ALBANIAN JOURNAL
OF MATHEMATICS
Volume 3, Number 4, Pages 175–188
ISSN 1930-1235: (2009)

SOME CLASSES OF GENERAL NONCONVEX VARIATIONAL INEQUALITIES

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Abstract. In this paper, we introduce and consider a new class of variational inequalities, which is called the general nonconvex variational inequality. We establish the equivalence between the general nonconvex variational inequalities and the fixed point problems as well as the Wiener-Hopf equations using the projection method. This alternative equivalent formulation is used to study the existence of a solution of the general convex variational inequalities. We also use this equivalence formulation to suggest some iterative methods. Convergence criteria of these new iterative is also discussed under suitable conditions. Our method of proofs is very simple as compared with other techniques.

1. Introduction

Variational inequalities theory, which was introduced by Stampacchia [39], can be viewed as a natural generalization and extension of the variational principles, the origin of can be traced back to Fermat, Newton, Leibniz, Bernoulli, Euler and Lagrange. It is tool of great power that can be applied to a wide variety of problems, which arise in almost all branches of pure, applied, physical, regional and engineering sciences. During this period, variational inequalities have played an important, fundamental and significant part as a unifying influence and as a guide in the mathematical interpretation of many physical phenomena. In fact, it has been shown that the variational inequalities provide the most natural, direct, simple and efficient framework for the general treatment of wide range of problems.Variational inequalities have been extended and generalized in several directions for studying a wide class of equilibrium problems arising in financial, economics, transportation, elasticity, optimization, pure and applied sciences, see [1-40] and the references therein. An important and useful generalization of variational inequalities is called the *general variational inequality* introduced by Noor [12] in 1988, which enables us to study the odd-order and nonsymmetric problems in a unified framework. See, for example 2.1 and example 2.2 for some applications of the general variational inequalities in differential equations and nonlinear optimization.

Received by the editors October 26, 2009 .
2000 *Mathematics Subject Classification.* Primary 49J40; Secondary 90C33.
Key words and phrases. Variational inequalities; nonconvex functions; fixed-point problem, Wiener-Hopf equations, convergence.

It is worth mentioning that almost all the results regarding the existence and iterative schemes for variational inequalities, which have been investigated and considered in the classical convexity. This is because all the techniques are based on the properties of the projection operator over convex sets, which may not hold in general for nonconvex sets. Motivated and inspired by the ongoing research in this area, we introduce and consider a new class of variational inequalities, which is called the general nonconvex variational inequality in conjunction with the uniformly prox-regular sets. It is well-known that the prox-regular are nonconvex sets and include the convex sets as a special case, see [7,37]. Using the idea and technique of Noor [26-30], we show that the projection technique can be extended for the general nonconvex variational inequalities. We establish the equivalence between the general nonconvex variational inequalities and fixed point problems using essentially the projection technique. This equivalent alternative formulation is used to discuss the existence of a solution of the nonconvex variational inequalities, which is Theorem 3.1. Theorem 3.1 extends the previous results for the general nonconvex variational inequalities. We use this alternative equivalent formulation to suggest and analyze an implicit type iterative methods for solving the nonconvex variational inequalities. In order to implement this new implicit method, we use the predictor-corrector technique to suggest a two-step method for solving the nonconvex variational inequalities, which is Algorithm 3.4. We also consider the convergence (Theorem 3.2) of the new iterative method under some suitable conditions. Some special cases are also discussed.

Related to the general nonconvex variational inequalities, we consider the problem of solving the nonconvex Wiener-Hopf equations. Using essentially the projection technique and Lemma 3.1, we show that the general nonconvex variational inequalities are equivalent to the Wiener-Hopf equations, which is Lemma 4.1. This alternative equivalent formulation is more general and flexible than the projection operator technique. This alternative equivalent formulation is used to suggest and analyze a number of iterative methods for solving the nonconvex variational inequalities. These iterative methods is the subject of Section 4. We also consider the convergence criteria of the proposed iterative methods under some suitable conditions. Several special cases are also discussed. Results obtained in this paper can be viewed as refinement and improvement of the previously known results for the variational inequalities and related optimization problems. We would like to point out that our methods of proof are very simple as compared with other techniques. It is an open problem to implement these methods numerically. The comparison of these new methods with other similar methods for solving nonconvex variational inequalities is also open problem and needs further research.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let K be a nonempty and convex set in H.

We, first of all, recall the following well-known concepts from nonlinear convex analysis and nonsmooth analysis [7,37].

