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A COMMON FIXED POINT THEOREM FOR A FAMILY OF SELFMAPPINGS SATISFYING A GENERAL CONTRACTIVE CONDITION OF OPERATOR TYPE

Cihangir Alaca
*Department of Mathematics,
Faculty of Science and Arts,
Sinop University, 57000 Sinop,
Turkey
cihangiralaca@yahoo.com.tr*

Abstract. In this paper, we prove a common fixed point theorem for a family of selfmappings satisfying a general contractive condition of operator type.

1. Introduction

The class of generalized contraction mappings, introduced and studied by Ćirić in [6], is very significant in a fixed point theory. As noted by Gornićki and Rhoades [8], a contractive condition (2.1) on a pair of generalized contractions. Jungck [9] proved a fixed point theorem for commuting maps generalizing the Banach's fixed point and further he [10] introduced more generalizing commutativity, so called compatibility, which is more general than that of weak commutativity defined by Sessa [12]. Lately, Branciari [4] obtained a fixed point results for a single mapping satisfying an analogue of Banach's contraction principle (see [3] and [5]) for an integral type inequality. Rhoades [11] proved two fixed point theorems involving more general contractive conditions. Vijayaraju et al. [13] established a general principle, which maked it possible to proved many fixed point theorems for a pair of maps of integral type. Aliouche [1] gave a common fixed point theorem for selfmappings of a symmetric space under a contractive condition of integral type. Altun and Turkoglu [2] proved a fixed point theorem for mappings satisfying a general contractive of operator type.

The main purpose of this paper is to give a common fixed point theorem for a family of selfmappings satisfying a general contractive condition of operator type.

2. Preliminaries

Let X be a nonempty set and let $\{T_\alpha\}_{\alpha \in J}$ be a family of selfmappings on X and J indexing set. A point $u \in X$ is called a common fixed point for a family

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$\{T_\alpha\}_{\alpha\in J}$ iff for each T_α. The following theorem was given by Ćirić [7] for a family of generalized contraction.

Theorem 1. *Let (X,d) be a complete metric space and let $\{T_\alpha\}_{\alpha\in J}$ be a family of selfmappings of X. If there exists fixed $\beta\in J$ such that for each $\alpha\in J$:*

$$d(T_\alpha x, T_\beta y)\le \lambda \max\left\{\begin{array}{c} d(x,y), d(x,T_\alpha x), d(y,T_\beta y), \\ \frac{1}{2}\left[d(x,T_\beta y)+d(y,T_\alpha x)\right]\end{array}\right\} \tag{2.1}$$

for some $\lambda=\lambda(\alpha)\in(0,1)$ and all x, $y\in X$, then all T_α have a unique common fixed point, which is a unique fixed point of each T_α, $\alpha\in J$.

The following theorem was given by Branciari [4] was to analyze the existence of fixed points for mappings of f defined on a complete metric space (X,d) satisfing a contractive condition of integral type.

Theorem 2. *Let (X,d) be a complete metric space, $c\in(0,1)$ and $f:X\to X$ be a mapping such that for each x, $y\in X$ one has*

$$\int_0^{d(fx,fy)}\varphi(t)dt\le c\int_0^{d(x,y)}\varphi(t)dt$$

where $\varphi:[0,+\infty)\to[0,+\infty)$ is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,+\infty)$, non-negative and such that for each $\varepsilon>0$, $\int_0^\varepsilon\varphi(t)dt>0$; then f has a unique fixed point $a\in X$ such that for each $x\in X$, $\lim_{n\to\infty}f^n x=a$.

The following concept of $O(f\ ;.)$ and its examples was given by Altun and Turkoglu [2].

Let $F([0,\infty))$ be class of all function $f:[0,\infty)\to[0,\infty]$ and let Θ be class of all operators

$$O(\bullet;.):F([0,\infty))\to F([0,\infty)),\ \ f\to O(f\ ;.)$$

satisfying the following conditions:

(i) $O(f\ ;t)>0$ for $t>0$ and $O(f\ ;0)=0$,
(ii) $O(f\ ;t)\le O(f\ ;s)$ for $t\le s$,
(iii) $\lim_{n\to\infty}O(f\ ;t_n)=O\left(f\ ;\ \lim_{n\to\infty}t_n\right)$,
(iv) $O(f\ ;\max\{t,s\})=\max\{O(f\ ;t),O(f\ ;s)\}$ for some $f\in F([0,\infty))$.

