ALBANIAN JOURNAL
OF MATHEMATICS
Volume 3, Number 4, Pages 161–174
ISSN 1930-1235: (2009)

MEAN STAIRCASES OF THE RIEMANN ZEROS: A COMMENT ON THE LAMBERT W-FUNCTION AND AN ALGEBRAIC ASPECT

DAVIDE A MARCA, STEFANO BELTRAMINELLI, AND DANILO MERLINI

ABSTRACT. In this note we discuss explicitly the structure of some simple sets of values on the critical line (the "trivial critical values") which are associated with the mean staircases emerging from the Zeta function. They are given as solutions of an equation involving the Lambert W-function. The argument of the latter function may then be set equal to a special $N \times N$ (classical) matrix (for every N) related to the Hamiltonian of the Mehta-Dyson model. In this way we specify a function of an hermitean operator whose eigenvalues are exactly these values. In the general case, the sets of such trivial critical values (zeros) are special solutions of a parametric equation involving a linear combination of $\mathrm{Re}\,\zeta(s)$ and $\mathrm{Im}\,\zeta(s)$ on the critical line.

This research note is dedicated to the international Swiss-Italian mathematician and physicist Professor Dr. Sergio Albeverio on the occasion of his seventieth birthday; a friend and long-standing scientific director of Cerfim (Research Center for Mathematics and Physics of Locarno, situated opposite the "Rivellino"[1]).

1. Introduction: a search for an hermitean operator associated with the Riemann Zeta Function

There is much interest in understanding the complexity related to the Riemann Hypothesis (RH) and concerned with the location and the structure of the nontrivial zeros of the Riemann Zeta function $\zeta(s)$, where $s = \rho + it$ is the complex variable.

Following a suggestion of Hilbert and Polya, in recent years many efforts have been devoted to a possible construction of an hermitean operator having as eigenvalues the imaginary parts t_n of the nth nontrivial zeros of ζ (ζ being meromorphic, the zeros are countable). These are given by the solutions of the equation $\zeta(\rho_n + it_n) = 0$, $n = 1, 2, \ldots$. If $\rho_n = \frac{1}{2}$ for all n, then all the zeros lie on the critical line (the RH is true); the program is then to find an hermitean "operator" T such that $T \cdot \varphi_n = t_n \cdot \varphi_n$ in some appropriate (Hibert) space (φ_n would be the nth eigenvector of T).

There are today many strategies in the direction of constructing such an operator and in the sequel we will shortly comment on some (among many others) very stimulating works on the subject.

1991 *Mathematics Subject Classification.* 11M26.
Key words and phrases. Riemann Zeta function, Lambert W-function, Riemann zeros, harmonic oscillator, Riemann Hypothesis.

[1]The Bastion "Il Rivellino", is 95% attribuable to Leonardo da Vinci (1507).

In [1], Pitkänen's heuristic work goes in the direction of constructing orthogonality relations between eigenfunctions of a non hermitean operator related to the superconformal symmetries; a different operator than the one just mentioned has also been proposed in [2] by Castro, Granik and Mahecha in terms of the Jacobi Theta series and an orthogonal relation among its eigenfunctions has also been found. In the rigorous work by Elizalde et al. [3] some problems with those approches have been pointed out.

In a work of some years ago Julia [4] proposed a fermionic version of the Zeta function which should be related to the partition function of a system of p-adic oscillators in thermal equilibrium.

In two others pioneering works of these years, Berry and Keating [5, 6] proposed an interesting heuristic operator to study the energy levels t_n (the imaginary parts of the nontrivial zeros of the Zeta function). The proposed Hamiltonian (in one dimension) has a very simple form given, on a dense domain, by: $H = p \cdot x + \frac{1}{2}$, where

$$p = \left(\frac{1}{i}\right) \frac{\partial}{\partial x}. \tag{1}$$

As explained by the authors, the difficulty is then to define appropriate spaces and boundary conditions to properly determine p and H as hermitean operators. In such an approach the heuristic appearance of "instantons" is also discussed.

In another important work Bump et al. [7] introduced a local RH and proved in particular that the Mellin transform of the Hermite polynomials (associated with the usual quantum mechanical harmonic oscillator) contains as a factor a polynomial $p_n(s)$, corresponding to the nth energy eigenstate of the oscillator, whose zeros are exactly located on the critical line $\sigma = \frac{1}{2}$. The relation of the polynomials $p_n(s)$ with some truncated approximation of the entire funcion $\xi(s)$ (the Xi-function), related to the Riemann Zeta function seems to be still lacking.

Other important mathematical results concerning the nontrivial Riemann zeros, have been obtained by many leading specialists (see among others the work by Connes [8], the work by Albeverio and Cebulla [9] and the recent work on the xp Hamiltonian by Sierra [10]).

Let us also mention that for the nontrivial zeros of the Zeta function an interesting equation has been proposed originally by Berry and Keating in [5]. In fact, remembering the definition of $\xi(s)$, given by:

$$\xi(s) := \frac{1}{2} s\,(s-1)\,\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta\,(s) = \xi(1-s) \tag{2}$$

an equation possibly giving an approximation to the zeros of ξ is proposed in [5] and given by:

$$\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} + \frac{\pi^{\frac{1-s}{2}}}{\Gamma\left(\frac{1-s}{2}\right)} = 0. \tag{3}$$

As stated by the authors, (3) could be considered as a "quantization condition". Unfortunately, as mentioned in [5], (3) possesses complex zeros and so can not be used to provide an hermitean operator which would generate the nontrivial zeros of ζ.

The content of our note is concerned with the "mean staircase" of the Riemann zeros. We first construct sets of solutions of a parametric equation involving a linear combination of $\operatorname{Re}\zeta\,(s)$ and $\operatorname{Im}\zeta\,(s)$ and point out an explicit characterization of them using the Lambert W-function. Then we introduce a specific argument (an $n \times n$ hermitean matrix H, describing a discrete harmonic oscillator with creation and annihilation "operators" a and a^* such that $[a, a^*] = -2$) into the Lambert W-function. So we obtain, for the above values, the same goal that the "Polya-Hilbert program" has for the nontrivial zeros of the Zeta function.

