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RANK 2 ARITHMETICALLY COHEN-MACAULAY VECTOR BUNDLES ON $K3$ AND ENRIQUES SURFACES

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Abstract. Here we study arithmetically Cohen-Macaulay rank 2 vector bundles with trivial determinant on $K3$ and Enriques surfaces.

1. Introduction

Let X be either an Enriques surface or a $K3$-surface defined over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) \neq 2$. Let η_+ denote the set of all ample line bundles on X. Let E be any vector bundle on X. We will say that E is WACM or that it is *weakly arithmetically Cohen-Macaulay* if $h^1(X, E \otimes L) = h^1(X, E \otimes L^*) = 0$ for all $L \in \eta_+$. We will say that E is ACM or that it is *arithmetically Cohen-Macaulay* if it is WACM and $h^1(X, E) = 0$. We will say that E is SACM or that it is *strongly arithmetically Cohen-Macaulay* if it is ACM and $h^1(X, E \otimes \omega_X) = 0$. Hence on a $K3$ surface a vector bundle is ACM if and only if it is SACM. This definition is very natural, but different from the usual one (unless X is a $K3$ surface with $\mathrm{Pic}(X) \cong \mathbb{Z}$) in which we fix an ample $H \in \mathbb{Z}$ and only require $h^1(X, E \otimes H^{\otimes t}) = 0$ for all $t \in \mathbb{Z}$ (see [6] and references therein for many papers using the classical definition on varieties with $\mathrm{Pic}(X) \neq \mathbb{Z}$). To state our results we introduce a few definitions. We recall that an Enriques surface X is said to be *nodal* if there is an integral curve T such that $T^2 < 0$. A generic Enriques surface is not nodal ([3], Th. 4).

Theorem 1. *Let X be a non-nodal Enriques surface and E a rank 2 ACM vector bundle on X such that $\det(E) \cong \mathcal{O}_X$. Then one of the following cases occurs.*

(i) *$c_1(E) = 1$ and E is a member of the family of ACM vector bundles described in Example 1;*
(ii) *E is an extension of a line bundle A^* by its dual A.*

In case (ii) $c_2(E) = -A^2$ is an even integer. If $E \neq A \oplus A^*$ and we are in case (ii), then $c_2(E) \in \{0, 2\}$.

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Roughly speaking, the family $\{E_1\}$ of ACM vector bundles described in Example 1 depends from two parameters: each E_1 uniquely determines a point $Z \in X$ and a very general point $Z \in X$ determines one of these vector bundles.

We will say that a $K3$-surface X has Property (+) if X contains no smooth rational curve, i.e. (adjunction formula) no integral curve T such that $T^2 = -2$. The adjunction formula shows that X has Property (+) if and only if there is no effective divisor D on X such that $D^2 < 0$. Hence X has Property (+) if and only if every effective divisor is nef. If $\mathbb{K} = \mathbb{C}$, then a global Torelli theorem makes easy to construct $K3$-surfaces with Property (+) (see [7], Lemma 4.3, for a construction of an elliptic $K3$ surface with $\rho = 2$ and Property (+)).

Theorem 2. *Let X be a $K3$-surface with Property (+) and not quasi-elliptic. Let δ be the minimal self-intersection of an ample line bundle on X. δ is a positive even integer. Let E be a rank 2 ACM vector bundle on X such that $\det(E) \cong \mathcal{O}_X$. Then one of the following cases occurs:*

- (i) *There is an integer t such that $2 \le t \le \delta/2 + 2$ and E is one of the vector bundles E_t described in Example 2; in this case $c_2(E) = t$;*
- (ii) *E is an extension of a line bundle A^* by its dual A.*

In case (ii) $c_2(E) = -A^2$ is an even integer. If $E \neq A \oplus A^*$ and we are in case (ii), then $c_2(E) \in \{0, 2, 4\}$.

If $\mathrm{char}(\mathbb{K}) \neq 2, 3$, then no surface is quasi-elliptic. Fix any integer t such that $2 \le t \le \delta/2 + 2$. Roughly speaking, the set $\{E_t\}$ of ACM vector bundles described in Example 2 for the integer t depends from $2t+(t-1)$ parameters: each E_t uniquely determines a length t zero-dimensional subschemes of X and a very general length t zero-dimensional subschemes of X determines a $(t-1)$-dimensional family of non-isomorphic bundles contained in the set $\{E_t\}$.

Remark 1. Let X be a $K3$-surface with Property (+). Assume that X has no elliptic pencil. Equivalently, assume that there is no integral curve T such that $T^2 \le 0$. If this condition is satisfied we will say that X has Property (++). Assume that X has Property (++). This assumption implies that every effective divisor $D \neq 0$ on X is nef and big. We have $h^0(X, D) \ge D^2/2 + 2$ and hence the linear system $|D|$ covers X. Fix any integral curve $T \subset X$. If T is not contained in a divisor of $|D|$, then $D \cdot T > 0$, because $|D|$ covers X. If T is contained in a divisor of $|D|$, then $D \cdot T > 0$, because $T^2 > 0$. Hence D is ample by Nakai criterion ([5], Th. 1.5.1). Use also Riemmann-Roch to see that if X has Property (++) and $L \in \mathrm{Pic}(X)$, then the following conditions are equivalent:

- (i) $L \in \eta_+$;
- (ii) $h^0(X, L) > 0$ and $L \neq \mathcal{O}_X$;
- (iii) $h^0(X, L) \ge 2$;
- (iv) $L^2 \ge 0$, $L \neq \mathcal{O}_X$, and $L^* \notin \eta_+$;
- (v) $L^2 > 0$ and $L^* \notin \eta_+$.