Example 1. *If $f:[0,\infty)\to[0,\infty)$ is a Lebesque integrable mapping which is finite integral on each compact subset of $[0,\infty)$, non-negative and such that for each $t>0$, $\int_0^t f(s)ds>0$, then the operator defined by*

$$O(f\ ;t)=\int_0^t f(s)ds$$

satisfies the conditions (i)-(iv).

Example 2. *If $f:[0,\infty)\to[0,\infty)$ non-decreasing, continuous function such that $f(0)=0$ and $f(t)>0$ for $t>0$, then the operator defined by*

$$O(f\ ;t)=\frac{f(t)}{1+f(t)}$$

satisfies the conditions (i)-(iv).

Example 3. *If $f : [0, \infty) \to [0, \infty)$ non-decreasing, continuous function such that $f(0) = 0$ and $f(t) > 0$ for $t > 0$, then the operator defined by*

$$O(f\ ; t) = \frac{f(t)}{1 + \ln(1 + f(t))}$$

satisfies the conditions (i)-(iv).

3. A common fixed point theorem and it's results

Now, we prove a common fixed point theorem for a family of selfmappings satisfying a general contractive condition of operator type in complete metric spaces.

Theorem 3. *Let (X, d) be a complete metric space and $\{T_\alpha\}_{\alpha \in J}$ be a family of selfmappings of X. If there exists a fixed $\beta \in J$ such that for each $\alpha \in J$:*

$$O(f; d(T_\alpha x, T_\beta y)) \leq \lambda O(f; m(x, y)) \tag{3.1}$$

where $O(\bullet; .) \in \Theta$ and

$$m(x, y) = \max\left\{d(x, y), d(x, T_\alpha x), d(y, T_\beta y), \frac{1}{2}\left[d(x, T_\beta y) + d(y, T_\alpha x)\right]\right\} \tag{3.2}$$

for some $\lambda = \lambda(\alpha) \in (0, 1)$ and all x, $y \in X$, then all T_α have a unique common fixed point, which is a unique fixed point of each T_α, $\alpha \in J$.

Proof. Let $\alpha \in J$ and $x \in X$ be arbitrary. Consider a sequence, defined inductively by

$$x_0 = x,\ x_{2n+1} = T_\alpha x_{2n},\ x_{2n+2} = T_\beta x_{2n+1}, \quad (n \geq 0).$$

For each integer $n \geq 0$, from (3.1),

$$\begin{aligned} O(f; d(x_{2n+1}, x_{2n+2})) &= O(f; d(T_\alpha x_{2n}, T_\beta x_{2n+1})) \\ &\leq \lambda O(f; m(x_{2n}, x_{2n+1})). \end{aligned} \tag{3.3}$$

Using (3.2), we have

$$m(x_{2n}, x_{2n+1}) = \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}.$$

Substituting into (3.3) and (*iv*), one obtains

$$\begin{aligned} &O(f; d(x_{2n+1}, x_{2n+2})) \\ \leq\ &\lambda O(f; \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}) \\ =\ &\lambda \max\{O(f; d(x_{2n}, x_{2n+1})), O(f; d(x_{2n+1}, x_{2n+2}))\}. \end{aligned} \tag{3.4}$$

If $O(f; d(x_{2n+1}, x_{2n+2})) \geq O(f; d(x_{2n}, x_{2n+1}))$, then from (3.4) we have

$$O(f; d(x_{2n+1}, x_{2n+2})) \leq \lambda O(f; d(x_{2n+1}, x_{2n+2}))$$

which is a contradiction ($\lambda < 1$). Thus $O(f; d(x_{2n+1}, x_{2n+2})) < O(f; d(x_{2n}, x_{2n+1}))$ and so from (3.4) one obtains

$$O(f; d(x_{2n+1}, x_{2n+2})) \leq \lambda O(f; d(x_{2n}, x_{2n+1})).$$

Similarly, we get that

$$O(f; d(x_{2n}, x_{2n+1})) \leq \lambda O(f; d(x_{2n-1}, x_{2n})).$$

Thus, for any $n \geq 1$ we have

$$\begin{aligned} O(f; d(x_n, x_{n+1})) &\leq \lambda O(f; d(x_{n-1}, x_n)) \\ &\leq \lambda^2 O(f; d(x_{n-2}, x_{n-1})) \\ &\vdots \\ &\leq \lambda^n O(f; d(x_0, x_1)). \end{aligned} \tag{3.5}$$

Taking the limit of (3.5), as $n \to \infty$, we have

$$\lim_{n\to\infty} O(f; d(x_n, x_{n+1})) = 0,$$

which, from (i), implies that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Therefore, $\{x_n\}$ is Cauchy sequence. (Similarly, see [2]).