2. The mean staircaise of the Riemann zeros and the trivial critical values on the critical line associated with it

Let $\xi(s)$ be the Xi-function given by (2) and $s = \frac{1}{2} + it$ a complex variable on the critical line. If $N(t)$ denotes the number of zeros of ξ in the critical strip of height smaller or equal to t, and if $S(t) := \frac{1}{\pi} \arg\left(\zeta\left(\frac{1}{2} + it\right)\right)$, then from [11] we have:

$$N(t) = \langle N(t) \rangle + S(t) + O\left(\frac{1}{t}\right) \tag{4}$$

where

$$\langle N(t) \rangle = \frac{t}{2\pi}\left(\ln\left(\frac{t}{2\pi}\right) - 1\right) + \frac{7}{8}. \tag{5}$$

$\langle N(t) \rangle$, the "bulk contribution" to N, is called the "mean staircase of the zeros". The fluctuations of the number of zeros around the mean staircase, are given by the function $S(t)$. It is known [11] that $S(t) = O(\ln t)$ without assuming RH while, assuming RH is true, it is known that $S(t) = O\left(\frac{\ln t}{\ln(\ln t)}\right)$.

We introduce a model related to (4) by replacing (4) by (6), which corresponds to setting $S(t) + O\left(\frac{1}{t}\right) = \lambda$ for a fixed real parameter λ. So (4) becomes:

$$N(t) = \frac{t}{2\pi}\left(\ln\left(\frac{t}{2\pi}\right) - 1\right) + \frac{7}{8} + \lambda. \tag{6}$$

For each fixed λ and for each $n \in \mathbb{N}$, we can define a set of real values $t_n(\lambda)$ which are solutions of:

$$N(t) = \frac{t}{2\pi}\left(\ln\left(\frac{t}{2\pi}\right) - 1\right) + \frac{7}{8} + \lambda = n. \tag{7}$$

We call the $t_n(\lambda)$ "trivial values on the critical line" or shortly "trivial critical values". Trivial because they are fully given by (7) and critical because they lie on the critical line. On the other hand, we will indicate the imaginary part of the nth nontrivial zero of ζ simply by t_n and call it a true zero (of ζ).

Notice that since $\arg\left(\zeta\left(\frac{1}{2} + it\right)\right) = \arctan\left(\frac{\operatorname{Im}\zeta\left(\frac{1}{2}+it\right)}{\operatorname{Re}\zeta\left(\frac{1}{2}+it\right)}\right)$, we obtain

$$\frac{\operatorname{Im}\zeta\left(\frac{1}{2} + it\right)}{\operatorname{Re}\zeta\left(\frac{1}{2} + it\right)} = \tan(\lambda\pi). \tag{8}$$

So for every λ and for $\operatorname{Re}\zeta\left(\frac{1}{2} + it\right) \neq 0$ the sequences $\{ t_n(\lambda) \}$ are defined by:

$$\{ t_n(\lambda) \} = \left\{ t \in \mathbb{R} \mid \operatorname{Im}\zeta\left(\frac{1}{2} + it\right) - \tan(\lambda\pi)\operatorname{Re}\zeta\left(\frac{1}{2} + it\right) = 0 \right\}. \tag{9}$$

Note that we may restrict the values of λ to the interval $\left]-\frac{1}{2}, \frac{1}{2}\right[$ and for $|\lambda| = \frac{1}{2}$ (8) is aquivalent to $\operatorname{Re}\zeta\left(\frac{1}{2} + it\right) = 0$ and $\operatorname{Im}\zeta\left(\frac{1}{2} + it\right) \neq 0$.

Two particular sets of interest are defined by the choices $\lambda = 0$ and $\lambda = \frac{1}{2}$. In the first case we have the set:

$$\{ t_n^* \} := \{ t_n(0) \} = \left\{ t \in \mathbb{R} \mid \operatorname{Re}\zeta\left(\frac{1}{2}+it\right) \neq 0 \wedge \operatorname{Im}\zeta\left(\frac{1}{2}+it\right) = 0 \right\} \tag{10}$$

or equivalently each t_n^* is the solution of

$$\frac{t}{2\pi}\left(\ln\left(\frac{t}{2\pi}\right) - 1\right) + \frac{7}{8} = n. \tag{11}$$

This set has been known for a long time and constitutes the "Gram points" [11].

The second set is given by the solutions t_n^{**} of

$$\frac{t}{2\pi}\left(\ln\left(\frac{t}{2\pi}\right) - 1\right) + \frac{7}{8} = n - \frac{1}{2}, \text{ or} \tag{12}$$

$$\{ t_n^{**} \} := \{ t_n(1/2) \} = \left\{ t \in \mathbb{R} \mid \operatorname{Re}\zeta\left(\frac{1}{2}+it\right) = 0 \wedge \operatorname{Im}\zeta\left(\frac{1}{2}+it\right) \neq 0 \right\} \tag{13}$$

We note that the values t_n^* and t_n^{**} are given by the abscissa of the intersection points between the staircase (5) and the two functions $\pi^{-1}\arg\left(\xi\left(\frac{1}{2}+it\right)\right)$ and $\pi^{-1}\arg\left(\xi\left(\frac{1}{2}+it\right)\right) - \frac{1}{2}$. The plot of Fig. 1 illustrates the situation for some low lying true zeros. The values for t_n^* lie mostly in between the exact value of the Riemann zeros t_{n-1} and t_n, but it is known that the "Gram law" [11] fails for the first time at $t = 282.4\ldots$ ("first instanton" according to [5]).

The solution of the above equations which give t_n^* and t_n^{**} using a very special function (the Lambert W-function, see [12]) is given below in Section 3.

