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ON A GENERALIZED CLASS OF ANALYTIC FUNCTIONS WITH BOUNDED TURNING

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Abstract. In this paper, we consider the classes of analytic functions which are defined by conditions joining ideas of analytic functions with generalized bounded turning and bounded boundary rotation. Inclusion and radii results for these classes are studied.

1. Introduction

Let A denote the class of functions f, given by,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

analytic in the unit disc $E = \{z : |z| < 1\}$. Let $P_k(\beta), k \geq 2, 0 \leq \beta < 1$, be the class of functions $p(z)$, with $p(0) = 1$, and defined as

$$p(z) = \left(\frac{k}{2} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{2} - \frac{1}{2}\right) p_2(z), \tag{2}$$

where $Re\{p_i(z)\} > 0, \quad i = 1, 2$, and $z \in E$.

The class $P_k(0) \equiv P_k$ was introduced in [4], and $P_2(0) \equiv P$ is the class of functions with positive real part.

Let

$$J(\alpha, f) = (1-\alpha) f'(z) + \alpha \left(1 + \frac{z f''(z)}{f'(z)}\right), \tag{3}$$

for α real and $f \in A$.

Then we define the classes $N_\alpha(k, \beta)$ and $P_k^{'}(\beta)$, for $0 \leq \beta < 1$, as follows.

$$\begin{aligned} N_\alpha(k,\beta) &= \{f \in A, \quad J(\alpha, f) \in P_k(\beta), \quad z \in E\} \\ P_k^{'}(\beta) &= \{f \in A, \quad f' \in P_k(\beta), \quad z \in E\}. \end{aligned}$$

We note that $N_1(k, 0) \equiv V_k$, the well-known class of functions with bounded boundary rotation and with $k = 2$, we obtain the class $N_\alpha(2, \beta) \equiv H_\alpha(\beta)$ discussed in

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[6]. The class $N_0(2,0) \equiv P_2^{'}(2)$ is the class $P^{'}$ consisting of functions with bounded turning [1, p 101].

Lemma 1.1[2]. Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and $\Psi(u,v)$ be a complex -valued function satisfying the conditions:
(i). $\Psi(u,v)$ is continuous in a domain D
(ii). $(1,0) \in D$ and $\Psi(1,0) > 0$.
(iii). $Re\{\Psi(iu_2, v_1) \leq 0$ whenever $(iu_2, v_1) \in D$ and $v_1 \leq \frac{-1}{2}(1+u_2^2)$.

Let $p(z) = 1 + c_1 z + c_2 z^2 + \dots$, regular in the unit disc E, such that $(p(z), zp'(z)) \in D, \quad \forall z \in E$. If $Re\{\Psi(p(z), zp'(z))\} > 0$ for $z \in E$, then $Rep(z) > 0, \quad z \in E$.

Lemma 1.2 [5]. Let p be analytic function in E with $p(0) = 1$ and $Re\{p(z)\} > 0, \quad z \in E$. Then, for $s > 0$ and $\nu \neq -1$(complex),

$$Re\left\{p(z) + \frac{szp'(z)}{p(z)+\nu}\right\} > 0,$$

for $|z| < r_0$, where r_0 is given by

$$\begin{aligned} r_0 &= \frac{|\nu+1|}{\sqrt{A + \sqrt{(A^2 - |\nu^2 - 1|^2}}}, \\ A &= 2(s+1)^2 + |\nu|^2 - 1, \end{aligned}$$

and this radius is best possible.

2. Main Results

Theorem 2.1. For $0 < \alpha \leq \gamma \leq \frac{3}{2}\alpha < 1, \quad N_\alpha(k,\gamma) \subset P_k^{'}(\beta)$, where

$$\beta = \beta(\alpha,\gamma)\left[\frac{2\alpha}{(2\alpha-\gamma) + \sqrt{(2\alpha-\gamma)^2 + 4\alpha(1-\alpha)}}\right]. \tag{4}$$

Proof. Let $f \in N_\alpha(k,\gamma)$. Then $J(\alpha, f) \in P_k(\gamma), \quad z \in E$. Let

$$f'(z)p(z) = (1-\beta)h(z) + \beta. \tag{5}$$

where $p(z), h(z)$ are analytic in E and $p(0) = h(0) = 1$.