Theorem 3. *Let X be a $K3$-surface with Property (++). Let E be a rank 2 vector bundle on X such that $\det(E) \cong \mathcal{O}_X$ and $c_2(E) \le 0$. Then one of the following cases is true:*

- (i) *$E \cong \mathcal{O}_X^{\oplus 2}$;*
- (ii) *E there is $L \in Pic(X)$ such that L is ample and ACM, $c_2(E) = -L^2 < 0$ and $E \cong L \oplus L^*$.*

2. X an Enriques surface

In this section X is an Enriques surface defined over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) \neq 2$. Hence $\omega_X \neq \mathcal{O}_X$ and $\omega_X^{\otimes 2} \cong \mathcal{O}_X$ ([4], p. 76). Since $\mathrm{char}(\mathbb{K}) \neq 2$, $h^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$, $\omega_X \neq \mathcal{O}_X$ (i.e. ω_X has order 2) and ω_X is the only non-trivial torsion line bundle on X ([4], p. 76). The intersection product on $NS(X)$ is a perfect pairing of $\mathbb{Z}$-modules ([4], p. 78).

Let $T \subset X$ be an integral curve such that $T^2 < 0$. Since $T \cdot \omega_X = 0$, $T^2 = -2$ and $p_a(T) = 0$, i.e. $T \cong \mathbf{P}^1$. X is said to be *nodal* if there is an integral curve T such that $T^2 < 0$. A generic Enriques surface is not nodal ([3], Th. 4).

For any $M \in \mathrm{Pic}(X)$ and any rank 2 vector bundle E on X Riemann-Roch says $\chi(M) = M^2/2 + 1$ and $\chi(E) = c_1(E)^2/2 - c_2(E) + 2$. Fix any $L \in \eta_+$. Kodaira vanishing gives $h^i(X, L^*) = 0$, $i = 0, 1$ (see [3], Th. 2.6, when L is nef and big). Nakai criterion of ampleness ([5], I.5.1) shows that $\omega_X \otimes L$ is ample. Hence Kodaira vanishing ([3], Th. 2.6) and Serre duality gives $h^i(X, L) = 0$, $i = 1, 2$. Hence Rieman-Roch gives $h^0(X, L) = 1 + L^2/2$. We just checked that both $\mathcal{O}_X$ and ω_X are SACM.

Remark 2. Fix any $A \in \mathrm{Pic}(X)$. Serre duality gives that A is SACM if and only if both A and A^* are ACM.

Example 1. Fix an integer $t \geq 2$ and $L \in \eta_+$ such that $L^2/2 + 1 \geq t$. We just saw that $h^0(X, L) = L^2/2 + 1$. Since $t \leq h^0(X, L)$ and $h^1(X, L) = 0$, we have $h^1(X, \mathcal{I}_Z \otimes L) = 0$ for a general $Z \subset X$ such that $\sharp(Z) = t$. Now assume that $\mathbb{K}$ is uncountable. Since $\mathrm{Pic}^0(X)$ is countable, there are only countably many ample line bundles on X. Hence there is a non-empty set W_t of the Hilbert scheme $\mathrm{Hilb}^t(X)$ of all zero-dimensional length t subschemes of Z such that $\mathrm{Hilb}^t(X) \backslash W_t$ is a union of countably many proper algebraic subsets of $\mathrm{Hilb}^t(X)$, each $Z \in W_t$ is locally a complete intersection and $h^1(X, \mathcal{I}_Z \otimes L) = 0$ for all $L \in \eta_+$ such that $L^2/2 + 1 \geq t$ and all $Z \in W_t$. Fix any $Z \in W_t$ and consider the general extension