Since X is complete, there is a $p \in X$ such that

$$\lim_{n\to\infty} x_n = p.$$

From (3.1) we have,

$$\begin{aligned} O(f; d(x_{2n+1}, T_\beta p)) &= O(f; d(T_\alpha x_{2n}, T_\beta p)) \\ &\leq \lambda \max \left\{ \begin{array}{c} d(x_{2n}, p), d(x_{2n}, T_\alpha x_{2n}), d(p, T_\beta p), \\ \frac{1}{2}[d(x_{2n}, T_\beta p) + d(p, T_\alpha x_{2n})] \end{array} \right\}. \end{aligned}$$

Taking the limit as $n \to \infty$ we get

$$O(f; d(p, T_\beta p)) \leq \lambda O(f; d(p, T_\beta p)),$$

which implies that

$$O(f; d(p, T_\beta p)) = 0,$$

which from (i), implies that $d(p, T_\beta p) = 0$; hence $T_\beta p = p$.

Now we show that p is a fixed point of all $\{T_\alpha\}_{\alpha \in J}$. Let $\alpha \in J$ be arbitrary. Then from (3.1) with $x = y = p = T_\beta p$ we have

$$\begin{aligned} O(f; d(T_\alpha p, p)) &= O(f; d(T_\alpha p, T_\beta p)) \leq \lambda(\alpha) O(f; m(p, p)) \\ &\leq \lambda(\alpha) \max \left\{ \begin{array}{c} O(f; d(p, p)), O(f; d(p, T_\alpha p)), O(f; d(p, T_\beta p)), \\ \frac{1}{2}[O(f; d(p, T_\beta p)) + O(f; d(p, T_\alpha p))] \end{array} \right\} \\ &= \lambda(\alpha) \max \left\{ O(f; d(p, T_\alpha p)), \frac{1}{2} O(f; d(p, T_\alpha p)) \right\}. \end{aligned}$$

Therefore, we get

$$O(f; d(T_\alpha p, p)) \leq \lambda(\alpha) O(f; d(p, T_\alpha p))$$

which implies that

$$O(f; d(T_\alpha p, p)) = 0,$$

which, from (i), implies that $d(T_\alpha p, p) = 0$ or $T_\alpha p = p$. Thus, all T_α have a common fixed point.

Now we prove the uniqueness of the fixed point p. Suppose that q is another a fixed point of T_β. Then it follows, as above, that q is a common fixed point of all $\{T_\alpha\}_{\alpha\in J}$. Thus, from (3.1) we have

$$\begin{aligned} O(f; d(p,q)) &= O(f; d(T_\alpha p, T_\beta q)) \\ &\leq \lambda O(f; m(p,q)) \\ &= \lambda O(f; d(p,q)), \end{aligned}$$

which implies that

$$O(f; d(p,q)) = 0,$$

which, from (i), implies that $d(p,q) = 0$. Hence $p = q$. Thus, p is a unique common fixed point of all $\{T_\alpha\}_{\alpha\in J}$. □

Remark 1. *It is clear that Theorem 3 is a generalization of Theorem 1 in [2].*

Remark 2. *We can have new result, if we combine Theorem 3 and some examples for $O(f;.)$.*

Remark 3. *Theorem 3 is a generalization of Theorem 1, in fact letting $f = I$ (identity map) and $O(f;t) = t$ in* (3.1) *(it is obvious that $O(f\ ;.) \in \Theta$) one has*

$$d(T_\alpha x, T_\beta y) = O(f; d(T_\alpha x, T_\beta y)) \leq \lambda O(f; m(x,y)) = \lambda m(x,y),$$

thus Ćirić's [6,7] generalized contraction also satisfies.

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