3. An exact solution for the sequence t_n^*, t_n^{**} and $t_n(\lambda)$

The equation corresponding to (11), may be written in the form

$$\left(\frac{t}{2\pi \mathrm{e}}\right)^{\frac{t}{2\pi\mathrm{e}}} = \exp\left(\frac{n-\frac{7}{8}}{\mathrm{e}}\right) \tag{14}$$

and the equation corresponding to (12) in the form

$$\left(\frac{t}{2\pi \mathrm{e}}\right)^{\frac{t}{2\pi\mathrm{e}}} = \exp\left(\frac{n-\frac{1}{2}-\frac{7}{8}}{\mathrm{e}}\right) \tag{15}$$

so that introducing the new variables $x = \frac{n-\frac{7}{8}}{\mathrm{e}}$ respectively $x = \frac{n-\frac{1}{2}-\frac{7}{8}}{\mathrm{e}}$ we obtain the equation (from (14) and (15), $x > 0$)

$$W(x)\exp\left(W(x)\right) = x. \tag{16}$$

The function $W(x)$ defined by the functional equation (16) is called the Lambert W-function and has been studied extensively in these recent years. In fact such an equation appears in many fields of science. In particular the use of such a function has appeared in the study of the wave equation in the double-well Dirac delta function model or in the solution of a jet fuel problem. See [12] for an important work on the subject. Moreover the Lambert W-function appears also in combinatorics as the generating function of trees and as explained in [12] it has many applications, although its presence often goes unrecognized.

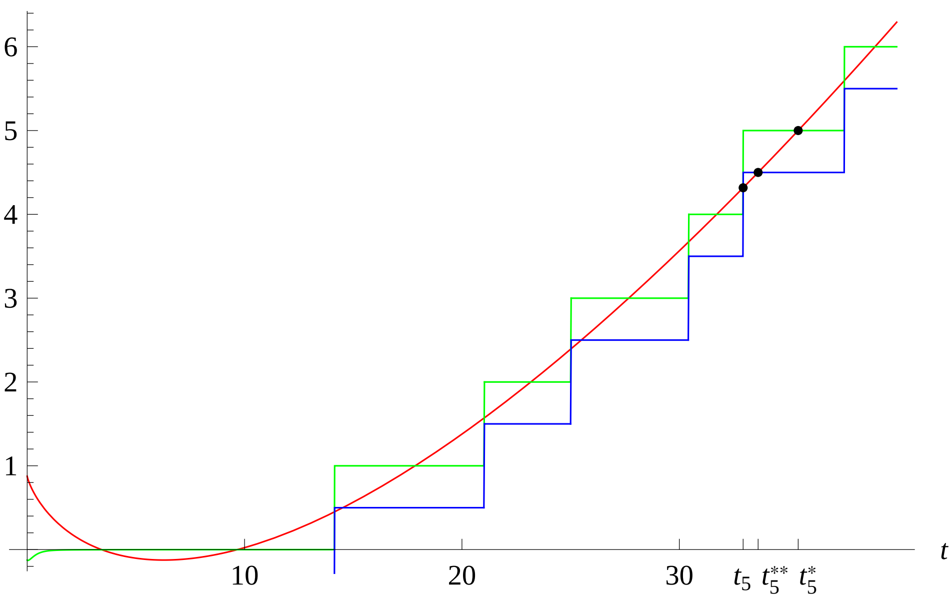

FIGURE 1. Plot of $\langle N(t)\rangle = \frac{t}{2\pi}\left(\ln\left(\frac{t}{2\pi}\right)-1\right)+\frac{7}{8}$ (red curve), $\langle N(t)\rangle + S(t)$ (green stair) and $\langle N(t)\rangle + S(t) - \frac{1}{2}$ (blue stair), where $S(t) := \frac{1}{\pi}\arg\left(\zeta\left(\frac{1}{2}+it\right)\right)$

The Lambert W-function has many complex branches; of interest here is the principal branch of W which is analytic at $x = 0$. The solutions of (11) and (12) are given by:

$$t_n^* = 2\pi\mathrm{e}\cdot\exp\left(W\left(\frac{n-\frac{7}{8}}{\mathrm{e}}\right)\right) \tag{17}$$

$$t_n^{**} = 2\pi\mathrm{e}\cdot\exp\left(W\left(\frac{n-\frac{1}{2}-\frac{7}{8}}{\mathrm{e}}\right)\right) \tag{18}$$

with W here understood as the principal branch of the Lambert W-function. We have thus constructed, with the help of the Lambert W-function, the sequences $\{t_n^*\}$ and $\{t_n^{**}\}$. In the general case where $S(t) + O\left(\frac{1}{t}\right) = \lambda$, we obtain the general set of $t_n(\lambda)$ values given by:

$$t_n(\lambda) = 2\pi\mathrm{e}\cdot\exp\left(W\left(\frac{n-\frac{7}{8}-\lambda}{\mathrm{e}}\right)\right). \tag{19}$$

Getting back now to the case $\lambda = 0$, it should be noted that, having replaced in (4) $S(t) + O\left(\frac{1}{t}\right)$ by 0 we cannot expect in (11) n, which would correspond to the exact value t_n of a true zero, to be an integer. For the first few low lying true zeros t_n of ζ, it may be observed numerically that the corresponding values of the index n, let say n^*, are randomly distributed mostly between two consecutive integers, but their mean values are nearby the integers plus $\frac{1}{2}$. A calculation with some known true zeros of ζ gives a mean value of 0.49. So, in average it seems that the behavior of the true zeros t_n "follows" more the pattern of the t_n^{**}. In a similar way the values of the first set, i.e. t_n^*, lie mostly in between two true zeros of ζ, but of course it is known that there are very complicated phenomena associated with the chaotic behavior of the non trivial zeros of the Riemann ζ function.