$$0 \to \mathcal{O}_X \to E_t \to \mathcal{I}_Z \to 0 \tag{1}$$

Since $h^0(X, \omega_X) = 0$, the Cayley-Bacharach condition is satisfied ([1], Th. 1.4) and hence E_t is locally free. Since $h^1(X, \mathcal{O}_X) = 0$, [1], Th. 1.4, gives that the set of all non-trivial extensions is parametrized by a $(t-1)$-dimensional projective space. Two non-proportional extensions gives non-isomorphic vector bundles, because $h^0(X, E_t) = 1$ and hence each E_t fits in a unique extension (1). In particular, if $t = 1$, then the point Z gives, up to isomorphisms, a unique vector bundle E_t. Now take any t. Since $Z \neq \emptyset$, $h^0(X, E_t) = 1$. Thus E_t uniquely determines Z as the scheme-theoretic locus at which any non-zero section of E_t drops rank. We have $\det(E_t) \cong \mathcal{O}_X$, $c_2(E_t) = t$ and E_t is slope properly semistable with respect to any polarization on X. Since $\mathcal{O}_X$ is spanned, $h^0(X, \mathcal{O}_X) = 1$ and $h^1(X, \mathcal{O}_X) = 0$, we have $h^1(X, \mathcal{I}_Z) = t - 1$. Hence (1) gives $h^1(X, E_1) = 0$ and $h^1(X, E_t) = t - 1 > 0$ if $t > 1$. Fix $L \in \eta_+$. We saw that $h^1(X, L) = 0$. Since $Z \in W_t$, $h^1(X, \mathcal{I}_Z \otimes L) = 0$. Hence $h^1(X, E_t \otimes L) = 0$. Since $\det(E_t) \cong \mathcal{O}_X$ and $\mathrm{rank}(E_t) = 2$, $E_t^* \cong E_t$. Hence $h^1(X, E \otimes L^*) = h^1(X, E \otimes (L \otimes \omega_X))$. Since $L \otimes \omega_X \in \eta_+$ by Nakai criterion of ampleness ([5], I.5.1), we get that E_t is WACM and it is ACM if and only if $t = 1$. Tensor the case $t = 1$ of (1) with ω_X. Since $h^0(X, \omega_X) = h^1(X, \omega_X) = 0$, we get $h^1(X, E_1 \otimes \omega_X) = 1$. Hence E_1 is not SACM. Obviously, if E_t is as above, then $E_t \otimes \omega_X$ is WACM. Since $\mathrm{rank}(E) = 2$ and

$\omega_X^{\otimes 2} \cong \mathcal{O}_X$, $\det(E_t \otimes \omega_X) \cong \mathcal{O}_X$. Hence $(E_t \otimes \omega_X)^* \cong E_t \otimes \omega_X$. By tensoring (1) with the numerically trivial line bundle ω_X we get $c_2(E_t \otimes \omega_X) = t$. Serre duality gives $h^1(X, E_t \otimes \omega_X) = h^1(X, (E_t \otimes \omega_X)^* \otimes \omega_X) = h^1(X, E_t)$. Hence $E_t \otimes \omega_X$ is ACM if and only if $t = 1$. $E_t \otimes \omega_X$ is properly semistable in the sense of Mumford-Takemoto with respect to any polarization of X. By tensoring (1) with ω_X we get that $h^0(X, E_t \otimes \omega_X) = 0$. Hence E_t and $E_t \otimes \omega_X$ are not isomorphic. Now assume $t \geq 2$. Fix any $Z \in W_t$ and consider a general extension

$$0 \to \omega_X \to G_t \to \mathcal{I}_Z \to 0 \tag{2}$$

Since $h^0(X, \mathcal{I}_{Z'}) = 0$ for any length $t-1$ subscheme Z' of Z, the Cayley-Bacharach condition is satisfied and hence G_t is locally free. We need to exclude the case $t = 1$, because in this case the Cayley-Bacharach condition is not satisfied and hence the middle term of any such extension is not locally free. $\det(G_t) \cong \omega_X$ and $c_2(G_t) = t$. As above we see that G_t is WACM, but not ACM. G_t is properly semistable with respect to any polarization of X. Again, each Z_t determines a $(t-1)$-dimensional family of vector bundles G_t and each of them uniquely determine Z as the scheme at which any non-zero section of $H^0(X, G_t \otimes \omega_X^*)$ drops rank. Fix $H \in \eta_+$.

Claim: E_t and G_t are not an extensions of two line bundles.

Proof of the Claim: We will only write down the proof for E_t, since the one for G_t requires only notational modifications (e.g. using $h^0(X, G_t \otimes \omega_X)$ instead of $h^0(X, E_t)$). In order to obtain a contradiction we assume that E is an extension of a line bundle M^* by M. Here we use $\det(E_t) \cong \mathcal{O}_X$. Set $z := M \cdot H$. Notice that E_t is properly H-semistable. Hence $z \leq 0$. Since $h^0(X, E_t) > 0$, either $h^0(X, M) > 0$ or $h^0(X, M^*) > 0$. First assume $z < 0$. Hence $h^0(X, M) = 0$. Thus $h^0(X, M^*) > 0$. However, any non-zero section σ of E drops rank exactly at the non-zero zero-dimensional scheme Z. Since $h^0(X, M) = 0$, σ drops rank on the zero locus D of the section σ' of M^* induced by σ. Since D has pure codimension one, we got a contradiction. Now assume $z = 0$. Since $H \in \eta_+$, $H \cdot M^* = -H \cdot M = 0$, and $h^0(X, M) + h^0(X, M^*) > 0$, M must be trivial. Thus $c_2(E_t) = 0$, contradiction.

Proposition 1. *Fix an integer $t \geq 2$ and $L \in \eta_+$. The following conditions are equivalent:*

(a) $t \leq L^2/2 + 1$;
(b) $h^1(X, E_t \otimes L) = 0$;
(c) $h^1(X, E_t \otimes L^*) = 0$;
(d) $h^1(X, G_t \otimes L) = 0$;
(e) $h^1(X, G_t \otimes L^*) = 0$.