Definition 2.1. The proximal normal cone of K at $u \in H$ is given by

$$N_K^P(u) := \{\xi \in H : u \in P_K[u + \alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in K : d_K(u) = \|u - u^*\|\}.$$

Here $d_K(.)$ is the usual distance function to the subset K, that is

$$d_K(u) = \inf_{v \in K} \|v - u\|.$$

The proximal normal cone $N_K^P(u)$ has the following characterization.

Lemma 2.1. Let K be a nonempty, closed and convex subset in H. Then $\zeta \in N_K^P(u)$,
if and only if, there exists a constant $\alpha > 0$ such that

$$\langle \zeta, v - u\rangle \leq \alpha \|v - u\|^2, \quad \forall v \in K.$$

Poliquin et al. [37] and Clarke et al [7] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions.

Definition 2.2. For a given $r \in (0, \infty]$, a subset K_r is said to be normalized uniformly r-prox-regular if and only if every nonzero proximal normal to K_r can be realized by an r-ball, that is, $\forall u \in K_r$ and $0 \neq \xi \in N_{K_r}^P(u)$, one has

$$\langle (\xi)/\|\xi\|, v - u\rangle \leq (1/2r)\|v - u\|^2, \quad \forall v \in K.$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [7,37]. It is clear that if $r = \infty$, then uniformly prox-regularity of K_r is equivalent to the convexity of K. It is known that if K_r is a uniformly prox-regular set, then the proximal normal cone $N_{K_r}^P(u)$ is closed as a set-valued mapping.

For a given nonlinear operator T, g, we consider the problem of finding $u \in H$: $g(u) \in K_r$ such that

$$\langle Tu, g(v) - g(u)\rangle \geq 0, \quad \forall v \in H : g(v) \in K_r, \tag{1}$$

which is called the *general nonconvex variational inequality.*

If $g \equiv I$, the identity operator, then problem (1) is equivalent to finding $u \in K_r$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in K_r, \tag{2}$$

which is known as the nonconvex variational inequality, studied and introduced by Noor [26].

If $K_r \equiv K$, the convex set in H, then problem (1) is equivalent to finding $u \in H : g(u) \in K$ such that

$$\langle Tu, g(v) - g(u)\rangle \geq 0, \forall v \in H : g(v) \in K, \tag{3}$$

which was introduced and studied by Noor [12] in 1988. To convey an idea of the applications of the general variational inequalities (3) in the differential equations, we have the following.

Example 2.1. For simplicity, we consider the third-order obstacle boundary value problem of finding u such that

$$\text{(4)} \qquad \begin{cases} -u''' \geq f(x) & \text{on } \Omega = [0,1] \\ \quad u \geq \psi(x) & \text{on } \Omega = [0,1] \\ \quad [-u''' - f(x)][u - \psi(x)] = 0 & \text{on } \Omega = [0,1] \\ \quad u(0) = 0, \quad u'(0) = 0, \quad u'(1) = 0. \end{cases}$$

where $f(x)$ is a continuous function and $\psi(x)$ is the obstacle function. We study the problem (4) in the framework of variational inequality approach. To do so, we first define the set K as

$$K = \{v : v \in H_0^2(\Omega) : v \geq \psi \quad \text{on } \Omega\},$$

which is a closed convex set in $H_0^2(\Omega)$, where $H_0^2(\Omega)$ is a Sobolev (Hilbert) space, see [9]. One can easily show that the energy functional associated with the problem (4) is

$$\begin{aligned} I[v] &= -\int_0^1 \left(\frac{d^3v}{dx^3}\right)\left(\frac{dv}{dx}\right) dx - 2\int_0^1 f(x)\left(\frac{dv}{dx}\right) dx, \quad \text{for all } \tfrac{dv}{dx} \in K \\ &= \int_0^1 \left(\frac{d^2v}{dx^2}\right)^2 dx - 2\int_0^1 f(x)\left(\frac{dv}{dx}\right) dx \\ &= \langle Tv, g(v)\rangle - 2\langle f, g(v)\rangle \end{aligned} \qquad \text{(5)}$$

where

$$\text{(6)} \qquad \begin{aligned} \langle Tu, g(v)\rangle &= \int_0^1 \left(\frac{d^2u}{dx^2}\right)\left(\frac{d^2v}{dx^2}\right) dx \\ \langle f, g(v)\rangle &= \int_0^1 f(x)\frac{dv}{dx} dx \end{aligned}$$

and $g = \dfrac{d}{dx}$ is the linear operator.

