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AUTOMORPHISMS AND DERIVATIONS ON THE CENTER OF A RING

V.K.Bhat
School of Mathematics,
SMVD University,
P/o Kakryal, Katra, J and K,
India- 182320
vijaykumarbhat2000@yahoo.com

Abstract. Let R be a ring, σ_1 an automorphism of R and δ_1 a σ_1-derivation of R. Let σ_2 be an automorphism of $O_1(R) = R[x; \sigma_1, \delta_1]$, and δ_2 be a σ_2-derivation of $O_1(R)$. Let $S \subseteq Z(O_1(R))$, the center of $O_1(R)$. Then it is proved that σ_i is identity when restricted to S, and δ_i is zero when restricted to S; $i = 1, 2$. The result is proved for iterated extensions also.

1. Introduction

A ring R means an associative ring with identity $1 \neq 0$. $Z(R)$ denotes the center of R. The set of positive integers is denoted by $\mathbb{N}$. Let A be a nonempty set and $\alpha : A \to A$ be a map and $B \subseteq A$. Then $\alpha \mid B$ means α restricted to B.

In this paper we investigate the nature of an automorphism σ and a σ-derivation δ of a ring R, when restricted to the center of R.

Recall that a σ-derivation of R is an additive map $\delta : R \to R$ such that

$$\delta(ab) = \delta(a)\sigma(b) + a\delta(b), \text{ for all } a, b \in R.$$

Let σ be an endomorphism of a ring R and $\delta : R \to R$ any map.
Let $\phi : R \to M_2(R)$ be a homomorphism defined by

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}, \text{ for all } r \in R.$$

Then δ is a σ-derivation of R.

In case σ is the identity map, δ is called just a derivation of R. For example let F be a field and $R = F[x]$. Then the formal derivative $\frac{d}{dx}$ is a derivation of R.

Recall that the Ore extension $R[x; \sigma, \delta] = \{f = \sum_{i=0}^{n} x^i a_i,\ a_i \in R,\ n \in \mathbb{N}\}$, subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take coefficients on the right as followed in McConnell and Robson [13]. Some authors take coefficients on the left as in Goodearl and Warfield [7]. We denote the Extension ring $R[x; \sigma, \delta]$ by $O_1(R)$. In case σ is the identity map, we denote $R[x; \delta]$ by $D_1(R)$ and in case δ is the zero map, we denote $R[x; \sigma]$ by $S_1(R)$.

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We also recall that the skew-Laurent extension $R[x, x^{-1}; \sigma] = \{\sum_{i=-m}^{n} x^i a_i$, $a_i \in R;\ m,\ n \in \mathbb{N}\}$, where multiplication is subject to the relation $ax = x\sigma(a)$ for all $a \in R$.

The rings that we deal with are the above mentioned rings and their iterations as given below:

(1) $S_t(R) = R[x_1; \sigma_1][x_2; \sigma_2]...[x_t; \sigma_t]$, the iterated skew-polynomial ring, where each σ_i is an automorphism of $S_{i-1}(R)$.
(2) $L_t(R) = R[x_1, x_i^{-1}; \sigma_1][x_2, x_2^{-1}; \sigma_2]...[x_t, x_t^{-1}; \sigma_t]$, the iterated skew-Laurent polynomial ring, where each σ_i is an automorphism of $L_{i-1}(R)$.
(3) $D_t(R) = R[x_1; \delta_1][x_2; \delta_2]...[x_t; \delta_t]$, the iterated differential operator ring, where each δ_i is a derivation of $D_{i-1}(R)$.
(4) $O_t(R) = R[x_1; \sigma_1, \delta_1][x_2; \sigma_2, \delta_2]...[x_t; \sigma_t, \delta_t]$, the iterated Ore extension, where σ_i is an automorphism of $O_{i-1}(R)$ and δ_i is a σ_i-derivation of $O_{i-1}(R)$.