Proof. We will do the proofs for E_t, since the proofs for G_t require only notational modifications. First assume $t \leq L^2/2 + 1$. We saw that $h^1(X, L) = 0$. Since $Z \in W_t$, $h^1(X, \mathcal{I}_Z \otimes L) = 0$. Hence $h^1(X, E_t \otimes L) = 0$, i.e. (a) implies (b). Since $\det(E_t) \cong \mathcal{O}_X$ and $\mathrm{rank}(E_t) = 2$, $E_t^* \cong E_t$. Hence $h^1(X, E \otimes L^*) = h^1(X, E \otimes (L \otimes \omega_X))$. Since $L \otimes \omega_X \in \eta_+$ by Nakai criterion of ampleness ([5], I.5.1) and $(L \otimes \omega_X)^2 = L^2$, the definition of the set W_t gives that (a) implies (c). Now assume $t \geq L^2/2 + 2 = 1 + h^0(X, L)$. Hence $h^1(X, \mathcal{I}_Z \otimes L) > 0$. Since $h^1(X, L) = 0$ ([3], Th. 2.6), tensoring (1) with L we get $h^1(X, L) = 0$. Since $(L \otimes \omega_X)^2 = L^2$, we also get $h^1(X, E \otimes (\omega_X \otimes L) > 0$. Since $E_t^* \cong E_t$, Serre duality gives $h^1(X, E_t \otimes L^*) > 0$. Since $L^2 = (L \otimes \omega_X)^2$, we also see that (a), (b) and (c) are equivalent. □

Proof of Theorem 1. Let E be a rank 2 ACM vector bundle on X such that $\det(E) \cong \mathcal{O}_X$. Since $\chi(\mathcal{O}_X) = 1$ and ω_X and $\det(E)$ are numerically trivial, Riemann-Roch gives $\chi(F) = c_1(E)^2/2 - c_2(E) + 2$. Since $h^1(X, E) = 0$ and $c_1(E)$ is numerically trivial, we get $h^0(X, E) + h^2(X, E) - c_2(E) + 2 \geq 0$. Fix $H \in \eta_+$ and let A be the rank 1 subsheaf of E such that $w := A \cdot H$ is maximal. The maximality of the integer w and the ampleness of H gives that A is saturated in E. Since $\det(E) \cong \mathcal{O}_X$, we get an exact sequence

$$0 \to A \to E \to \mathcal{I}_Z \otimes A^* \to 0 \tag{3}$$

with Z a zero-dimensional subscheme of X and $c_2(E) = \text{length}(Z) - A^2$. Since $h^1(X, E) = 0$, we get $h^1(X, \mathcal{I}_Z \otimes A^*) \leq h^2(X, A)$ and $h^1(X, A) \leq h^0(X, \mathcal{I}_Z \otimes A^*)$. Serre duality gives $h^2(X, A) = h^0(X, A^* \otimes \omega_X)$.

(a) Here we assume $w = 0$. Since H is ample and ω_X has order 2, $h^0(X, A^* \otimes \omega_X) > 0$ if and only if $A \cong \omega_X$. Hence $h^1(X, \mathcal{I}_Z \otimes A^*) = 0$ if $A \neq \omega_X$. For the same reason $h^0(X, A) + h^0(X, A^*) > 0$ if and only if $A \in \{\mathcal{O}_X, \omega_X\}$. First assume $A \notin \{\mathcal{O}_X, \omega_X\}$. We get $h^1(X, \mathcal{I}_Z \otimes A^*) = 0$. Hence $h^1(X, A^*) = 0$ and $\text{length}(Z) \leq h^0(X, A^*) = 0$. Thus E is an extension of A^* by A.

(a1) Here we assume $h^1(X, A^{\otimes 2}) > 0$. If $h^0(X, A^{\otimes 2}) = h^2(X, A^{\otimes 2}) = 0$, then Riemann-Roch gives $A^2 < 0$ and hence $A^2 = -2$. Now assume $h^0(X, A^{\otimes 2}) + h^2(X, A^{\otimes 2}) > 0$ and that X is not nodal. Since X has no curve with negative self-intersection, every effective divisor is nef. Since $h^0(X, A^{\otimes 2}) + h^2(X, A^{\otimes 2}) > 0$ and ω_X is numerically trivial, we get that $A^{\otimes 2}$ is nef. Hence $A^2 \geq 0$. Riemann-Roch gives that either $h^0(X, A) > 0$ or $h^0(X, A^* \otimes \omega_X) > 0$. Hence either $h^0(X, A^{\otimes 2}) > 0$ or $h^0(X, A^{\otimes -2}) > 0$. Since $w = 0$ any of these inequalities implies $A^{\otimes 2} \in \{\mathcal{O}_X, \omega_X\}$. We cannot have $A^{\otimes 2} \cong \omega_X$, because $\text{Tors}(X) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by ω_X. Hence $A^{\otimes 2} \cong \mathcal{O}_X$, contradicting the assumption $h^1(X, A^{\otimes 2}) > 0$. In summary, if $w = 0$, $A \notin \{\mathcal{O}_X, \omega_X\}$ and $E \neq A \oplus A^*$, then $A^2 = -2$.

