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GENERALIZED COMPLEMENTARITY PROBLEMS IN BANACH SPACES

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Abstract. In this paper, a class of complementarity problem and three classes of variational inequalities in real Banach spaces are introduced, and the equivalence among them are established under certain conditions. Several coercivity conditions are introduced for the existence of solutions of the generalized complementarity problem. Our results can be viewed as extension and generalization of the recent paper [N. J. Huang, J. Li, D. O'Regan, Generalized f-complementarity problems in Banach spaces, vol. **142,** pg. 3828-3840, 2008].

1. Introduction and Preliminaries

The complementarity problem (for short, CP) was introduced first by Cottle and Dantzig [6] in 1968. It is well known that (CP) is closely related to optimization problems, variational inequalities, equilibrium problems, fixed point theory, operations research, game theory, economics and finance, as well as applied sciences. Since 1960s, (CP) has been studied extensively by many authors (see, for instance, [1]-[6], [8]-[16] and the references therein).

In this paper let X be a real Banach space with dual X^*, and K a nonempty, closed and convex cone of X. Denote $\langle t, x\rangle$ value of the linear continuous function $t \in X^*$ at x. In 2001, Yin, Xu and Zhang [19] introduced and studied a class of

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f-complementarity problems (for short, f-CP), which consists of finding $\overline{x} \in K$ such that

$$\langle T\overline{x}, \overline{x}\rangle + f(\overline{x}) = 0 \text{ and } \langle T\overline{x}, y\rangle + f(y) \geq 0, \ \forall y \in K.$$

The (f-CP) has been extended to the vector f-complementarity problem by Fang and Huang [9], the vector f-implicit complementarity problem by Li and Huang [15] and (the latest extension) the generalized f-complementarity problem (for short, Gf-CP) by Huang, Li and O'Regan [12] which consists of finding $\overline{x} \in K$ and $\bar{t} \in F(\overline{x})$ such that

$$\langle \bar{t}, \overline{x}\rangle + f(\overline{x}) = 0 \text{ and } \langle \bar{t}, y\rangle + f(y) \geq 0, \ \forall y \in K,$$

where $F : K \to 2^{X^*} \backslash \{\emptyset\}$.

In this paper we consider the following problem which consists of finding $\overline{x} \in K$ and $\bar{t} \in F(\overline{x})$ such that

$$\text{(GCP)} \qquad G(\bar{t}, \overline{x}) = 0 \text{ and } G(\bar{t}, y) \geq 0, \ \forall y \in K,$$

where $G : X^* \times K \to \mathbb{R}$. We call this problem generalized complementarity problem (GCP) and denote it by S_c the solution set of (GCP).

Remark that if we define $G(t, x) = \langle t, x\rangle + f(x)$, where $f : K \to (-\infty, \infty)$, then (GCP) reduces to the (Gf-CP).

We also study the following three classes of variational inequalities:

$(GVI)_1$ Find $\overline{x} \in K$ such that

$$\exists\ \bar{t} \in F(\overline{x}) : \ G(\bar{t}, y) \geq G(\bar{t}, \overline{x}), \ \forall\ y \in K;$$

$(GVI)_2$ Find $\overline{x} \in K$ such that

$$\forall\ y \in K, \ \exists\ \bar{t} \in F(\overline{x}) \ : \ G(\bar{t}, y) \geq G(\bar{t}, \overline{x});$$

$(GVI)_3$ Find $\overline{x} \in K$ such that

$$\forall\ y \in K, \ \forall\ t \in F(y) \ : \ G(t, y) \geq G(t, \overline{x}).$$

We denote the solution set of $(GVI)_1$, $(GVI)_2$ and $(GVI)_3$ by S_1, S_2 and S_3, respectively.

2. EQUIVALENCE AMONG (G-CP) $(GVI)_1$, $(GVI)_2$ AND $(GVI)_3$

In this section, we investigate the equivalence among (G-CP) $(GVI)_1$, $(GVI)_2$ and $(GVI)_3$. First we recall some definitions. Let $G : X^* \times K \to \mathbb{R}$ and $t \in X^*$. The mapping $x \to G(t, x)$ is said to be