As an example, the first of the instantons [5] corresponding to $n = 127$, is located at the value of $t_{127} = 282.4651\ldots$. In Table 1 we give the values of t_n^* and of t_n (a true zero) around $t = 280$.

$$
\begin{aligned}
t_{126} &= 279.22925\\
t_{126}^* &= 280.80246\\
t_{127}^* &= 282.4547596\\
t_{127} &= 282.4651147\\
t_{128} &= 283.211185\\
t_{128}^* &= 284.1045158\\
t_{129} &= 284.8359639
\end{aligned}
$$

TABLE 1

From such numerical computations we see that two consecutive zeros of $\operatorname{Im}\zeta$ alone are followed by two consecutive true zeros, in particular t_{127}^* anticipates t_{127}. The difference between the two subsequent t values (i.e. $t_{127} - t_{127}^*$) is very small and given by $\Delta t = 0.0103$. The phase change is given by $i\pi$ as illustrated on the plot of $\operatorname{Im}\ln\left(\zeta\left(\frac{1}{2}+it\right)\right)$ (step curve) and that of $\operatorname{Im}\zeta\left(\frac{1}{2}+it\right)$.

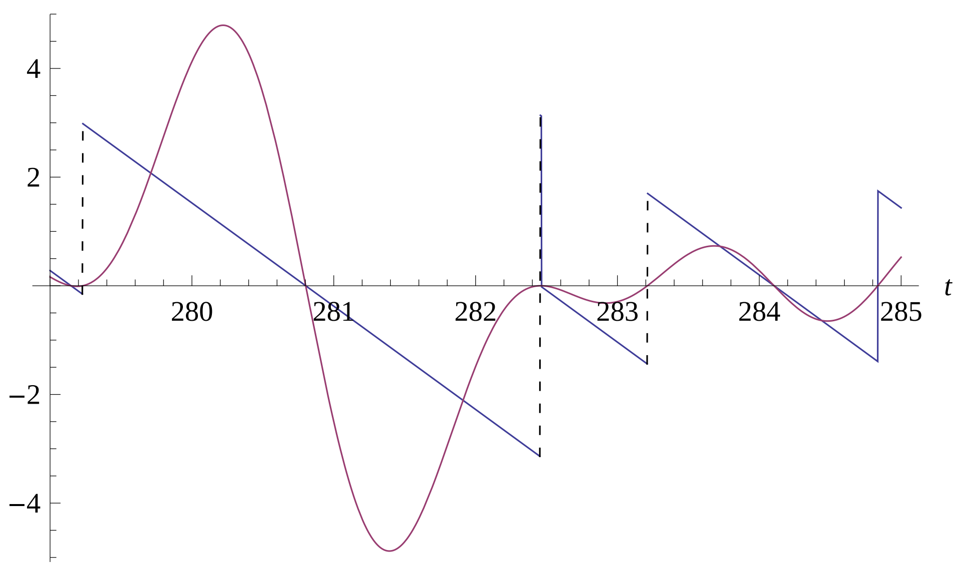

FIGURE 2. Plot of $\operatorname{Im}\zeta\left(\frac{1}{2}+it\right)$ (continuous curve) and $\operatorname{Im}\ln\left(\zeta\left(\frac{1}{2}+it\right)\right) = \pi S(t)$ (step curve).

For the first 500 energy levels, that is for values of t from 0 to $t = 811.184\ldots$ (level number $n = 500$), it may be seen that there are 13 instantons (in the language of [5]), all with a Maslov phase change of $+i\pi$ or of $-i\pi$. The width Δt is usually small but it is larger for the instanton located at $t = 650.66$ (n corresponding to 379), where $\Delta t = 0.31\ldots$. Returning now to the two sets $\{t_n^*\}$ and $\{t_n^{**}\}$, we note the elementary relation which follows from (11) and (12) given by:

$$\frac{t_n^* + t_{n+1}^*}{2} = t_{n+1}^{**} \tag{20}$$

and

$$\frac{t_{n-1}^{**} + t_{n+1}^{**}}{2} = t_n^*. \tag{21}$$

(20) and (21) say that the zeros of the real part of $\zeta\left(\frac{1}{2} + it\right)$ alone are obtained by those of the imaginary part alone by simple average and viceversa. The two sequences are regularly spaced and the mean distance between two trivial critical values at the height t, as the mean staircase indicates (5), is given approximatively by:

$$\frac{t}{\langle N(t)\rangle} = \frac{2\pi}{\ln\frac{t}{2\pi}} = \frac{2\pi}{\ln n} \tag{22}$$

for t and n large.

Before proposing an hermitean operator for the sequences of the trivial critical values it is important to investigate a possible "quantization condition" for the nontrivial zeros. For this we start with the functional equation of the ζ function.

From the exact relation for the ξ-function given by:

$$\begin{aligned}\xi(s) &= \frac{1}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)s(s-1)\\ &= \xi(1-s) = \frac{1}{2}\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)(1-s)(1-s-1)\end{aligned} \tag{23}$$

$s \in \mathbb{C}$, we have that

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{24}$$

In equation (24) we limit ourselves to consider the values $s = \frac{1}{2} + \varepsilon + it$, t, $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, and thus $1 - s = \frac{1}{2} - \varepsilon - it$; moreover we are interested in high values of t so that we may use Stirling's formula for the Gamma function given (in the sense of asymptotic equivalence) by:

$$\Gamma(x) \cong (2\pi)^{\frac{1}{2}}\, x^{x-\frac{1}{2}}\mathrm{e}^{-x} \tag{25}$$

as $x \to \infty$. From (24) and (25) we then obtain (asymptotically for $t \to \infty$)

$$\begin{aligned}&\exp\left(i\pi\left(\frac{t}{2\pi}\left(\ln\left(\frac{t}{2\pi}\right) - 1\right) - \frac{1}{8}\right) + i\arg\left(\zeta\left(\frac{1}{2+\varepsilon} + it\right)\right)\right) =\\ &\exp\left(-i\pi\left(\frac{t}{2\pi}\left(\ln\left(\frac{t}{2\pi}\right) - 1\right) - \frac{1}{8}\right) + i\arg\left(\zeta\left(\frac{1}{2-\varepsilon} - it\right)\right)\right).\end{aligned} \tag{26}$$