(a2) Here we assume $h^1(X, A^{\otimes 2}) = 0$. Hence (4) splits. Hence both A and A^* are ACM. Remark 2 gives that both A and A^* are SACM. Hence E is SACM.

(a3) Here we assume $A \in \{\mathcal{O}_X, \omega_X\}$. First assume $Z \neq \emptyset$. Since $\text{length}(Z) \leq h^2(X, A)$, we get $A \cong \mathcal{O}_X$ and that Z is a point. Hence E is one of the vector bundles E_1 described in Example 1. If $Z = \emptyset$, then $E \cong A \oplus A^*$, because $h^1(X, \mathcal{O}_X) = 0$.

(b) Here we assume $w > 0$. Hence $h^0(X, A^*) = 0$. Serre duality gives $h^2(X, A) = 0$. Hence $Z = \emptyset$ and $h^1(X, A) = h^1(X, A^*) = 0$. Thus Riemann-Roch gives $h^0(X, A) = A^2/2 + 1$ and $h^2(X, A^*) = A^2/2 + 1$. Hence $A^2 \geq -2$. Since $h^0(X, A^*) = h^2(X, A) = 0$, (4) gives $h^0(X, E) = h^0(X, A)$ and $h^2(X, E) = h^2(X, A^*)$. Since $Z = \emptyset$, (4) gives $c_2(E) = -A^2$. Since $\det(E) \cong \mathcal{O}_X$, $\chi(E) = -c_2(E) + 2 = A^2 + 2$.

(b1) Here we assume $h^1(X, A^{\otimes 2}) > 0$. As in case (a1) we get $A^2 = -2$ if $h^0(X, A^{\otimes 2}) = h^2(X, A^{\otimes 2}) = 0$. Now assume $A^2 \geq 0$ and that X is not nodal. Riemma-Roch gives $h^0(X, A^{\otimes 2}) + h^2(X, A^{\otimes 2}) > 0$. Hence either $h^0(X, A^{\otimes 2})) > 0$ or $h^0(X, A^{\otimes -2} \otimes \omega_X) > 0$. The latter inequality cannot occur, because $w > 0$. Hence $A^{\otimes 2}$ is effective. Since X is not nodal, $A^{\otimes 2}$ is nef. Hence the assumption $h^1(X, A^{\otimes 2}) > 0$ and the vanishing theorem [3], Theorem 2.6, for nef and big effective divisors gives $A^2 = 0$.

(b2) Here we assume $h^1(X, A^{\otimes 2}) = 0$. Hence (4) splits. Hence both A and A^* are ACM. Remark 2 gives that both A and A^* are SACM. Hence E is SACM.

(c) Here we assume $w < 0$. Hence E is H-stable in the sense of Mumford and Takemoto. Since $c_1(E) \cdot H = 0$, this implies $h^0(X, E) = 0$. Since E is H-stable,

$E^* \otimes \omega_X$ is an H-stable with trivial determinant. Hence $h^0(X, E \otimes \omega_X) = 0$, i.e. $h^2(X, E) = 0$. Since E is ACM, Riemman-Roch gives $-c_2(E) + 2 = \chi(E) = 0$, i.e. $-A^2 + \mathrm{length}(Z) = 2$. Riemann-Roch for A gives that A^2 is an even integer. Since $w < 0$, $h^2(X, A) = 0$. Hence $h^1(X, E) = 0$ implies $h^1(X, \mathcal{I}_Z \otimes A^*$. Hence $h^1(X, A^*) = 0$ and $h^0(X, A^*) \geq \mathrm{length}(Z)$. Thus $Z = \emptyset$ if $h^0(X, A^*) = 0$. Hence $h^0(X, A^*) = 0$ implies $c_2(E) = -A^2 = 2$.

(c1) Here we assume that X is not nodal. Assume $h^0(X, A^*) > 0$. Since X is not nodal, $A^2 \geq 0$. Hence if X is not nodal and $Z \neq \emptyset$, then $\mathrm{length}(Z) = 2$ and $A^2 = 0$. However, $h^1(X, A^*) = 0$ and $A^2 = 0$, gives $h^0(X, A^*) \leq 1 < \mathrm{length}(Z)$. Hence if X is not nodal, then $Z = \emptyset$, $c_2(E) = 2 = A^2$ and E is an extension of A^* by A. □

Remark 3. Fix $A, B \in \mathrm{Pic}(X)$. Let E be the middle term of an extension ϵ of B by A. If $\epsilon = 0$ and $A \cong B$, then $h^0(X, End(E)) = 4$. If $\epsilon = 0$ and $A \neq B$, then $h^0(X, End(E)) = 2$ and any element of $H^0(X, End(E))$ may be put is a diagonal form. Now assume $\epsilon \neq 0$ and $h^0(X, E \otimes A^*) = 1$. The latter condition is satisfied if there is an ample line bundle H such that either $A \cdot H > B \cdot H$ or $A \neq B$ and $A \cdot H = B \cdot H$. Then $h^0(X, End(E)) = 1 + h^0(X, A \otimes B^*)$ and every element of $H^0(X, End(E))$ may be put in a triangular form with the same constant on the two diagonal elements and an element of $H^0(X, A \otimes B^*)$ as the $(1, 2)$-entry.