- positively homogeneous, if for all $\alpha > 0$ and $x \in K$,
$$G(t, \alpha x) = \alpha G(t, x);$$
- convex, if for all pairs $(x, y) \in K \times K$ and all $\lambda \in [0, 1]$,
$$G(t, \lambda x + (1 - \lambda)y) \leq \lambda G(t, x) + (1 - \lambda)G(t, y);$$
- sublinear, if it is convex and homogeneous;
- subadditive, if for all pairs $(x, y) \in K \times K$,
$$G(t, x + y) \leq G(t, x) + G(t, y);$$

• lower semicontinuous (l.s.c.), if for every $x \in K$,

$$\liminf_{y \to x} G(t,y) \geq G(t,x).$$

Theorem 2.1. *Let $F : K \to 2^{X^*} \setminus \{\emptyset\}$ and $G : X^* \times K \to \mathbb{R}$. Then the following statements are valid:*

(i) $S_c \subseteq S_1$.
(ii) *If $G(x,.)$ is positively homogeneous, then $S_1 \subseteq S_c$.*

Proof. (i) is trivial.

(ii) Let $\overline{x} \in S_1$. Then $\overline{x} \in K$ and

$$\exists \overline{t} \in F(\overline{x}): \; G(\overline{t},y) \geq G(\overline{t},\overline{x}), \; \forall y \in K. \tag{2.1}$$

By letting $y = 2\overline{x}$ and $y = \frac{\overline{x}}{2}$,respectively, in 2.1 (note that K is convex cone and the mapping $x \to G(t,x)$ is positively homogeneous) we get

$$G(\overline{t},\overline{x}) \geq 0 \;\text{ and } G(\overline{t},\overline{x}) \leq 0,$$

and hence

$$G(\overline{t},\overline{x}) = 0. \tag{2.2}$$

Now (2.1) and (2.2) imply that

$$G(\overline{t},\overline{x}) = 0 \;\text{ and } G(\overline{t},y) \geq 0.$$

Hence $\overline{x} \in S_c$. The proof is complete. □

Definition 2.2. Let $F : K \to 2^{X^*} \setminus \{\emptyset\}$ and $G : X^* \times K \to \mathbb{R}$. The F is said to be

• upper semi-continuous (u.s.c) at $x \in K$ if, for every open set V containing $F(x)$, there exists an open set U containing x such that $F(U) \subseteq V$, where X^* is equipped with the w^* -topology;
• upper hemi-continuous (u.h.c) if the restriction of F on straight lines is upper semi-continuous;
• G-monotone if, for every $x, y \in K$

$$G(t_x,x) + G(t_y,y) \geq G(t_x,y) + G(t_y,x) \; \forall t_x \in F(x), \forall t_y \in F(y);$$

• strictly G-monotone if, for every $x, y \in K$

$$G(t_x,x) + G(t_y,y) > G(t_x,y) + G(t_y,x) \; \forall t_x \in F(x), \forall t_y \in F(y);$$

Lemma 2.3. *([18]) Let A be a nonempty convex set in a vector space and let B be a nonempty compact convex set in a Hausdorff topological vector space. Suppose that g is a real valued function on $A \times B$ such that for each $a \in A$, $g(a,.)$ is l.s.c and convex on B, and for each fixed $b \in B$, $g(.,b)$ is concave on A. Then*

$$\min_{b \in B} \sup_{a \in A} g(a,b) = \sup_{a \in A} \min_{b \in B} g(a,b).$$

Theorem 2.4. *Let $F : K \to 2^{X^*} \setminus \{\emptyset\}$ and $G : X^* \times K \to \mathbb{R}$. Then the following hold:*

(i) $S_1 \subseteq S_2$.
(ii) *If F is G-monotone, then $S_2 \subseteq S_3$.*
(iii) *If F is u.h.c, for each fixed $t \in X^*$, $x \to G(t,x)$ is convex, and for each fixed $x \in K$, $t \to G(t,x)$ is u.s.c, then $S_3 \subseteq S_2$.*

(iv) *If F has w^*-compact and convex values, for each fixed $x \in K$, $t \to G(t,x)$ is concave, l.s.c, and for each fixed $t \in X^*$, $x \to G(t,x)$ is convex, then $S_1 = S_2$.*

Proof. (i) It is trivial.
(ii) Let $\overline{x} \in S_2$. Then

$$\forall y \in K,\ \exists \overline{t} \in F(\overline{x})\ :\ G(\overline{t}, y) \geq G(\overline{t}, \overline{x}). \qquad (2.3)$$

