopen.

**Theorem 3.28.** *A regular open and rsg-closed set is clopen.*

*Proof.* Let  $A$  be regular open then  $A$  is semi-regular. This gives that  $\text{cl}(A) \subset A$ . But  $A \subset \text{cl}(A)$ . Therefore  $A$  is closed.

**Theorem 3.29.** *In a locally indiscrete space  $X$ , every semi-closed set is rsg-closed.*

*Proof.* Let  $A$  be semi-closed. Then  $X - A \in SO(X)$ . Since  $X$  is locally indiscrete,  $SO(X) = RO(X)$  ([9], Theorem 3.3). This shows that  $X - A$  is regular open in  $X$  or  $A$  is regular-closed in  $X$ . Therefore  $A$  is rsg-closed.

**Definition 3.30.** *The intersection of all semi-regular subsets of a space  $X$  containing a set  $A$  is called the semi-regular kernel of  $A$  and is denoted by  $srker(A)$ .*

**Lemma 3.31.** *A subset  $A$  of a space  $X$  is rsg-closed if and only if  $cl(A) \subset srker(A)$ .*

*Proof.* Assume that  $A$  is an rsg-closed set in  $X$ . Then  $cl(A) \subset G$  whenever  $A \subset G$  and  $G$  is semi-regular in  $X$ . This implies  $cl(A) \subset \bigcap \{G : A \subset G \text{ and } G \in SR(X)\} = srker(A)$

Conversely. Assume that  $cl(A) \subset srker(A)$ . This implies  $cl(A) \subset \bigcap \{G : A \subset G \text{ and } G \in SR(X)\}$ . This shows that  $cl(A) \subset G$  for any semi-regular set  $G$  containing  $A$ . This proves that  $A$  is rsg-closed.

**Lemma 3.32.** (Jankovic and Reilly [10]). *Let  $x$  be a point of a space  $X$ . Then  $\{x\}$  is either nowhere dense or preopen.*

**Theorem 3.33.** *Arbitrary intersection of rsg-closed sets in a locally indiscrete space  $X$  is rsg-closed.*

*Proof.* Let  $\{A_\alpha : \alpha \in I\}$  be an arbitrary collection of rsg-closed sets in a space  $X$  and let  $A = \bigcap_{\alpha \in I} A_\alpha$ . Let  $x \in cl(A)$ . In view of Lemma 3.32, we consider the following two cases.

Case I. Let  $\{x\}$  be nowhere dense. If  $x \notin A$ , then for some  $j \in I$ , we have  $x \notin A_j$ . Since nowhere dense subsets are semi-closed and  $X$  is locally indiscrete, therefore  $X - \{x\}$  is a regular open set containing  $A_j$ . Hence  $x \notin srker(A_j)$ . On the other hand, by Lemma 3.31, since  $A_j$  is rsg-closed,  $x \in cl(A) \subset cl(A_j) \subset srker(A_j)$ . By contradiction,  $x \in A$  and hence  $x \in srker(A)$ .

Case II. Let  $\{x\}$  be preopen. Set  $F = Int(cl(\{x\}))$ . Assume that  $x \notin srker(A)$ . Then there exists a semi-regular set  $C$  containing  $x$  such that  $C \cap A = \phi$ . Now by ([5], Theorem 1.2)  $x \in F = Int(cl(\{x\})) \subset Int(cl(C)) \subset C$ . Since  $F$  is an open set containing  $x$  and  $x \in cl(A)$ , therefore  $F \cap A \neq \phi$ . Since  $F \subset C$ ,  $C \cap A \neq \phi$ . By contradiction  $x \in srker(A)$ . Thus in both cases  $x \in srker(A)$ . By Lemma 3.31,  $A$  is rsg-closed.

**Corollary 3.34.** *For a locally indiscrete space  $X$ , the family of all rsg-open sets of  $X$  is a topology for  $X$ .*

*Proof.* This is an immediate consequence of Theorems 3.6 and 3.33.