3. X a $K3$-surface

In this section X is a smooth and projective $K3$-surface. Hence $\omega_X \cong \mathcal{O}_X$, $h^1(X, \mathcal{O}_X) = 0$, $b_2(X) = 22$ and $\mathrm{Pic}(X) \cong \mathbb{Z}^\rho$ for some integer ρ such that $1 \leq \rho \leq 22$. If $\mathrm{char}(\mathbb{K}) = 0$, then $\rho \leq 20$. For any $L \in \mathrm{Pic}(X)$ and any rank 2 vector bundle on X we have $\chi(L) = L^2/2 + 2$ and $\chi(E) = \det(E)^2/2 - c_2(E) + 4$ (Rieman-Roch). Hence L^2 is always an even integer. Now assume $L \in \eta_+$. Hence $h^0(X, L^*) = h^2(X, L) = 0$. Kodaira vanishing gives $h^1(X, L) = h^1(X, L^*) = 0$. In positive characteristic we use [8] to get Kodaira vanishing. However, to apply [8], Cor. 8, we need to assume that X is not quasi-elliptic. We just recall that no surface is quasi-elliptic if $\mathrm{char}(\mathbb{K}) \neq 2, 3$. Hence $h^0(X, L) = L^2/2 + 2$ for every $L \in \eta_+$ if $\mathrm{char}(\mathbb{K}) \neq 2, 3$.

Example 2. Set $\delta := \min\{L^2 : L \in \eta_+\}$. δ is a positive even integer. Fix an integer t such that $2 \leq t \leq \delta/2 + 2$. Fix $L \in \eta_+$. Since $L^2 \geq \delta$, we have $h^0(X, L) = L^2/2 + 1 \geq t$. Since $h^0(X, L) \geq t$ and $h^1(X, L) = 0$, we have $h^1(X, \mathcal{I}_Z \otimes L) = 0$ for a general $Z \subset X$ such that $\sharp(Z) = t$. Now assume that $\mathbb{K}$ is uncountable. Since $\mathrm{Pic}^0(X)$ is countable, there are only countably many ample line bundles on X. Hence there is a non-empty set W_t of the Hilbert scheme $\mathrm{Hilb}^t(X)$ of all zero-dimensional length t subschemes of Z such that $\mathrm{Hilb}^t(X) \backslash W_t$ is a union of countably many proper algebraic subsets of $\mathrm{Hilb}^t(X)$, each $Z \in W_t$ is locally a complete intersection and $h^1(X, \mathcal{I}_Z \otimes L) = 0$ for all $L \in \eta_+$ such that $L^2/2 + 2 \geq t$ and all $Z \in W_t$. Fix any $Z \in W_t$ and consider the general extension (1). Since $h^0(X, \omega_X) = 0$ and $t \geq 2$, the Cayley-Bacharach condition is satisfied ([1], Th. 1.4) and hence E_t is locally free. We have $\det(E_t) \cong \mathcal{O}_X$, $c_2(E_t) = t$ and E_t is slope properly semistable with respect to any polarization on X. Since $h^1(X, \mathcal{O}_X) = 0$, [1], Th. 1.4, gives that the set of all non-trivial extensions is parametrized by a $(t-1)$-dimensional projective space. Since $Z \neq \emptyset$, $h^0(X, E_t) = 1$. Thus E_t uniquely determines Z as the scheme-theoretic locus at which any non-zero section of E_t drops rank. Since

$\mathcal{O}_X$ is spanned, $h^0(X, \mathcal{O}_X) = 1$ and $h^1(X, \mathcal{O}_X) = 0$, we have $h^1(X, \mathcal{I}_Z) = t - 1$. Hence (1) gives $h^1(X, E_1) = 0$ and $h^1(X, E_t) > 0$ if $t > 1$. Fix $L \in \eta_+$. We saw that $h^1(X, L) = 0$. Since $Z \in W_t$ and $t \leq h^0(X, L)$, $h^1(X, \mathcal{I}_Z \otimes L) = 0$. Hence $h^1(X, E_t \otimes L) = 0$. Since $\det(E_t) \cong \mathcal{O}_X$ and $\mathrm{rank}(E_t) = 2$, $E_t^* \cong E_t$. Hence $h^1(X, E \otimes L^*) = h^1(X, E \otimes L)$. Thus E_t is WACM, but not ACM. E_t is properly semistable in the sense of Mumford-Takemoto with respect to any polarization of X. As in the case of an Enriques surface we see that E is not an extension of two line bundles. Conversely, take a zero-dimensional scheme $Z \subset X$, $Z \neq \emptyset$ and take any extension (1) with locally free middle term, F. set $t := \mathrm{length}(Z)$. Since F is locally free, the Cayley-Bacharach condition must be satisfied and hence $t \geq 2$. Now assume that F is WACM. Fix $L \in \eta_+$. Since $h^2(X, L) = 0$ and $h^1(X, F \otimes L) = 0$, we get $h^1(X, \mathcal{I}_Z \otimes L) = 0$. Hence $t \geq h^0(X, L)$. Taking L with minimal self-intersection, we get $t \leq \delta/2 + 2$. Since $h^1(X, \mathcal{I}_Z \otimes L) = 0$ for all $L \in \eta_+$, we see that all WACM non-trivial vector bundles E with $\det(E) \cong \mathcal{O}_X$, $h^0(X, E) > 0$, $h^0(X, E(-D)) = 0$ for every divisor $D > 0$ are given by our construction for some integer $t := c_2(E)$ such that $2 \leq t \leq \delta/2 + 2$.