Since F is G-monotone, for every $y \in K$ and $t \in F(y)$, we have

$$G(t,y) + G(\overline{t}, \overline{x}) \geq G(t, \overline{x}) + G(\overline{t}, y). \qquad (2.4)$$

It follows from (2.3) and (2.4) that

$$G(t,y) - G(t,\overline{x}) \geq G(\overline{t}, y) - G(\overline{t}, \overline{x}) \geq 0\ \forall y \in K, \forall t \in F(y),$$

and so $\overline{x} \in S_3$.
(iii) Suppose that the conclusion is not true. Then there exists $\overline{x} \in K$ such that $\overline{x} \in S_3$ and $\overline{x} \notin S_2$. It follows from $\overline{x} \notin S_2$, that there exists $y \in K$ for which,

$$G(t,y) < G(\overline{t}, \overline{x}),\ \forall t \in F(\overline{x}).$$

Hence, setting $x_\lambda = \lambda y + (1-\lambda)\overline{x}$ and taking λ close to 0, we have

$$G(t_\lambda, y) < G(\overline{t}, \overline{x}) \qquad \forall t_\lambda \in F(x_\lambda), \qquad (2.5)$$

(note $\{t \in X^* : G(t,y) < G(\overline{t}, \overline{x})\}$ is a w^*-open neighborhood of $F(\overline{x})$ and F is u.h.c). From the convexity, for each fixed $t \in X^*$, the mapping $x \to G(t,x)$ gives

$$G(t_\lambda, x_\lambda) \leq \lambda G(t_\lambda, y) + (1-\lambda) G(t_\lambda, \overline{x}). \qquad (2.6)$$

Now (2.5) and (2.6) imply that

$$G(t_\lambda, \overline{x}) < G(\overline{t}, \overline{x}),\ \forall t_\lambda \in F(x_\lambda),$$

which contradicts $\overline{x} \in S_3$. Thus, $\overline{x} \in S_2$ and (iii) is true.
(iv) From conclusion (i), it suffices to show that $S_2 \subseteq S_1$. Let $\overline{x} \in S_2$. Then

$$\forall y \in K,\ \exists \overline{t} \in F(\overline{x})\ :\ G(\overline{t}, y) \geq G(\overline{t}, \overline{x}). \qquad (2.7)$$

Define $g : K \times F(\overline{x}) \to \mathbb{R}$ by

$$g(a,b) = G(b, \overline{x}) - G(b,a)$$

One can easily see that g satisfies all assumptions of Lemma 2.3. Hence it follows from Lemma 2.3 that

$$\min_{b \in F(\overline{x})} \sup_{a \in K} (g(a,b) = G(b,\overline{x}) - G(b,a)) = \sup_{a \in K} \min_{b \in F(\overline{x})} g(a,b) \leq 0, \qquad (2.8)$$

(note (2.7) guarantees the inequality in (2.8)). Thus,

$$\exists \overline{t} \in F(\overline{x}) :\ \sup_{a \in K}\ g(a, \overline{t}) \leq 0,$$

and so

$$G(\overline{t}, \overline{x}) \leq G(\overline{t}, y),\ \forall y \in K.$$

Therefore $\overline{x} \in S_1$. This completes the proof. □

Theorem 2.5. *Let $G : X^* \times K \to \mathbb{R}$ such that $G(a,.)$ is sublinear. Assume that $F : K \to 2^{X^*} \backslash \{\emptyset\}$ is G-monotone, u.h.c., and has w^*-compact convex values. Then*

(i) $S_c = S_1 = S_2 = S_3$.

(ii) *If F is strictly G-monotone and $S_i \neq \emptyset (i = 1, 2, 3)$, then S_c consists of one point.*

Proof. Conclusion (i) follows directly from Theorem 2.4.
(ii) It is sufficient, by (i), to show that S_2 consists of one point. Assume that F is strictly G-monotone and $S_2 \neq \emptyset$. Let $\overline{x_1}$ and $\overline{x_2}$ solve $(GVI)_2$. Then, by (i), $\overline{x_1}, \overline{x_2} \in S_1$ and so

$$\exists\, \bar{t}_1 \in F(\overline{x}_1) \ : \ G(\bar{t}_1, \overline{x}_2) \geq G(\bar{t}_1, \overline{x}_1), \tag{2.9}$$

and

$$\exists\, \bar{t}_2 \in F(\overline{x}_2) \ : \ G(\bar{t}_2, \overline{x}_1) \geq G(\bar{t}_2, \overline{x}_2). \tag{2.10}$$