It is clear that the operator T defined by (6) is linear, g-symmetric, that is, $\langle Tu, g(v)\rangle = \langle Tv, g(u),\rangle \quad \forall u, v \in H$ and g-positive, that is, $\langle Tu, g(u) \geq 0, \quad \forall u \in H$. Using the technique of Noor [20], one can easily show that the minimum $u \in H$ of the functional $I[v]$ defined by (5) associated with the problem (4) on the closed convex set K can be characterized by the inequality of type

$$\langle Tu, g(v) - g(u)\rangle \geq \langle f, g(v) - g(u)\rangle, \quad \forall g(v) \in K,$$

which is exactly the general variational inequality (3). It is worth mentioning that a wide class of unrelated odd-order and nonsymmetric obstacle, unilateral, contact, free, moving, and equilibrium problems arising in regional, physical, mathematical, engineering and applied sciences can be studied in the unified and general framework of the general variational inequalities (1), see [2-5, 14-24,32-34] and the references therein.

Example 2.2. We now show that the minimum of a class of differentiable nonconvex functions on g-convex set K in H can be characterized by general variational inequality (3). For the sake of completeness and to convey an idea of the applications, we give all the details.

For this purpose, we recall the following well known concepts, see [8].

Definition 2.3. Let K be any set in H. The set K is said to be g-convex, if there exist a function $g : H \longrightarrow H$ such that

$$g(u) + t(g(v) - g(u)) \in K, \quad \forall u, v \in H : g(u), g(v) \in K, \quad t \in [0, 1].$$

Note that every convex set is g-convex, but the converse is not true, see [8]. We would like to mention that the g-convex set K was introduced by Noor [12] in 1998 implicitly. See also Youness [40] for other properties of the g-convex set.

Definition 2.4. The function $F : K \longrightarrow H$ is said to be g-convex, if there exists a function g such that

$$\begin{aligned} F(g(u) + t(g(v) - g(u))) \quad &\leq \quad (1-t)F(g(u)) + tF(g(v)), \\ &\forall u, v \in H : g(u), g(v) \in K, \quad t \in [0, 1]. \end{aligned}$$

Clearly every convex function is g-convex, but the converse is not true, see [8,40].

We now show that the minimum of a differentiable g-convex function on the g-convex set K in H can be characterized by the general variational inequality (1) and this is the main motivation of our next result, which is due to Noor [16].

Lemma 2.2[16]. Let $F : K \longrightarrow H$ be a differentiable g-convex function. Then $u \in H : g(u) \in K$ is the minimum of g-convex function F on K, if and only if, $u \in H : g(u) \in K$ satisfies the inequality

$$\langle F'(g(u)), g(v) - g(u) \rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{7}$$

where $F'(u)$ is the differential of F at $g(u) \in K$.

Proof. Let $u \in H : g(u) \in K$ be a minimum of g-convex function F on K. Then

$$F(g(u)) \leq F(g(v)), \quad \forall v \in H : g(v) \in K. \tag{8}$$

Since K is a g-convex set, so, for all $u, v \in H : g(u), g(v) \in K, t \in [0, 1], g(v_t) = g(u) + t(g(v) - g(u)) \in K$. Setting $g(v) = g(v_t)$ in (8), we have

$$F(g(u)) \leq F(g(u) + t(g(v) - g(u))).$$

Dividing the above inequality by t and taking $t \longrightarrow 0$, we have

$$\langle F'(g(u)), g(v) - g(u) \rangle \geq 0, \quad \forall v \in H : g(v) \in K, \langle F'(g(u)), g(v) - g(u) \rangle \geq 0, \quad \forall v \in H : g(v) \in K,$$

which is the required result(7).

Conversely, let $u \in H : g(u) \in K$ satisfy the inequality (7). Since F is a g-convex function, $\quad \forall u, v \in H : g(u), g(v) \in K, \quad t \in [0, 1], \quad g(u) + t(g(v) - g(u)) \in K$ and

$$F(g(u) + t(g(v) - g(u))) \leq (1-t)F(g(u)) + tF(g(v)),$$

which implies that

$$F(g(v)) - F(g(u)) \geq \frac{F(g(u) + t(g(v) - g(u))) - F(g(u))}{t}.$$

Letting $t \longrightarrow 0$, and using (7), we have

$$F(g(v)) - F(g(u)) \geq \langle F'(h(u)), g(v) - g(u) \rangle \geq 0,$$

which implies that

$$F(g(u)) \leq F(g(v)), \quad \forall v \in H : g(v) \in K$$

showing that $u \in H : g(u) \in K$ is the minimum of F on K in H. □

Lemma 2.2 implies that g-convex programming problem can be studied via the general variational inequality (1) with $Tu = F'(g(u))$. In a similar way, one can show that the general variational inequality (1) is the Fritz-John condition of the inequality constrained optimization problem.