#### 4. RSG-REGULAR SPACES

In this section, we define an rsg-regular space and investigate some of its fundamental properties.

**Definition 4.1.** *A space  $(X, \tau)$  is said to be s-regular [13] if for each closed set  $F$  and any point  $x \in X - F$ , there exist disjoint semi-open sets  $U$  and  $V$  in  $X$  such that  $x \in U$  and  $F \subset V$ .*

**Definition 4.2.** *A space  $(X, \tau)$  is said to be rsg-regular if for every rsg-closed set  $F$  and  $x \in X - F$  there exist disjoint open sets  $U$  and  $V$  in  $X$  such that  $x \in U$  and  $F \subset V$ .*

**Remark 4.3.** Every rsg-regular space is regular as well as  $s$ -regular but the converse is not true in general.

**Example 4.4.** Let  $X = Y \cup Z$  where  $Y \cap Z = \phi$  and  $Y, Z$  are infinite sets. Let  $\tau = \{\phi, Y, Z, X\}$  then  $(X, \tau)$  is a regular space. If  $\phi \neq A \subset Y$  and  $x \in Y - A$ , then  $A$  is an rsg-closed set but  $A$  and  $x$  can not be separated by disjoint open sets. Hence  $(X, \tau)$  fails to be an rsg-regular space.

**Theorem 4.5.** The following are equivalent for a space  $(X, \tau)$ :

- (1):  $(X, \tau)$  is rsg-regular.
- (2): For every rsg-open set  $U$  containing  $x \in X$ , there exists an open set  $G$  in  $X$  such that  $x \in G \subset \text{cl}(G) \subset U$ .

*Proof.* (1)  $\Rightarrow$  (2) Let  $U$  be any rsg-open set containing  $x \in X$ . Then  $x \notin X - U$ , where  $X - U$  is rsg-closed in  $X$ . Hence there exist disjoint open sets  $G$  and  $H$  such that  $x \in G$  and  $X - U \subset H$  or  $x \in G \subset \text{cl}(G) \subset X - H \subset U$ . This proves (2).

(2)  $\Rightarrow$  (1) Let  $F$  be an rsg-closed set and  $x \in X - F$ . By hypothesis, there exists an open set  $G$  in  $X$  such that  $x \in G \subset \text{cl}(G) \subset X - F$  or  $x \in G$  and  $F \subset X - \text{cl}(G)$  where  $G \cap (X - \text{cl}(G)) = \phi$ . This proves that  $X$  is rsg-regular.

**Definition 4.6.** A space  $(X, \tau)$  is said to be rsg-regular at a point  $x \in X$  if every rsg-open neighbourhood of  $x$  contains a closed neighbourhood of  $x$ .

**Theorem 4.7.** A space  $(X, \tau)$  is rsg-regular if and only if it is rsg-regular at each of its points.

*Proof.* Suppose  $X$  is rsg-regular and  $x \in X$ . Let  $U$  be any rsg-open neighbourhood of  $x \in X$ . Then  $X - U$  is rsg-closed and  $x \notin X - U$ . Since  $X$  is rsg-regular, there exist disjoint open sets  $G$  and  $H$  such that  $x \in G$  and  $X - U \subset H$ . Now  $G \cap H = \phi$  implies  $x \in G \subset X - H \subset U$ . This proves that  $X$  is rsg-regular at each of its points.

Conversely, let  $X$  be rsg-regular at each of its points. Let  $F$  be an rsg-closed set and  $x \in X - F$ , where  $X - F$  is an rsg-open neighbourhood of  $x$ . By hypothesis there exists an open set  $V$  of  $X$  such that  $x \in V \subset \text{cl}(V) \subset X - F$ . By Theorem 4.5,  $X$  is rsg-regular.