If $\overline{x}_1 \neq \overline{x}_2$, then the strict G-monotonicity of F implies that

$$G(\overline{t_1}, \overline{x_1}) + G(\overline{t_2}, \overline{x_2}) > G(\overline{t_1}, \overline{x_2}) + G(\overline{t_2}, \overline{x_1}). \tag{2.11}$$

Now from (2.9) and (2.11) we get

$$G(\bar{t}_2, \overline{x}_2) > G(\bar{t}_2, \overline{x}_1),$$

which contradicts (2.10). Hence $\overline{x}_1 = \overline{x}_2$, and so S_2 consists of one point. This completes the proof. □

3. Coercivity Conditions for (GCP)

Denote by $\mathcal{K}$ the set of all weakly compact convex subset of K. We consider the following three classes of coercivity conditions:

(C_1) $\exists E \in \mathcal{K}, \forall x \in K \backslash E, \forall t \in F(x), \exists y \in E : \ G(t, y) < G(t, x);$

(C_2) $\exists E \in \mathcal{K}, \forall x \in K \backslash E, \exists y \in E, \forall t \in F(x) : \ G(t, y) < G(t, x);$

(C_3) $\exists E \in \mathcal{K}, \forall x \in K \backslash E, \exists y \in E, \exists t \in F(y) : \ G(t, y) < G(t, x).$

Theorem 3.1. *Let $F : K \to 2^{X^*} \backslash \{\emptyset\}$ and $G : X^* \times K \to \mathbb{R}$. Then the following statements hold:*

(i) *Condition (C_1) (resp., (C_2) and (C_3)) implies that S_1 (resp., S_2 and S_3) is contained in the weakly compact set E.*
(ii) *Condition (C_2) implies (C_1).*
(iii) *If F has convex values, for each fixed $t \in X^*$, $x \to G(t, x)$ is convex l.s.c. and for each fixed $x \in K$ $t \to G(t, x)$ is concave, then condition (C_1) implies (C_2).*
(iv) *If F is G-monotone, then condition (C_3) implies (C_2).*

Proof. Statements (i), (ii) and (iv) are obvious. It suffices to show that conclusion (iii) holds. Assume that condition (C_1) holds. Then

$$\exists E \in \mathcal{K}, \forall x \in K \backslash E, \forall t \in F(x), \exists y \in E : \ G(t, y) < G(t, x).$$

Assume F has convex values and $G(t, .)$ is convex and l.s.c. (w.l.s.c). As the proof in (iv) of Theorem 2.4, for any given $x \in K \backslash E$, define $h : F(x) \times E \to \mathbb{R}$ by

$$h(a, b) = G(a, b) - G(a, x).$$

It is easy to see that h satisfies all the assumptions of Lemma 2.3. Hence Lemma 2.3 implies that

$$\min_{b \in E} \sup_{a \in F(x)} h(a, b) = \sup_{a \in F(x)} \min_{b \in E} h(a, b) < 0$$

and so

$$\exists y \in E : \sup_{a \in F(x)} h(a, y) < 0.$$

Therefore

$$G(t, y) < G(t, x), \ \forall t \in F(x),$$

which implies that condition (C_2) holds. This completes the proof. □

Let P be a nonempty subset of a topological vector space Y. A set-valued mapping $G : P \to 2^Y$ is called a KKM-mapping if, for every finite subset $\{y_1, y_2, ..., y_k\}$ of P,

$$co\{y_1, y_2, ..., y_k\} \subseteq \bigcup_{i=1}^{k} G(y_i).$$

Lemma 3.2. *([8]) Let P a nonempty subset of a Hausdorff topological vector space Y. Let $G : P \to 2^Y$ be a KKM-mapping such that, for any $y \in Y$, $G(y)$ is closed and, for some $y^* \in P$, $G(y^*)$ is compact . Then*

$$\bigcap_{y \in P} G(y) \neq \emptyset.$$

Theorem 3.3. *Let $G : X^* \times K \to \mathbb{R}$ and $F : K \to 2^{X^*} \backslash \{\emptyset\}$ be u.h.c., and G-monotone. Let, for each fixed $t \in X^*$, the mapping $x \to G(t, x)$ be a convex and l.s.c. If condition C_2 holds, then S_2 is nonempty.*