If $g \equiv I$, the identity operator, then problem (3) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle \geq 0, \quad v \in K, \tag{9}$$

which is known as the classical variational inequality, introduced and studied by Stampacchia [39] in 1964. It turned out that a number of unrelated obstacle, free, moving, unilateral and equilibrium problems arising in various branches of pure and applied sciences can be studied via variational inequalities, see [1-39] and the references therein.

It is well-known that problem (9) is equivalent to finding $u \in K$ such that

$$0 \in Tu + N_K(u), \tag{10}$$

where $N_K(u)$ denotes the normal cone of K at u in the sense of convex analysis. Problem (10) is called the variational inclusion associated with variational inequality (9).

Similarly, if K_r is a nonconvex (uniformly prox-regular) set, then problem (1) is equivalent to finding $u \in K_r$ such that

$$0 \in \rho Tu + g(u) - g(u) + \rho N^P_{K_r}(g(u)), \tag{11}$$

where $N^P_{K_r}(g(u))$ denotes the normal cone of K_r at $g(u)$ in the sense of nonconvex analysis. Problem (11) is called the nonconvex variational inclusion problem associated with nonconvex variational inequality (1). This implies that the variational inequality (1) is equivalent to finding a zero of the sum of two monotone operators (11). This equivalent formulation plays a crucial and basic part in this paper. We would like to point out this equivalent formulation allows us to use the projection operator technique for solving the general nonconvex variational inequality (1).

We now recall the well known proposition which summarizes some important properties of the uniform prox-regular sets.

Lemma 2.3. Let K be a nonempty closed subset of H, $\quad r \in (0, \infty]$ and set $K_r = \{u \in H : d(u, K) < r\}$. If K_r is uniformly prox-regular, then

i. $\quad \forall u \in K_r, P_{K_r}(u) \neq \emptyset$.

ii. $\quad \forall r' \in (0, r)$, P_{K_r} is Lipschitz continuous with constant $\frac{r}{r-r'}$ on $K_{r'}$.

We now consider the problem of solving the nonlinear Wiener-Hopf equations. To be more precise, let $Q_{K_r} = I - P_{K_r}$, where P_{K_r} is the projection operator, and I is the identity operator. For given nonlinear operators T, g, consider the problem of finding $z \in H$ such that

$$Tg^{-1}P_{K_r}z + \rho^{-1}Q_{K_r}z = 0, \tag{12}$$

where g^{-1} is the inverse of the operator g. Equations of the type (12) are called the general nonconvex Wiener-Hopf equations. Note that, if $r = \infty$, Then the nonlinear Wiener-Hopf equations are exactly the same Wiener-Hopf equations associated with the general variational inequalities (3), which were introduced and studied by Noor [14]. For $g \equiv I$, the identity operator and $r = \infty$, one can obtain the original Wiener-Hopf equations which were introduced and studied by Shi [38] in conjunction with the variational inequalities. This shows that the original Wiener-Hopf equations are the special case of the general nonconvex Wiener-Hopf equations (12). The Wiener-Hopf equations technique has been used to study and develop several iterative methods for solving variational inequalities and related optimization problems, see [14-33].

Definition 2.6. An operator $T : H \to H$ is said to be:

(i) *strongly monotone*, if and only if, there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \alpha ||u - v||^2, \quad \forall u, v \in H.$$

(ii) *Lipschitz continuous*, if and only if, there exists a constant $\beta > 0$ such that

$$||Tu - Tv|| \leq \beta ||u - v||, \quad \forall u, v \in H.$$

3. Projection Iterative Algorithms

In this section, we establish the equivalence between the general nonconvex variational inequality (1) and the fixed point problem using the projection operator technique. This alternative formulation is used to discuss the existence of a solution of the problem (1) and to suggest some new iterative methods for solving the general nonconvex variational inequality (1).

Lemma 3.1. $u \in H : g(u) \in K_r$ is a solution of the general nonconvex variational inequality (1) if and only if $u \in H : g(u) \in K_r$ satisfies the relation

$$g(u) = P_{K_r}[g(u) - \rho Tu], \tag{13}$$

where P_{K_r} is the projection of H onto the uniformly prox-regular set K_r.