Since

$$\begin{aligned}\exp\left(i\arg\left(\zeta\left(\frac{1}{2} - \varepsilon - it\right)\right)\right) &= \exp\left(i\arg\left(\zeta\left(\frac{1}{2} + \varepsilon + it\right) + i\pi\right)\right)\\ &= -\exp\left(i\arg\left(\zeta\left(\frac{1}{2} + \varepsilon + it\right)\right)\right)\end{aligned}$$

we then have, taking the limit $\varepsilon \to 0$ i.e. at $\rho = \frac{1}{2}$, that:

$$\cos\Psi = 0 \ \text{ where } \ \Psi = \frac{t}{2}\left(\ln\left(\frac{t}{2\pi}\right) - 1\right) - \frac{\pi}{8} + \arg\left(\zeta\left(\frac{1}{2} + it\right)\right). \tag{27}$$

Thus $\Psi = \pi\left(n + \frac{1}{2}\right)$ with $n \in \mathbb{N}$. We then obtain:

$$\frac{t}{2\pi}\left(\ln\left(\frac{t}{2\pi}\right) - 1\right) - \frac{1}{8} + \frac{1}{\pi}\arg\left(\zeta\left(\frac{1}{2} + it\right)\right) = n - \frac{1}{2}$$

hence

$$\frac{t}{2\pi}\left(\ln\left(\frac{t}{2\pi}\right) - 1\right) + \frac{7}{8} + \frac{1}{\pi}\arg\left(\zeta\left(\frac{1}{2} + it\right)\right) = n + \frac{1}{2}. \tag{28}$$

(28) may be seen as a "quantum condition" for the nontrivial zeros of ζ and it is a consequence of the functional equation (24). In fact, if in (27) we neglect the last term $\arg(\zeta)$, then (27) has as a solution the second set of trivial critical values $\{t_n^{**}\}$. It is true, as remarked by Berry and Keating, that their equation stated above as (3) has complex zeros which are not the nontrivial zeros of ζ, but it should be remarked that if in (3) we set $\operatorname{Re} s = \frac{1}{2}$ then (3) reduces to (27) without the fluctuation term $\arg(\zeta)$; so the solution of (3) for $\operatorname{Re} s = \frac{1}{2}$ is the same as the second set of trivial critical values $\{t_n^{**}\}$.

Below the plots of $\cos\Psi$, with and without the term $\arg\left(\zeta\left(\frac{1}{2} + it\right)\right)$, are given. As an illustration, we may observe on that plot the first instanton discussed above and the second one located around $t = 295$. In fact the maximum of the function which gives t_n^* ((27) without the term $\arg(\zeta)$) is outside the plot of the step function given by (27). This is visible on the plot near $t = 282$ and near $t = 296$ (the second instanton). This concludes our remark on (3) and (27). In the next Section, we shall costruct an hermitean operator whose eigenvalues are the trivial critical values of the Zeta function defined above in (9).

Now in (19) a trivial critical value $t_n(\lambda)$ is given through its index n by means of the Lambert W-function so that such values are related in a non linear way to the integers n, i.e. in principle to the spectrum of an harmonic oscillator. So, for the trivial critical values, no boundary condition is needed here, since they are obtained by means of (19) in the large t limit. At this moment we are free to introduce an hermitean matrix which may generate the trivial critical values.

4. An Hermitean operator (matrix) associated with the mean staircase (trivial critical values) of the Riemann Zeta function

As remarked above, in (19) the only "quantal number" is the index n of the trivial critical values and the construction may be given using, for any n, an hermitean $n \times n$ matrix H related to the classical one dimensional many body system whose fluctuation spectrum around the equilibrium positions is that of the harmonic oscillator. In fact, the one dimensional Mehta-Dyson model of random matrices (which may be seen as a classical Coulomb system with n particles) has, at low temperature, an energy fluctuation spectrum given by the integers and it is possible to introduce correspondingly annihilation and creation operators, as studied in [13] (a short discussion is presented in the Appendix). The matrix elements of the associated hermitean matrix are then functions of the zeros of the Hermite polynomials; in this case we do not have a Hilbert space and no Schrödinger equation will be associated with the Lambert W-function.

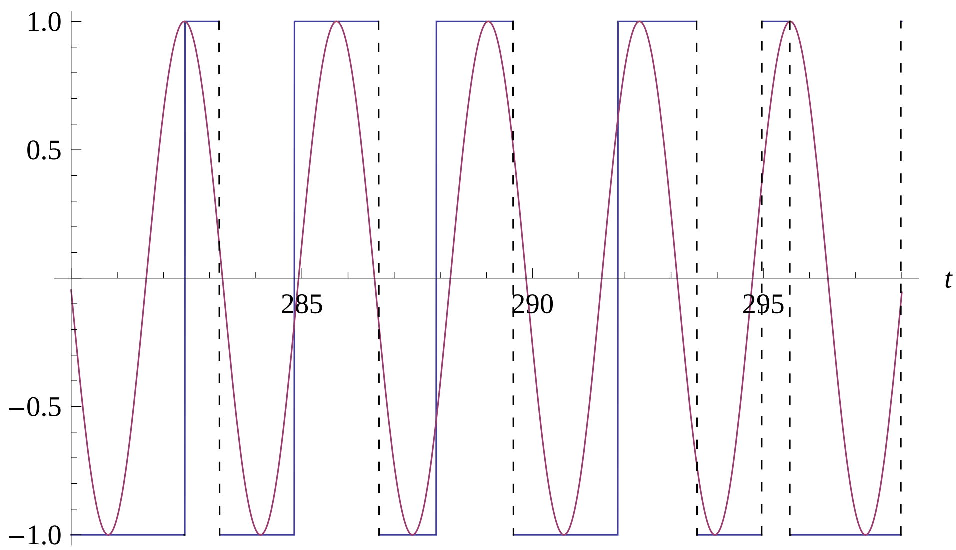

FIGURE 3. Plot of the function $\cos\Psi$ of (27) with the term $\arg\left(\zeta\left(\frac{1}{2}+it\right)\right)$ (step function) and without that term.