Proof. Suppose that condition C_2 holds, i.e.,

$$\exists E \in \mathcal{K}, \forall x \in K \backslash E, \exists y \in E, \forall t \in F(x) : \ G(t, y) < G(t, x).$$

Define $H_0 : K \to 2^E$ by

$$H_0(y) = \{x \in E : G(t, y) \geq G(t, x), \forall t \in F(y)\} \qquad \forall y \in K.$$

It is clear that $\bigcap_{y \in K} H_0(y) \subseteq S_3$, and from Theorem 2.5 one has $\bigcap_{y \in K} H_0(y) \subseteq S_3 = S_2$. We now show that $\bigcap_{y \in K} H_0(y) \neq \emptyset$. Since the mapping $x \to G(t, x)$ is convex and l.s.c, it is easy to show that $H_0(y)$ is weakly closed, for each $y \in K$, and so $\bigcap_{y \in K} H_0(y)$ is weakly closed. Hence, since E is w-compact, $H_0(y)$ and $\bigcap_{y \in K} H_0(y)$ are compact. Thus, it suffices to prove that the family $\{H_0(y)\}_{y \in K}$ has the finite intersection property. Let $\{y_1, y_2, ..., y_k\}$ be any finite subset of K, and set $M = \overline{co}(E \cup \{y_1, y_2, ..., y_k\})$, where $\overline{co}$ denotes the closed convex hull. Then M is nonempty, w-compact and convex. Define $H : M \to 2^M$ by

$$H(y) = \{x \in M : G(t, y) \geq G(t, x), \forall t \in F(y)\}, \qquad \forall y \in M.$$

Clearly $y \in H(y)$, and the convexity of the mapping $x \to G(t, x)$ and l.s.c, respectively, imply that $H(y)$ is convex and closed, respectively. Since M is weakly compact, so is $H(y)$. Next, we prove that H is a KKM-mapping. Suppose to the contrary that there exist a finite subset $\{u_1, ..., u_n\}$ of M and $\lambda_i \geq 0, i = 1, 2, ..., n$ with $\sum_{i=1}^{n} \lambda_i = 1$, such that

$$u = \sum_{i=1}^{n} \lambda_i u_i \notin \bigcup_{j=1}^{n} H(u_j).$$

Then, for each $i = 1, 2, ..., n$, there exists $t_i \in F(u_i)$ such that

$$G(t_i, u_i) < G(t_i, u).$$

Since F is G-monotone, we have

$$G(t_i,u)+G(t,u_i)\le G(t_i,u_i)+G(t,u),$$

and so

$$G(t,u_i)\le G(t,u).$$

From the convexity of the mapping $x\to G(t,x)$, we have

$$0=G(t,u)-G(t,u)=G(t,\sum_{i=1}^{n}\lambda_i u_i)-G(t,u)\le$$

$$\sum_{i=1}^{n}\lambda_i(G(t,u_i)-G(t,u))<0,$$

which is a contradiction. Hence H is a KKM-mapping. Therefore H satisfies all the assumptions of Lemma 3.2 and hence $\bigcap_{y\in M}H(y)\neq\emptyset$. So

$$\exists\ \overline{x}\in M\ \forall\ y\in M,\ \forall\ t\in F(y)\ :\ G(t,y)\ge G(t,\overline{x}).$$

Hence $\overline{x}$ is a solution of $(GVI)_3$ in M. From Theorem 2.5 (i), we get that $\overline{x}$ is a solution of $(GVI)_2$ in M and so

$$\forall\ y\in M,\ \exists\ \bar{t}\in F(\overline{x})\ :\ G(\bar{t},y)\ge G(\bar{t},\overline{x}).$$

Since condition C_2 holds, one has $\overline{x}\in E$. Moreover, $\overline{x}\in H_0(y_j)$, for $j=1,2,...,k$, which implies that $\{H_0(y)\}_{y\in K}$ has the finite intersection property. This completes the proof. □

Theorem 3.4. *Let $G:X^*\times K\to\mathbb{R}$ and $F:K\to 2^{X^*}\backslash\{\emptyset\}$ is G-monotone, u.h.c., and has w^* compact convex values. Let $G(t,.)$ be sublinear and l.s.c. If each of conditions (C_1), (C_2) and (C_3) holds, then S_c is nonempty and bounded.*

Proof. The conclusion follows directly from Theorem 2.5 and Theorem 3.3. □

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