Proof. Let $u \in H : g(u) \in K_r$ be a solution of (1). Then, for a constant $\rho > 0$,

$$\begin{aligned} 0 &\in g(u) + \rho N^P_{K_r}(g(u)) - (g(u) - \rho Tu) = (I + \rho N^P_{K_r})(g(u)) - (g(u) - \rho Tu) \\ &\Longleftrightarrow \\ g(u) &= (I + \rho N^P_{K_r})^{-1}[g(u) - \rho Tu] = P_{K_r}[g(u) - \rho Tu], \end{aligned}$$

where we have used the well-known fact that $P_{K_r} \equiv (I + N^P_{K_r})^{-1}$. □

Lemma 3.1 implies that the general nonconvex variational inequality (1) is equivalent to the fixed point problem (13). This alternative equivalent formulation is very useful from the numerical and theoretical point of views.

We rewrite the the relation (13) in the following form

$$F(u) = u - g(u) + P_{K_r}[g(u) - \rho Tu], \tag{14}$$

which is used to study the existence of a solution of the general nonconvex variational inequality (1).

We now study those conditions under which the general nonconvex variational inequality (1) has a solution and this is the main motivation of our next result.

Theorem 3.1. Let P_{K_r} be the Lipschitz continuous operator with constant $\delta = \frac{r}{r-r'}$. Let T, g be strongly monotone with constants $\alpha > 0, \sigma > 0$ and Lipschitz continuous with constants $\beta > 0, \delta > 0$, respectively. If there exists a constant $\rho > 0$ such that

$$|\rho - \frac{\alpha}{\beta^2}| \quad < \quad \frac{\sqrt{\delta^2\alpha^2 - \beta^2(1 - (1+\delta)k))^2)}}{\delta\beta^2}, \tag{15}$$

$$\delta\alpha \quad > \quad \beta\sqrt{k(1+\delta)(2 - k(1+\delta))}, \quad k < \frac{2}{1+\delta}, \tag{16}$$

where

$$k = \sqrt{1 - 2\sigma + \delta^2}, \tag{17}$$

then there exists a solution of the general nonconvex variational inequality (1).

Proof. From Lemma 3.1, it follows that problems (13) and (1) are equivalent. Thus it is enough to show that the map $F(u)$, defined by (14), has a fixed point. For all $u \neq v \in K_r$, we have

$$\begin{aligned} \|F(u) - F(v)\| &= \|u - v - (g(u) - g(v))\| + \|P_{K_r}[g(u) - \rho Tu] - P_{K_r}[v - \rho Tv]\| \\ &\leq \|u - v - (g(u) - g(v))\| + \delta\|g(u) - g(v) - \rho(Tu - Tv)\|, \\ &\leq \|u - v - (g(u) - g(v))\| + \delta\|u - v - \rho(Tu - Tv)\| \\ &\quad +\delta\|u - v - (g(u) - g(v))\|, \end{aligned} \tag{18}$$

where we have used the fact that the operator P_{K_r} is a Lipschitz continuous operator with constant δ.

Since the operator T is strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, it follows that

$$\begin{aligned} \|u - v - \rho(Tu - Tv)\|^2 &\leq \|u - v\|^2 - 2\rho\langle Tu - Tv, u - v\rangle + \rho^2\|Tu - Tv\|^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u - v\|^2. \end{aligned} \tag{19}$$

In a similar way, we have

$$\|u - v - (g(u) - g(v))\| \leq \sqrt{1 - 2\sigma + \delta^2}\|u - v| = k\|u - v\|, \tag{20}$$

where $\sigma > 0$ is the strongly monotonicity constant and $\delta > 0$ is the Lipschitz continuity constant of the operator g respectively.

From (18), (19) and (20), we have

$$\begin{aligned} \|F(u) - F(v)\| &\leq \left\{k + \delta\left\{k + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2)}\right\} \|u - v\|\right\| \\ &= \theta\|u - v\|, \end{aligned}$$

where

$$\theta = k + \delta\left\{\sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\right\}. \tag{21}$$

From (15) and (16), it follows that $\theta < 1$, which implies that the map $F(u)$ defined by (14), has a fixed point, which is a unique solution of (1). □

This fixed point formulation (13) is used to suggest the following iterative method for solving the nonconvex variational inequality (1).

Algorithm 3.1. For a given $u_0 \in K_r$, find the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned} u_{n+1} &= (1 - \alpha_n)u_n - \alpha_n\{u_n - g(u_n) \\ &\quad +P_{K_r}[g(u_n) - \rho Tu_n]\}, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{22}$$

where $\alpha_n \in [0, 1], \forall n \geq 0$ is a constant. Algorithm 3.1 is also called the Mann iteration process.