We note that if σ is an automorphism of a ring R and δ is a σ-derivation of R, then σ can be extended to an automorphism of $R[x; \sigma, \delta]$ by taking $\sigma(x) = x$, i.e. $\sigma(xa) = x\sigma(a)$, for all $a \in R$. Also δ can be extended to a σ-derivation of $R[x; \sigma, \delta]$ by taking $\delta(x) = 0$, i.e. $\delta(xa) = x\delta(a)$, for all $a \in R$.

In view of this, we note that each σ_i is an automorphism of $S_t(R)$ and $O_t(R)$. Also each δ_i is a derivation (respectively σ-derivation) of $D_t(R)$ (respectively $O_t(R)$).

2. Automorphisms and derivations

We prove the following:

(1) Let $L \subseteq Z(K_t(R))$, where $K_t(R)$ is any of $S_t(R)$ or $L_t(R)$. Then $\sigma_i \mid L$ is the identity map; for all i, $1 \leq i \leq t$.
(2) Let $T \subseteq Z(D_t(R))$, where R is an integral domain. Then $\delta_i \mid T$ is the zero map; for all i, $1 \leq i \leq t$.
(3) Let $S \subseteq Z(O_t(R))$. Then $\delta_i \mid\ S$ is the identity map, and $\delta_i \mid S$ is the zero map; for all i, $1 \leq i \leq t$.

For more details on Ore extensions, and the basic results, the reader is referred to chapters (1) and (2) of [7]. Ore-extensions including skew-polynomial rings and differential operator rings have been of interest to many authors. For example [1, 5, 7, 8, 9, 10, 11].

Prime ideals (in particular minimal prime ideals and associated prime ideals) of these extensions have been characterized in [1, 4, 6, 14].

Recall that a ring R is said to be 2-primal if the prime radical (i.e. the intersection of prime ideals of R) coincides with the set of all nilpotent elements of R. This property has been discussed in [2, 3, 12].

We begin with the following Proposition:

Proposition 2.1. *Let R be a ring σ be an automorphism of R. Then $\sigma \mid Z(R)$ is an automorphism.*

Proof. It is enough to show that $a \in Z(R)$ implies that $\sigma(a) \in Z(R)$. Let $a \in Z(R)$ and $r \in R$. Then $\sigma(a)r = \sigma(a\sigma^{-1}(r)) = \sigma(\sigma^{-1}(r)a) = r\sigma(a)$ Therefore, $\sigma(a) \in Z(R)$. □

We now have the following proposition which is used to prove Proposition (2.5) and Theorem (2.6).

Proposition 2.2. *Let R be an integral domain. Then $O_1(R)$ is an integral domain.*

Proof. The proof is easy. We give a sketch. Let f, $g \in O_1(R)$ be such that $fg = 0$. Say $f = \sum_{i=0}^{n} x^i a_i$, and $g = \sum_{i=0}^{m} x^i b_i$, m, $n \in \mathbb{N}$. Suppose that $g \neq 0$.

To prove the result, we use induction on m, n. For $m = n = 0$, the result is trivial. For $m = n = 1$, we have $f = xa_1 + a_0$ and $g = xb_1 + b_0$. Now $fg = 0$ implies that

$$x[x\sigma(a_1) + \delta(a_1)]b_1 + [x\sigma(a_0) + \delta(a_0)]b_1 + xa_1b_0 + a_0b_0 = 0;$$

i.e.

$$x^2\sigma(a_1)b_1 + x\delta(a_1)b_1 + x\sigma(a_0)b_1 + xa_1b_0 + \delta(a_0)b_1 + a_0b_0 = 0,$$

and so we have $\sigma(a_1)b_1 = 0$, $\delta(a_1)b_1 + \sigma(a_0)b_1 + a_0b_0 = 0$, $\delta(a_0)b_1 + a_0b_0 = 0$. Now $g \neq 0$. Therefore, there are three possibilities:

(1) $b_1 \neq 0$, $b_0 \neq 0$. In this case $\sigma(a_1)b_1 = 0$ implies that $\sigma(a_1) = 0$; i.e. $a_1 = 0$. Now $\delta(a_1)b_1 + \sigma(a_0)b_1 + a_1b_0 = 0$ implies that $\sigma(a_0)b_1 = 0$. Therefore $\sigma(a_0) = 0$; i.e. $a_0 = 0$. Thus $f = 0$.