**Theorem 4.8.** Every open and  $s^*g$ -closed subspace of an rsg-regular space is rsg-regular.

*Proof.* Suppose  $X$  is an rsg-regular space and  $Y$  is an open and  $s^*g$ -closed subspace of  $X$ . Let  $A$  be an rsg-closed set in  $Y$ . By Theorem 3.15,  $A$  is an rsg-closed set in  $X$ . Let  $x \in Y - A$ , then  $x \in X - A$  implies that there exist open sets  $U$  and  $V$  in  $X$  such that  $x \in U$ ,  $A \subset V$  and  $U \cap V = \phi$ ; hence  $x \in U \cap Y$ ,  $A \subset V \cap Y$ , where  $U \cap Y$  and  $V \cap Y$  are disjoint open sets in  $Y$ . This proves that  $Y$  is an rsg-regular space.

**Lemma 4.9.** In an rsg-regular space every rsg-open set is the union of open sets.

*Proof.* Let  $U$  be an rsg-open subset of an rsg-regular space  $X$  such that  $x \in U$ . If  $A = X - U$ , then  $A$  is an rsg-closed set and  $x \in X - A$ . By hypothesis there exist disjoint open sets  $W_x$  and  $W$  of  $X$  such that  $x \in W_x$  and  $A \subset W$ . It follows that  $x \in W_x \subset U$ . This completes the proof.

**Corollary 4.10.** In an rsg-regular space every rsg-closed set is the intersection of closed sets.

**Definition 4.11.** A space  $(X, \tau)$  is called a  $T_r$ -space if every rsg-closed subset of  $X$  is closed.

**Lemma 4.12.** A space  $(X, \tau)$  is rsg-regular if and only if  $(X, \tau)$  is a regular and  $T_r$ -space.

*Proof.* Let  $X$  be an rsg-regular space, then  $X$  is a regular space. Let  $A$  be an rsg-closed subset of  $X$ . Let  $x \in cl(A)$ . If  $x \notin A$ , then by hypothesis, there exist disjoint open sets  $U$  and  $V$  containing  $x$  and  $A$ , respectively. This contradicts that  $x \in cl(A)$ . Therefore  $x \in A$  and hence  $A$  is closed.

Conversely, let  $(X, \tau)$  be a regular and  $T_r$ -space. Let  $A$  be an rsg-closed subset of  $X$  and  $x \in X - A$ . By definition 4.11,  $A$  is closed and by regularity of  $X$ , there exist disjoint open sets  $U$  and  $V$  containing  $x$  and  $A$ , respectively. This proves that  $X$  is an rsg-regular space.

**Theorem 4.13.** For a space  $(X, \tau)$ , the following are equivalent:

- (1):  $(X, \tau)$  is a  $T_r$ -space.
- (2): Every singleton subset of  $X$  is either open or semi-regular.

*Proof.* (1)  $\Rightarrow$  (2) Let  $x \in X$ . Suppose  $\{x\}$  is not a semi-regular subset of  $X$ . This gives  $X - \{x\}$  is not semi-regular and therefore  $X$  is the only semi-regular super set of  $X - \{x\}$ . Trivially  $X - \{x\}$  is rsg-closed. By hypothesis,  $X - \{x\}$  is closed or  $\{x\}$  is open.

(2)  $\Rightarrow$  (1) Let  $A$  be an rsg-closed subset of  $X$ . Let  $x \in cl(A)$ . By hypothesis  $\{x\}$  is either open or semi-regular. If  $\{x\}$  is open, then  $\{x\} \cap A \neq \emptyset$  implies  $x \in A$ . If  $\{x\}$  is semi-regular and  $x \notin A$ , then  $x \in cl(A) - A$ . This implies that  $cl(A) - A$  contains a nonempty semi-regular set. This contradicts Theorem 3.10. Hence  $x \in A$ . This proves (1).

**Remark 4.14.** In  $T_r$ -space, closed sets,  $s^*$ -g-closed sets and rsg-closed sets coincide.

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