For $\alpha_n = 1$, Algorithm 3.1 collapse to:

Algorithm 3.2. For a given $u_0 \in K_r$, find the approximate solution u_{n+1} by the iterative scheme

$$g(u_{n+1}) = P_{K_r}[g(u_n) - \rho Tu_n], \quad n = 0, 1, 2, \ldots$$

We again use the fixed formulation to suggest and analyze an iterative method for solving the nonconvex variational inequalities (1) as:

Algorithm 3.3. For a given $u_0 \in K_r$, find the approximate solution u_{n+1} by the iterative scheme

$$g(u_{n+1}) = P_{K_r}[g(u_{n+1}) - \rho T u_{n+1}], \quad n = 0, 1, 2, \ldots$$

Algorithm 3.3 is an implicit type iterative method, which is difficult to implement. To implement Algorithm 3.3, we use the predictor-corrector technique. Here we use the Algorithm 3.1 as a predictor and Algorithm 3.3 as a corrector. Consequently, we have the following iterative method

Algorithm 3.4. For a given $u_0 \in K_r$, find the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} g(y_n) &= P_{K_r}[g(u_n) - \rho T u_n] \\ g(u_{n+1}) &= P_{K_r}[g(y_n) - \rho T y_n], \quad n = 0, 1, 2, \ldots \end{aligned}$$

which is called the two-step or splitting type iterative method for solving the general nonconvex variational inequalities (1). It is worth mentioning that Algorithm 3.4 can be suggested by using the updating the technique of the solution.

In this paper, we suggest and analyze the following two-step iterative method for solving the general nonconvex variational inequalities (1).

Algorithm 3.5. For a given $u_0 \in K_r$, find the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} y_n &= (1 - \beta_n)u_n + \beta_n\{y_n - g(y_n) + P_{K_r}[g(u_n) - \rho T u_n]\} \\ u_{n+1} &= (1 - \alpha_n)u_n + \alpha_n\{u_n - g(u_n) + P_{K_r}[g(y_n) - \rho T y_n]\}, \quad n = 0, 1, 2, \ldots, \end{aligned}$$

where $\alpha_n, \beta_n \in [0, 1], \quad \forall n \geq 0$.

Clearly for $\alpha_n = \beta_n = 1$, Algorithm 3.5 reduces to Algorithm 3.4. It is worth mentioning that, if $r = \infty$, then the nonconvex set K_r reduces to a convex set K. Consequently Algorithms 3.1- 3.5 collapse to the following algorithms for solving the general variational inequalities (6). We would like to point that Algorithm 3.4 appears to be a new one for solving the variational inequalities (2)

We now consider the convergence analysis of Algorithm 3.1 and this is the main motivation of our next result. In a similar way, one can consider the convergence criteria of other Algorithms.

Theorem 3.2. Let P_{K_r} be the Lipschitz continuous operator with constant $\delta = \frac{r}{r-r'}$. Let the operators $T, g : H \longrightarrow H$ be strongly monotone with constants $\alpha > 0, \sigma > 0$ and Lipschitz continuous with constants with $\beta > 0, \delta > 0$, respectively. If (15), (16) hold and $\alpha_n, \in [0, 1], \quad \forall n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the approximate solution u_n obtained from Algorithm 3.1 converges to a solution $u \in K_r$ satisfying the nonconvex variational inequality (1).

Proof. Let $u \in H : g(u) \in K_r$ be a solution of the general nonconvex variational inequality (2.1). Then, using Lemma 3.1, we have

$$u = (1 - \alpha_n)u + \alpha_n\{u - g(u) + P_{K_r}[g(u) - \rho T u]\}, \tag{23}$$

where $0 \leq \alpha_n \leq 1$ is a constant.

From (19), (20), (17), (22), (23) and using the Lipschitz continuity of the projection P_{K_r} with constant δ, we have