(2) $b_1 \neq 0$, $b_0 = 0$. This could be treated similarly as above.

(3) $b_1 = 0$, $b_0 \neq 0$. In this case $\delta(a_1)b_1 + \sigma(a_0)b_1 + a_1b_0 = 0$ implies that $a_1b_0 = 0$, and therefore, $a_1 = 0$. Also $\delta(a_0)b_1 + a_0b_0 = 0$ implies that $a_0b_0 = 0$, and so $a_0 = 0$. Thus $f = 0$. So, in all cases we have $f = 0$.

Therefore, the result is true for $m = n = 1$. Suppose the result is true for $m = k$ and $n = 1$. We shall prove for $m = k + 1$. Now for $m = k + 1$ and $n = 1$, $fg = 0$ implies that

$$(x^{k+1}a_{k+1} + x^k a_k + ... + a_0)(xb_1 + b_0) = 0,$$

i.e.

$$x^{k+2}\sigma(a_{k+1})b_1 + x^{k+1}\delta(a_{k+1})b_1 + x^{k+1}\sigma(a_k)b_1 + x^{k+1}a_{k+1}b_0 + ...+$$
$$x\sigma(a_0)b_1 + \delta(a_0)b_1 + a_0b_0 = 0.$$

Now for $g \neq 0$, there are three possibilities:

(1) $b_1 \neq 0$, $b_0 \neq 0$. In this case $\sigma(a_{k+1})b_1 = 0$ implies that $\sigma(a_{k+1}) = 0$; i.e. $a_{k+1} = 0$. Therefore $fg = 0$ reduces to $(x^k a_k + x^{k-1}a_{k-1} + ... + a_0)(xb_1 + b_0) = 0$, and induction hypothesis implies that $f = 0$.

(2) $b_1 \neq 0$, $b_0 = 0$. This could be treated similarly as above.

(3) $b_1 = 0$, $b_0 \neq 0$. In this case $\delta(a_{k+1})b_1 + \sigma(a_k)b_1 + a_{k+1}b_0 = 0$ implies that $a_{k+1}b_0 = 0$, and therefore, $a_{k+1} = 0$. Therefore $fg = 0$ reduces to $(x^k a_k + x^{k-1}a_{k-1} + ... + a_0)(xb_1 + b_0) = 0$, and induction hypothesis implies that $f = 0$.

Therefore, in all the cases $f = 0$. In a similar way the result could be proved for higher degrees of g. Hence $O_1(R)$ is an integral domain. □

Proposition 2.3. *Let R be a ring and consider $S_t(R)$. Let $L \subseteq S_t(R)$. Then $\sigma_i \mid L$ is the identity map for all i, $1 \leq i \leq t$.*

Proof. Consider $S_1(R)$ and its automorphism σ_2. Let $a \in L$. Now $af_1 = f_1a$ for all $f_1 = \sum_{i=0}^{n} x_1^i b_i \in S_1(R)$, $n \in \mathbb{N}$,

$$a(x_1^n b_n + ... + b_0) = (x_1^n b_n + ... + b_0)a.$$

So we have

$$(x_1^n\sigma_1^n(a)b_n + ... + x_1\sigma_1(a)b_1 + ab_0 = (x_1^n b_n a + ... + x_1b_1a + b_0a).$$

Therefore $\sigma_1(a) = a$.

Now consider $S_2(R)$ and its automorphism σ_3. Let $a \in L$. Then $af_2 = f_2a$ for all $f_2 \in S_2(R)$. Let $f_2 = x_2^k f_k + ... + x_2f_1 + f_0$, where $f_i \in S_1(R)$. Then $af_2 = f_2a$ implies that

$$a(x_2^k f_k + ... + x_2f_1 + f_0) = (x_2^k f_k + ... + x_2f_1 + f_0)a;$$

i.e.

$$x_2^k\sigma_2^k(a)f_k + ... + x_2\sigma_2(a)f_1 + af_0 = x_2^k f_k a + ... + x_2f_1a + f_0a.$$

Therefore, $\sigma_2(a)f_1 = f_1a = af_1$ as $a \in Z(S_2(R))$. Hence $\sigma_2(a) = a$. With the same process, we can see that $\sigma_i \mid L$ is the identity map for all i, $1 \le i \le t$. □

Remark 2.4. The above result holds if $S_t(R)$ is replaced by $L_t(R)$, and the proof follows on the same lines.