Another direction, i.e. that of introducing a Schrödinger equation to describe the trivial critical values may in principle be obtained as an application of the results given by Nash [14]; this is so because for large n, as it is known, (19) gives the behavior ([11], page 214) related to the asymptotic behavior of the Lambert W-function:

$$t_n(\lambda) = \frac{2\pi n}{\ln n}\left(1 - \frac{7/8+\lambda}{n}\right) \tag{29}$$

and thus the spectrum appears in fact as one for which the associated Schrödinger equation contains a Gaussian type of potential [14].

Here we will consider the matrix formulation: the point may seem to be somewhat artificial but the hermitean matrix we will use (specified in the Appendix) is related to the Mehta-Dyson model, the "starting point" of the random matrix theory and we are free to choose such a matrix (of course other choices are possible). To do this, we begin to write (19) in a slighly different form using the Stirling formula for the Gamma function of real argument given by (in the sense of the asymptotic equivalence):

$$\Gamma(x) \cong (2\pi)^{\frac{1}{2}}\, x^{x-\frac{1}{2}}\mathrm{e}^{-x} \quad \text{as} \quad x \to \infty.$$

We then have that, as $t \to \infty$,

$$\begin{aligned} \ln\left(\Gamma\left(\frac{t}{2\pi}+\frac{1}{2}\right)\right) &\cong \frac{t}{2\pi}\ln\left(\frac{t}{2\pi}-1\right)+\frac{7}{8}+\frac{1}{2}\ln(2\pi)-\frac{7}{8}-\lambda \\ &= \langle N(t)\rangle + \frac{1}{2}\ln(2\pi) - \frac{7}{8} - \lambda = \langle N(t)\rangle + \theta(\lambda) \end{aligned} \tag{30}$$

where $\theta(\lambda) = \frac{1}{2}\ln(2\pi) - \frac{7}{8} - \lambda$.

Thus introducing the operator $T=T(H)$ whose eigenvalues should be the trivial critical values (as defined in the general case by (19)) as well as H, the hermitean matrix given in the Appendix and related to the Mehta-Dyson model, we may write following (30) the heuristic matrix equation:

$$\Gamma\left(\frac{T}{2\pi}+\frac{I}{2}\right)=\mathrm{e}^{H+\theta(\lambda)} \tag{31}$$

where I is the unit matrix. (31) is the equation for T, giving the trivial critical values. The inversion of this formula (if it is possible to take it) yields heuristically:

$$T(\lambda)=T(H)=2\pi\left(\Gamma^{-1}\left(\mathrm{e}^{H+\theta}\right)-\frac{I}{2}\right). \tag{32}$$

To conclude, if $H\varphi_n=\left(n+\frac{1}{2}\right)\varphi_n$, where φ_n is the nth eigenfunction of H, then (33)

$$T\varphi_n=2\pi\left(\Gamma^{-1}\left(\mathrm{e}^{H+\theta(\lambda)}\right)-\frac{I}{2}\right)\varphi_n=2\pi\left(\Gamma^{-1}\left(\mathrm{e}^{n+\theta(\lambda)}\right)-\frac{1}{2}\right)\varphi_n=t_n(\lambda)\varphi_n.$$

Of course (31) for the operator T is more appealing than (19) (where n is replaced by H and $t_n(\lambda)$ is replaced by matrix T) due to the combinatorial nature of the Gamma function, but the eigenvalues of the operators T are the same in the "termodynamic limit", dim $H=n\to\infty$.

This completes the second part of our note i.e. the algebraic aspect in the construction of the trivial critical values using the two creation and annihilation operators a, a^* with $[a,a^*]=aa^*-a^*a=-2I$ connected with the Mehta-Dyson model.

Remark: If one considers the map $s=\sigma+it\to 1-\frac{1}{s}=z$ then the critical line $s=\frac{1}{2}+it$ ($t\in\mathbb{R}$) is mapped onto the unit circumference $|z|=1$; the set $\{\,\frac{1}{2}+it_n(\lambda)\,\}$ has as accumulation point $z=1$ (as $n\to\infty$), which is the same accumulation point for the trivial zeros of the ζ function given by $z_n=1-\frac{1}{-2n}=1+\frac{1}{2n}$, as $n\to\infty$ (see Fig. 4). Neglecting the trivial zeros $\{z_n\}$, Fig. 4 illustrate by means of the set of trivial critical values $\{t_n(\lambda)\}$ an analogon of the Lee-Yang Theorem [15] for the zeros of the partition function for some general spin lattice system studied in statistical mechanics. If RH is true, then all nontrivial zeros of $\zeta(z)$ shall be located at the same circumference $|z|=1$, with $z=1$ as accumulation point.

Remark: Of course, (19) in matrix form or equivalently (31) or (32) give also the true zeros t_n, if λ is the corresponding value of the true zero. In fact a true zero (for example the first one given by $t_1=14.13472514\ldots$ where $\lambda_1=-0.449\ldots$, i.e. where $\tan(\pi\lambda_1)=-6.28\ldots$) may be seen as the groundstate of the sequence of trivial critical values given by the Lambert W-function with $\lambda=\lambda_1$. The same for all the other true zeros. Thus as λ is varying in $\left]-\frac{1}{2},\frac{1}{2}\right[$ we obtain a continuous spectrum including the imaginary part of the nontrivial zeros which are on the critical line. If λ is such that $\mathrm{Re}\,\zeta$ and $\mathrm{Im}\,\zeta$ are not both zero, then we obtain a continuous spectrum where the true zeros are missing. It is also interesting to analyse the two sequences $t_n(\lambda_1)$ and $t_n(-\lambda_1)$ and keeping those values of the two sequences which are closer to a true zero as given on Table 2 for n up to 19. From this Table we may compute the mean value of the absolute percentual error and we find the value 0.7%. Notice that such a value is smaller than the one computed with the sequence where $\lambda=\frac{1}{2}$ (i.e. with the t_n^{**}).