$$\begin{aligned}
\|u_{n+1} - u\| &= \|(1-\alpha_n)(u_n - u) + \alpha_n\{P_{K_r}[g(u_n) - \rho T u_n] - P_{K_r}[g(u) - \rho T u]\}\| \\
&\quad +\alpha_n\|u_n - u - (g(u_n) - g(u)\| \\
&\leq (1-\alpha_n)\|u_n - u\| + \alpha_n\|P_{K_r}[g(u_n) - \rho T u_n] - P_{K_r}[g(u) - \rho T u]\| + \alpha_n k\|u_n - u\| \\
&\leq (1-\alpha_n)\|u_n - u\| + \alpha_n\delta\|g(u_n) - g(u) + \rho(Tu_n - Tu)\| + \alpha_n k\|u_n - u\| \\
&\leq (1-\alpha_n)\|u_n - u\| + \alpha_n k\|u_n - u\| \\
&\quad \delta\|u_n - u - (g(u_n) - g(u))\| + \delta\|u_n - u - \rho(Tu_n - Tu)\| \\
&\leq (1-\alpha_n)\|u_n - u\| + \alpha_n\left\{k + \delta\left[k + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\right]\right\}\|u_n - u\| \\
&= [1 - \alpha_n(1-\theta)]\,\|u_n - u\| \\
&\leq \prod_{i=0}^{n}[1 - \alpha_i(1-\theta)]\,\|u_0 - u\|,
\end{aligned}$$

where, using (15), we have

$$\theta = k + \delta\sqrt{1 - 2\rho\alpha + \beta^2\rho^2} < 1.$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta > 0$, we have $\lim_{n\to\infty}\{\prod_{i=0}^{n}[1-(1-\theta)\alpha_i]\} = 0$. Consequently the sequence $\{u_n\}$ convergences strongly to u. This completes the proof. □

4. Wiener-Hopf Equations Technique

In this section, we first establish the equivalence between the general nonconvex variational inequality (1) and the Wiener-Hopf equations (12) using essentially the projection method. This equivalence is used to suggest and analyze some iterative methods for solving the general nonconvex variational inequality (1).

Using Lemma 3.1, we show that the general nonconvex variational inequality (1) is equivalent to the Wiener-Hopf equations (12).

Lemma 4.1. The general nonconvex variational inequality (1) has a solution $u \in H : g(u) \in K_r$ if and only if the Wiener-Hopf equations (12) have a solution $z \in H$, provided

$$g(u) = P_{K_r} z \tag{24}$$

$$z = g(u) - \rho T u, \tag{25}$$

where $\rho > 0$ is a constant.

Proof. Let $u \in H : g(u) \in K_r$ be a solution of (1). Then, from Lemma 3.1, we have

$$g(u) = P_{K_r}[g(u) - \rho T u]. \tag{26}$$

Taking $z = g(u) - \rho T u$ in (26), we have

$$g(u) = P_{K_r} z. \tag{27}$$

From (26) and (27), we have

$$z = g(u) - \rho T u = P_{K_r} z - \rho T g^{-1} P_{K_r} z,$$

which shows that $z \in H$ is a solution of the Wiener-Hopf equations (12). This completes the proof. □

From Lemma 4.1, we conclude that the general nonconvex variational inequality (1) and the Wiener-Hopf equations (12) are equivalent. This alternative formulation plays an important and crucial part in suggesting and analyzing various iterative methods for solving variational inequalities and related optimization problems. In this paper, by suitable and appropriate rearrangement, we suggest a number of new iterative methods for solving the general nonconvex variational inequality (1).

I. The Wiener-Hopf equations (12) can be written as

$$P_{K_r}z = -\rho T g^{-1} P_{K_r} z,$$

which implies that, using(4.2)

$$z = P_{K_r}z - \rho T g^{-1} P_{K_r} z = g(u) - \rho T u.$$

This fixed point formulation enables us to suggest the following iterative method for solving the general nonconvex variational inequality (1).

Algorithm 4.1. For a given $z_0 \in H$, compute z_{n+1} by the iterative schemes

$$g(u_n) = P_{K_r} z_n \tag{28}$$

$$z_{n+1} = (1-\alpha_n)z_n + \alpha_n\{g(u_n) - \rho T u_n,\} \quad n = 0, 1, 2, \ldots\ldots, \tag{29}$$

where $0 \leq \alpha_n \leq 1$, for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

II. The Wiener-Hopf equations (12) may be written as

$$\begin{aligned} z &= P_{K_r}z - \rho T g^{-1} P_{K_r} z + (1-\rho^{-1})Q_{K_r} z \\ &= g(u) - \rho T u + (1-\rho^{-1})Q_{K_r} z. \end{aligned}$$

Using this fixed point formulation, we suggest the following iterative method.

Algorithm 4.2. For a given $z_0 \in H$, compute z_{n+1} by the iterative schemes

$$\begin{aligned} g(u_n) &= P_{K_r} z_n \\ z_{n+1} &= (1-\alpha_n)z_n + \alpha_n\{g(u_n) - \rho T u_n + (1-\rho^{-1})Q_{K_r} z_n,\} \quad n = 0, 1, 2, \ldots, \end{aligned}$$

where $0 \leq \alpha_n \leq 1$, for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

III. If the operator T is linear and T^{-1} exists, then the Wiener-Hopf equation (12) can be written as

$$z = (I - \rho^{-1}T^{-1})Q_{K_r} z,$$

which allows us to suggest the iterative method.