Proposition 2.5. *Let R be an integral domain and consider $D_t(R)$. If $T \subseteq Z(D_t(R))$. Then $\delta_i \mid T$ is the zero map, for all i, $1 \le i \le t$.*

Proof. Let $a \in T$. Consider $D_1(R)$. Let $f_1 = x_1b + c$, $b \neq 0$. Then $af_1 = f_1a$; i.e. $a(x_1 1b + c) = (x_1b + c)a$, which implies that

$$x_1ab + \delta_1(a)b + ac = x_1ba + ca.$$

Now $a \in Z(D_1(R))$ implies that $\delta_1(a)b + ac = ca = ac$, and $\delta_1(a)b = 0$. Thus $\delta_1(a) = 0$. Polynomials of higher degree could be treated in a similar way.

Now consider $D_2(R)$. Let $f_2 = x_2g_1 + g_0$, where $g_1 \neq 0$; g_1, $g_0 \in D_1(R)$. Then $af_2 = f_2a$ implies that

$$a(x_2g_1 + g_0) = (x_2g_1 + g_0)a,$$

or,

$$x_2ag_1 + \delta_2(a)g_1 + ag_0 = x_2g_1a + g_0a.$$

Now a $\in Z(D_2(R))$ implies that

$$\delta_2(a)g_1 + ag_0 = g_0a = ag_0.$$

Therefore $\delta_2(a)g_1 = 0$, and so Proposition (2.2) implies that $\delta_2(a) = 0$. With the same process it can be shown that $\delta_i \mid T$ is the zero map, for all i, $1 \le i \le t$. □

Theorem 2.6. *Let R be an integral domain and consider $O_t(R)$. If $S \subseteq Z(O_t(R))$, then $\sigma_i \mid S$ is the identity map and $\delta_i \mid S$ is the zero map, for all i, $1 \le i \le t$.*

Proof. Let $a \in S$. Let $f_1 = x_1b + c \in O_1(R)$, $b \neq 0$. Then $af_1 = f_1a$, and we have $a(x_1b + c) = (x_1b + c)a$, which implies that

$$x_1\sigma_1(a)b + \delta_1(a)b + ac = x_1ba + ca.$$

Therefore $\sigma_1(a)b = ba = ab$ as $a \in Z(O_t(R))$. So we have $\sigma_1(a) = a$. Also, $\delta_1(a)b + ac = ca = ac$. Thus $\delta_1(a)b = 0$, and so $\delta_1(a) = 0$. Polynomials of higher degree can be treated similarly.

Now let $f_2 = x_2g_1 + g_0 \in O_2(R)$, $g_1 \neq 0$. Then $af_2 = f_2a$ implies that

$$a(x_2g_1 + g_0) = (x_2g_1 + g_0)a.$$

Therefore

$$x_2\sigma_2(a)g_1 + \delta_2(a)g_1 + ag_0 = x_2g_1a + g_0a$$

Now $a \in Z(O_t(R))$ implies that

$$x_2\sigma_2(a)g_1 = g_1a + ag_1.$$

Thus $\sigma_2(a) = a$. Also $\delta_2(a)g_1 + ag_0 = g_0a = ag_0$ as $a \in Z(O_t(R))$. Therefore $\delta_2(a)g_1 = 0$ and thus Proposition (2.2) implies that $\delta_2(a) = 0$. Polynomials of higher degree can be treated similarly.

With the same process it can be shown that $\sigma_i \mid S$ is the identity map for all i, $1 \leq i \leq t$ and $\delta_i \mid S$ is the zero map for all i, $1 \leq i \leq t$. $\square$

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