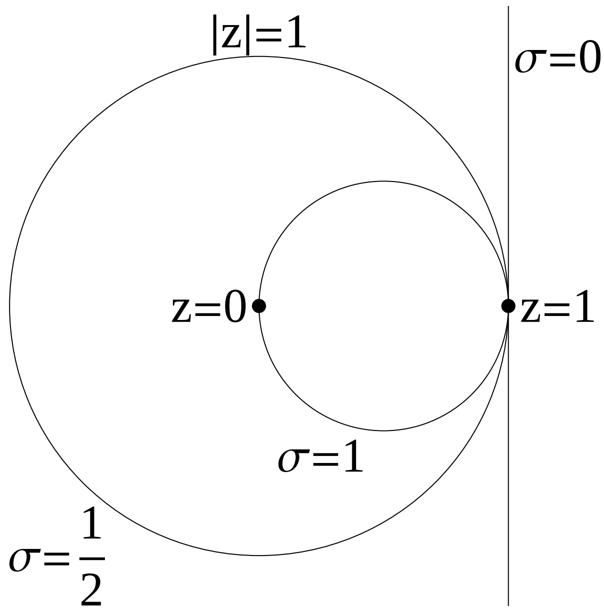

FIGURE 4. z-plane

t_n	$t_n(\lambda_1)$ or $t_n(-\lambda_1)$	t_n	$t_n(\lambda_1)$ or $t_n(-\lambda_1)$
14.134	14.138	52.970	53.066
21.022	20.919	56.446	56.263
25.010	25.266	59.347	59.097
30.424	29.942	60.831	61.595
32.424	33.435	65.112	64.593
37.586	37.434	67.079	66.999
40.918	40.869	69.546	69.632
43.327	43.831	72.067	72.225
48.005	47.321	75.704	75.043
49.773	50.081		

TABLE 2

5. CONCLUSION

In this note we have first obtained $t_n(\lambda)$, the explicit solution of (9) for the case where $S(t)+O(t) = \lambda$ i.e. the zeros of the function $\operatorname{Im} \zeta\left(\frac{1}{2}+it\right) - \tan(\pi\lambda) \operatorname{Re} \zeta\left(\frac{1}{2}+it\right)$ by means of the Lambert W-function.

The particular case $\lambda = 0$ gives the well known sequence of Gram points and for each value of λ, all the sequences of such trivial critical values are regularly spaced contrary to that of the nontrivial zeros which have so far been found numerically to lie on the critical line.

In the second part of our work concerning the algebraic aspect of $t_n(\lambda)$ we have formally given the construction of an operator equation for the operator $T(\lambda)$, related to an Hamiltonian H emerging from the one dimensional Mehta-Dyson model of N point charges. $T(\lambda)$ appears to be related to H in a strong nonlinear way, formally given by (19) or equivalently by (31), in operator form and has as eigenvalues the sets of trivial critical values given by (9). So in the construction, the sets of trivial critical values are related to the eigenvalues $n = 1, ..., N$ of a discrete harmonic oscillator furnished by the matrices a, a^*, i.e. the two discrete annihilation and creation operators of H (the only quantal number we have used is the index n of the corresponding trivial critical value given by (9)).

To the best of our knowledge, we do not know of any explicit existing treatment along these lines for the sets of the trivial critical values given by (9).

In a subsequent note we intend to treat (at least numerically) another sequence of values possibly "‘more related"’ to the nontrivial zeros of the Zeta function but not obtainable by a Lambert W-function or a Gamma function.

Updates

(1) Sierra and Townsend [16] introduced and studied an interesting physical model (a charged particle in the plane in the presence of an electrical and a magnetic potential). In particular, the lowest Landau level is connected with the smoothed counting function that gives the average number of zeros, i.e. the staircase which here we have studied, by means of a classical one-dimensional model of N interacting charged particles.

(2) Very recently Schumayer et al. [17] constructed (in particular) a quantum mechanical potential for the zeros of $\zeta(s)$, using the first 10^5 energy eigenvalues (nontrivial zeros). It is expected that the same form of a quantum mechanical potential would appear using only the values $t_n(\lambda)$ we have found in this note. For the construction of an Hamiltonian whose spectrum coincides with the primes, see also the interesting work of Sekatskii [18].

6. Appendix

We shall discuss the hermitean matrix H associated with the Mehta-Dyson model and the discrete annihilation and creation operators associated with H whose spectrum is given by the set of integers $(1, 2, \ldots, N)$, for any finite N.

In [13, 19] the one dimentional Mehta-Dyson model defined by the potential energy $E = \sum_{i=0}^{N} \left(\frac{1}{2} y_i^2 - \sum_{i<j\leqslant N} \log\left(|y_i - y_j|\right) \right)$ was studied, where y_i is the position of the ith particle on the line. The fluctuation around the equilibrium positions (these are given by the zeros of the Hermite polynomials of degree N, where N is the number of particles on the line, for every finite N), i.e. the harmonic fluctuation spectrum, is given by the eigenvalues of the hermitean $N \times N$ real matrix whose elements are given by:

$$\begin{cases} H_{\mathrm{ij}} = \frac{-1}{|x_i - x_j|^2} & i \neq j \\ \\ H_{\mathrm{ij}} = 1 + \sum\limits_{k \neq i} \frac{1}{|x_i - x_k|^2} & i = j \end{cases}$$

$i, j = 1, \ldots, N$, where the x_i are the "equilibrium positions" i.e. the zeros of the Hermite polynomials of degree N.