Algorithm 4.3. For a given $z_0 \in H$, compute z_{n+1} by the iterative scheme

$$z_{n+1} = (1-\alpha_n)z_n + \alpha_n\{(I - \rho^{-1}T^{-1})Q_{K_r} z_n,\} \quad n = 0, 1, 2 \ldots,$$

where $0 \leq \alpha_n \leq 1$, for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

We would like to point out that one can obtain a number of iterative methods for solving the general nonconvex variational inequality (1) for suitable and appropriate choices of the operators T, h and the space H. This shows that iterative methods suggested in this paper are more general and unifying ones.

We now study the convergence analysis of Algorithm 4.1. In a similar way, one can analyze the convergence analysis of other iterative methods.

Theorem 4.1. Let the operators T, A satisfy all the assumptions of Theorem 3.1. If the condition (15) holds and $\alpha_n \in [0, 1]$, $\forall n \geq 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the approximate solution $\{z_n\}$ obtained from Algorithm 4.1 converges to a solution $z \in H$ satisfying the Wiener-Hopf equation (12) strongly.

Proof. Let $u \in H$ be a solution of (1). Then, using Lemma 4.1, we have

$$z = (1-\alpha_n)z + \alpha_n\{g(u) - \rho Tu\}, \tag{30}$$

where $0 \leq \alpha_n \leq 1$, and $\sum_{n=0}^{\infty} a_n = \infty$.

From(29), (30), (19) and (20), we have

$$\begin{aligned} \|z_{n+1} - z\| &\leq (1-\alpha_n)\|z_n - z\| + \alpha_n\|g(u_n) - g(u) - \rho(Tu_n - Tu)\| \\ &\leq (1-\alpha_n)\|z_n - z\| + \alpha_n\{\|u_n - u - (g(u_n) - g(u))\| + \|u_n - u - \rho(Tu_n - Tu)\|\} \\ &\leq (1-\alpha_n)\|z_n - z\| + \alpha_n\left\{k + \sqrt{1 - 2\rho\alpha + \beta^2\rho^2}\right\}\|u_n - u\|. \end{aligned} \tag{31}$$

Also from (28), (24) and the Lipschitz continuity of the projection operator P_{K_r} with constant δ, we have

$$\begin{aligned} \|u_n - u\| &= \|u_n - u - (g(u_n) - g(u))\| + \|P_{K_r}z_n - P_{K_r}z\| \\ &= k\|u_n - u\| + \delta\|z_n - z\| \end{aligned}$$

from which, we have

$$\|u_n - u\| \leq \frac{\delta}{1-k}\|z_n - z\|. \tag{32}$$

Combining (31), and (32), we have

$$\begin{aligned} \|z_{n+1} - z\| &\leq (1-\alpha_n)\|z_n - z\| + \alpha_n\delta\frac{k + \sqrt{1 - 2\rho\alpha + \beta^2\rho^2}}{1-k}\|z_n - z\| \\ &= (1-\alpha_n)\|z_n - z\| + \alpha_n\theta_1\|z_n - z\|, \end{aligned} \tag{33}$$

where

$$\theta_1 = \delta\frac{k + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}}{1-k}$$

From (15) and (16), we see that $\theta_1 < 1$ and consequently

$$\begin{aligned} ||z_{n+1} - z|| &\leq (1-\alpha_n)||z_n - z|| + \alpha_n\theta_1||z_n - z|| \\ &= [1 - (1-\theta_1)\alpha_n]||z_n - z|| \\ &\leq \prod_{i=0}^{n}[1 - (1-\theta_1)\alpha_i]||z_0 - z||. \end{aligned}$$

Since $\sum_{n=0}^{\infty} \alpha_n$ diverges and $1 - \theta_1 > 0$, we have $\lim_{n\to\infty} \prod_{i=0}^{n}[1 - (1-\theta_1)\alpha_i] = 0$. Consequently the sequence $\{z_n\}$ convergences strongly to z in H, the required result. □

Acknowledgement. The author would like to express his gratitude to Dr. M. Junaid Zaidi, Rector, CIIT, for providing excellent research facilities. The authhor would like to thank Prof. Dr. Enkelejd Hashorva for the kind invitation.

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