Proof of Theorem 2. Let E be a rank 2 ACM vector bundle on X. Fix $H \in \eta_+$ and let A be the rank 1 subsheaf of E such that $w := A \cdot H$ is maximal. The maximality of the integer w and the ampleness of H gives that A is saturated in E. Since $\det(E) \cong \mathcal{O}_X$, we get an exact sequence

$$0 \to A \to E \to \mathcal{I}_Z \otimes A^* \to 0 \tag{4}$$

with Z a zero-dimensional subscheme of X and $c_2(E) = \mathrm{length}(Z) - A^2$. Since $h^1(X, E) = 0$, we get $h^1(X, \mathcal{I}_Z \otimes A^*) \leq h^2(X, A)$ and $h^1(X, A) \leq h^0(X, \mathcal{I}_Z \otimes A^*)$. Serre duality gives $h^2(X, A) = h^0(X, A^*)$.

(a) Here we assume $w = 0$. Since H is ample, $h^0(X, A^*) > 0$ if and only if $A \cong \mathcal{O}_X$. Hence $h^1(X, \mathcal{I}_Z \otimes A^*) = 0$ if $A \neq \mathcal{O}_X$. For the same reason $h^0(X, A) + h^0(X, A^*) > 0$ if and only if $A \cong \mathcal{O}_X$. First assume $A \neq \mathcal{O}_X$. We get $h^1(X, \mathcal{I}_Z \otimes A^*) = 0$. Hence $h^1(X, A^*) = 0$ and $\mathrm{length}(Z) \leq h^0(X, A^*) = 0$. Thus E is an extension of A^* by A if $A \neq \mathcal{O}_X$.

(a1) Here we assume $A \neq \mathcal{O}_X$ and $h^1(X, A^{\otimes 2}) > 0$. If

$$h^0(X, A^{\otimes 2}) = h^2(X, A^{\otimes 2}) = 0,$$

then Riemann-Roch gives $A^2 < 0$ and hence $A^2 \in \{-4, -2\}$. Now assume

$$h^0(X, A^{\otimes 2}) + h^2(X, A^{\otimes 2}) > 0$$

and that X has Property (+). Since X has no curve with negative self-intersection, every effective divisor is nef. Since $h^0(X, A^{\otimes 2}) + h^2(X, A^{\otimes 2}) > 0$ and $\omega_X \cong \mathcal{O}_X$, we get that $A^{\otimes 2}$ is nef. Hence $A^2 \geq 0$. Assume $A^2 > 0$. Riemann-Roch gives that either $h^0(X, A) > 0$ or $h^0(X, A^* \otimes \omega_X) > 0$. Hence either $h^0(X, A^{\otimes 2}) > 0$ or $h^0(X, A^{\otimes -2}) > 0$. Since $w = 0$ any of these inequalities implies $A^{\otimes 2} \cong \mathcal{O}_X$, contradicting the assumption on A and the fact that $\mathrm{Pic}(X)$ has no torsion.

(a2) Here we assume $h^1(X, A^{\otimes 2}) = 0$. Hence (4) splits. Hence both A and A^* are ACM.

(a3) Here we assume $A \cong \mathcal{O}_X$. Since $\mathrm{length}(Z) \leq h^0(X, A^*) = 1$, Z is a point. Since $\omega_X \cong \mathcal{O}_X$ and Z is a point, we get the Cayley-Bacharach condition is not satisfied and hence the middle term of any extension (4) with $\mathcal{O}_X$ and Z a point is not locally free, contradiction. Hence $Z = \emptyset$ if $w = 0$.

(b) Here we assume $w > 0$. Hence $h^0(X, A^*) = 0$. Serre duality gives $h^2(X, A) = 0$. Hence $Z = \emptyset$ and $h^1(X, A) = h^1(X, A^*) = 0$. Thus Riemann-Roch gives $h^0(X, A) = A^2/2 + 2$ and $h^2(X, A^*) = A^2/2 + 2$. Since $h^0(X, A^*) = h^2(X, A) = 0$, (4) gives $h^0(X, E) = h^0(X, A)$ and $h^2(X, E) = h^2(X, A^*)$. Since $Z = \emptyset$, (4) gives $c_2(E) = -A^2$. Since $\det(E) \cong \mathcal{O}_X$, $\chi(E) = -c_2(E) + 4 = A^2 + 4$. Since $h^1(X, E) = 0$, $\chi(E) \geq 0$. Hence $A^2 \geq -4$. Riemann-Roch gives that A^2 is an even integer.