The spectrum of H is given exactly by the integers (1, 2, ..., N) for every finite N and the eigenfunctions are given in terms of the Mehta-Dyson polynomials of order 1 up to N. The Hamiltonian describing the harmonic fluctuations takes then the form [13]:

$$H = N \cdot I - \frac{1}{2} a a^*$$

where I is the unit matrix of order N and a, resp a^*, are the discrete annihilation and creation operators (matrices of order $N \times N$) of H, which satisfy the commutation relation $[a, a^*] = -2I$. Moreover $[H, a^*] = a^*$ and $[H, a] = -a$.

If X_k is the kth eigenvector of H with eigenvalue the integer k, one has:

$$a^* X_{k+1} = X_{k+2}$$

and

$$a X_{k+1} = 2\,(N - k)\, X_k.$$

Explicitly, if $X_k = (\varphi_{1k}\,(x_1)\,, \ldots, \varphi_{\mathrm{Nk}}\,(x_N))$ is the kth eigenvector, where $\varphi_k(x)$ is the kth Mehta-Dyson polynomial of argument x, then

$$a\varphi_{k+1}\,(x_1) = \frac{d}{dx_1}\,(\varphi_{k+1}\,(x_1)) = \sum_{i \neq 1}^{N} \frac{\varphi_{k+1}\,(x_1) - \varphi_{k+1}\,(x_i)}{(x_1 - x_i)}$$

and

$$a^* \varphi_{k+1}\,(x_1) = \left(2x_1 - \frac{d}{dx_1}\right) \varphi_{k+1}\,(x_1)\,.$$

H as above, with $N = n$ may be used to give the first n trivial critical values of the first set in (17) i.e. t_1^*, …, t_n^* while $H + \frac{1}{2}$ may be used for obtaining the first n trivial critical values of the second set in (18) i.e. t_1^{**}, …, t_n^{**} in the discussion on the mean staircases given in Section 2. The same holds true for the general case λ different from 0 and $\pm\frac{1}{2}$.

References

[1] Pitkänen M. A strategy for proving Riemann Hypothesis. *Acta Math. Univ. Comenianae*, 72(1):1–13, 2003.

[2] Castro C., Granik A., and Mahecha J. On susy-qm, fractal strings and steps towards a proof of the Riemann Hypothesis, arXiv:hep-th/0107266v3, September 2001.

[3] Elizalde E., Moretti V., and Zerbini S. On recent strategies proposed for proving the Riemann hypothesis. *Int. J. Mod. Phys*, A18:2189–2196, 2003.

[4] Julia B. On the statistics of primes. *Journal de physique*, 50(12):1371–1375, 1989.

[5] Berry M.V. and Keating J.P. H = xp and the Riemann zeros. In *Supersymmetry and Trace Formulae: Chaos and Disorder*, edited by Lerner et al., page 355–367, New York, Kluwer Academic Publishers, 1999.

[6] Berry M.V. and Keating J.P. The Riemann zeros and eigenvalue asymptotics. *SIAM Review*, 41(2):236–266, 1999.

[7] Bump D., Choi Kwok-Kwong, Kurlberg P., and Vaaler J. A local Riemann hypothesis, I. *Mathematische Zeitschrift*, 233(1):1–18, 2000.

[8] Connes A. Trace formula in noncommutative geometry and the zeros of the Riemann zeta function. *Selecta Mathematica, New Series*, 5(1), May 1999.

[9] Albeverio S. and Cebulla C. Müntz formula and zero free regions for the Riemann zeta function. *Bulletin des Sciences Mathématiques*, 131(1):12–38, 2007.

[10] Sierra G. A quantum mechanical model of the Riemann zeros. *New Journal of Physics*, 10, 033016, 2008.

[11] Titchmarsh E.C. The Theory of the Riemann Zeta-Function. *Oxford University Press*, 2 edition, 1986.
[12] Corless R.M., Gonnet G.H., Hare D.E.G., Jeffrey D.J., and Knuth D.E. On the Lambert W-function. *Advances in Computational Mathematics*, 5:329–359, 1996.
[13] Merlini D., Rusconi L., and Sala N. I numeri naturali come autovalori di un modello di oscillatori classici a bassa temperatura. *Bollettino della Società ticinese di Scienze naturali*, 87:29–32, 1999.
[14] Nash C. The spectrum of the Schrödinger operator and the distribution of primes. *Advances in Applied Mathematics*, 6(4):436–446, December 1985.
[15] Yang C.N., Lee T.D. Statistical Theory of Equations of State and Phase Transitions. I. Theory of Condensation, *Physical Review Letters*, 87:404–409, 1952.
[16] Sierra G. and Townsend P.K. Landau levels and Riemann zeros. *Physical Review Letters*, 101, 110201, 2008.
[17] Schumayer D., van Zyl B., and Hutchinson D. Quantum mechanical potentials related to the prime numbers and Riemann zeros. *Physical Review E*, 78, 056215, 2008.
[18] Sekatskii S.K. On the hamiltonian whose spectrum coincides with the set of primes. arXiv:0709.0364v1 [math-ph], 2007.
[19] Bernasconi A., Merlini D., and Rusconi L. Complex aspects of the Riemann Hypothesis: a computational approach. In Losa G. et al., editors, *Fractals in Biology and Medicine*, volume 3, page 333. Birkhäuser, 2002.

D. a Marca, CERFIM, Research Center for Mathematics and Physics, PO Box 1132, 6600 Locarno, Switzerland
E-mail address: davide.amarca@ti.ch

S. Beltraminelli, CERFIM, Research Center for Mathematics and Physics, PO Box 1132, 6600 Locarno, Switzerland
E-mail address: stefano.beltraminelli@ti.ch

D. Merlini, CERFIM, Research Center for Mathematics and Physics, PO Box 1132, 6600 Locarno, Switzerland
E-mail address: merlini@cerfim.ch