(b1) Here we assume $h^1(X, A^{\otimes 2}) > 0$. As in case (a1) we get $-4 \leq A^2 \leq -2$ if $h^0(X, A^{\otimes 2}) = h^2(X, A^{\otimes 2}) = 0$. Now assume $A^2 \geq 0$, that X has Property (+) and that X is not quasi-elliptic. Riemann-Roch gives $h^0(X, A^{\otimes 2}) + h^2(X, A^{\otimes 2}) > 0$. Hence either $h^0(X, A^{\otimes 2})) > 0$ or $h^0(X, A^{\otimes -2} \otimes \omega_X) > 0$. The latter inequality cannot occur, because $w > 0$. Hence$A^{\otimes 2}$ is effective. Since X has Property (+), $A^{\otimes 2}$ is nef. Hence the assumption $h^1(X, A^{\otimes 2}) > 0$ and (assuming X not quasi-elliptic) the vanishing theorem [8], Cor. 8, for nef and big line bundles gives $A^2 = 0$.

(b2) Here we assume $h^1(X, A^{\otimes 2}) = 0$. Hence (4) splits. Hence both A and A^* are ACM.

(c) Here we assume $w < 0$. Hence E is H-stable in the sense of Mumford and Takemoto. Since $c_1(E) \cdot H = 0$, this implies $h^0(X, E) = 0$. Since E is H-stable, E^* is H-stable. Hence $h^0(X, E) = 0$, i.e. $h^2(X, E) = 0$. Since E is ACM, Riemman-Roch gives $-c_2(E) + 4 = \chi(E) = 0$, i.e. $-A^2 + \text{length}(Z) = 4$. Riemann-Roch for A gives that A^2 is an even integer. Since $w < 0$, $h^2(X, A) = 0$. Hence $h^1(X, E) = 0$ implies $h^1(X, \mathcal{I}_Z \otimes A^*)$. Hence $h^1(X, A^*) = 0$ and $h^0(X, A^*) \geq \text{length}(Z)$. Thus $Z = \emptyset$ if $h^0(X, A^*) = 0$

(c1) Here we assume that X has Property (+). Assume $Z \neq \emptyset$. Hence $h^0(X, A^*) > 0$. Since X has Property (+), $A^2 \geq 0$. Hence if X has Property (+) and $Z \neq \emptyset$, then $\text{length}(Z) = 4$ and $A^2 = 0$. However, $h^1(X, A^*) = 0$, $h^2(X, A^*) = h^0(X, A) = 0$ and $A^2 = 0$ give $h^0(X, A^*) = 2 < 4 = \text{length}(Z)$, contradiction. Since $Z = \emptyset$, $c_2(E) = -A^2 = 4$. □

Remark 4. Let X be a $K3$-surface such that $\text{Pic}(X) \cong \mathbb{Z}$. Let δ be the self-intersection of a generator of $\text{Pic}(X)$. Every line bundle on X is ACM. Hence the proof of Theorem 2 shows that a rank 2 vector bundle on X such that $\det(E) \cong \mathcal{O}_X$ is ACM if and only if one of the following conditions is satisfied:

(i) $E \cong A \otimes A^*$ for some $A \in \text{Pic}(X)$;
(ii) there is an integer t such that $2 \leq t \leq \delta/2 + 2$ such that E is one of the vector bundles E_t described in Example 2.

Proposition 2. *Let X be a projective $K3$ surface. The following conditions are equivalent:*

(i) *$Pic(X) \cong \mathbb{Z}$;*
(ii) *every line bundle on X is ACM;*
(iii) *every line bundle on X is WACM;*
(iv) *every ample line bundle on X is ACM;*
(v) *every ample line bundle on X is WACM.*

Proof. The first part of Remark 4 gives that (i) implies (ii). Hence it is sufficient to show that if $\rho \geq 2$, then there is an ample line bundle on X which is not WACM. Since $\rho \geq 2$, the intersection form on $\text{Pic}(X)$ is not definite positive by Hodge Index theorem. Hence there is $A \in \text{Pic}(X)$ such that $A^2 < 0$. Set $B := A^{\otimes 2}$. Since A^2 is an even integer $B^2 \leq -4$. Hence $\chi(B) = B^2 + 2 < 0$. Hence $h^1(X, B) > 0$. Since every Cartier divisor on a projective variety is the difference of two very ample divisors, there are ample R, L such that $B := R \otimes L^*$. Since $h^1(X, B) > 0$, R is not WACM. $\square$

Proof of Theorem 3. Since X has Property (++), it is not quasi-elliptic and hence we may use Kodaira vanishing on X ([8], Cor. 8). Take E given by an extension (4). We saw in the proof of Theorem 2 that $Z = \emptyset$ and hence $c_2(E) = -A^2$. First assume $A^2 = 0$, i.e. $c_2(E) = 0$. Since $\chi(A) = 2$, either A or A^* must have a section. Since $A^2 = 0$ and X has Property (++), we get $A \cong \mathcal{O}_X$ and hence $E \cong \mathcal{O}_X^{\oplus 2}$. Now assume $A^2 > 0$, i.e. $c_2(E) < 0$. Hence either A is ample or A^* is ample. In both cases we have $h^1(X, A^{\otimes 2}) = 0$ by Kodaira vanishing and Serre duality. Hence $E \oplus A \oplus A^*$, i.e. we are in case (ii). $\square$

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