From definitions and (5), we have

$$\left[\frac{1-\alpha}{1-\gamma}p(z) + \frac{\alpha}{(1-\gamma)}\left\{1 + \frac{zp'(z)}{p(z)}\right\} - \frac{\gamma}{1-\gamma}\right] \in P_k, \quad z \in E. \tag{6}$$

We can write

$$\begin{aligned} \left[\tfrac{1-\alpha}{1-\gamma}p(z) + \tfrac{\alpha}{1-\gamma}\right] &= \tfrac{1-\alpha}{1-\gamma}\left[p(z) + \frac{\frac{\alpha}{1-\alpha}zp'(z)}{p(z)}\right] \\ &= \tfrac{(1-\alpha)(1-\beta)}{(1-\gamma)}\left[h(z) + \tfrac{\alpha}{(1-\alpha)(1-\beta)}\frac{zh'(z)}{h(z)+\frac{\beta}{1-\beta}} + \tfrac{\beta}{1-\beta}\right]. \end{aligned} \tag{7}$$

Let

$$\begin{aligned}\frac{\alpha}{(1-\alpha)(1-\beta)} &= \alpha_1, \quad \frac{\beta}{1-\beta} = \beta_1, \\ h(z) &= \left(\frac{k}{4}+\frac{1}{2}\right) - \left(\frac{k}{4}-\frac{1}{2}\right) \Longrightarrow \\ \Longrightarrow h(z) &= \left(\frac{k}{4}+\frac{1}{2}\right) h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) h_g(z)\end{aligned}$$

Define

$$\Phi_{\alpha_1,\beta-1}(z) = \frac{1}{1-\beta_1}\frac{z}{(1-z)^{\alpha_1+1}} + \frac{\beta_1}{1+\beta_1}\frac{z}{(1-z)^{\alpha_1+2}}.$$

Then, using convolution technique, we have

$$\begin{aligned}\left(h \star \frac{\Phi_{\alpha_1,\beta_1}}{z}\right)(z) &= \left[h(z) + \frac{\alpha_1 z h'(z)}{h(z)+\beta_1}\right] \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{h_1(z) + \frac{\alpha_1 z h_1'(z)}{h_1(z)+\beta_1}\right\} \\ &\quad - \left(\frac{k}{4}-\frac{1}{2}\right)\left\{h_2(z) + \frac{\alpha_1 z h_2'(z)}{h_2(z)+\beta_1}\right\}. \qquad (8)\end{aligned}$$

Thus, using (7) and (8), we can write (6) as

$$\frac{(1-\alpha)(1-\beta)}{(1-\gamma)}\left[\left(\frac{k}{4}+\frac{1}{2}\right)\left\{h_1(z) + \frac{\alpha_1 z h_1'(z)}{h_1(z)+\beta_1} + \frac{\alpha-\gamma+\beta(1-\alpha)}{(1-\alpha)(1\beta)}\right\}\right]$$
$$-\frac{(1-\alpha)(1-\beta)}{(1-\gamma)}\left[\left(\frac{k}{4}-\frac{1}{2}\right)\left\{h_2(z) + \frac{\alpha_1 z h_2'(z)}{h_2(z)+\beta_1} + \frac{\alpha-\gamma+\beta(1-\alpha)}{(1-\alpha)(1-\beta)}\right\}\right]$$

and therefore it follows that

$$Re\left\{h_i + \frac{\alpha_1 z h_i'(z)}{h_i+\beta_1} + \frac{\alpha-\gamma+\beta(1-\alpha)}{(1-\alpha)(1-\beta)}\right\} > 0, \quad z \in E. \qquad (9)$$

We now formulate the functional $\Psi(u,v)$ by taking $u = h_i, \quad v = zh_i'$ in (9) and note that the first two conditions of Lemma 1.1 are clearly satisfied. We verify condition (iii) as follows.

$$\begin{aligned}Re\{\Psi(iu_2, v_1)\} &= \frac{\frac{\alpha}{(1-\alpha)(1-\beta)}\left(\frac{\beta}{1-\beta}\right)v_1}{\left(\frac{\beta}{1-\beta}\right)^2 + u_2^2} + \frac{(\alpha-\beta)+\beta(1-\alpha)}{(1-\alpha)(1-\beta)} \\ &\leq \frac{\frac{-\alpha\beta}{1-\beta}\left(1+u_2^2\right) + 2\left[\frac{\beta^2}{(1-\beta)^2}+u_2^2\right][\alpha-\gamma+\beta(1-\alpha)]}{2(1-\alpha)(1-\beta)\left[\left(\frac{\beta}{1-\beta}\right)^2+u_2^2\right]} \\ &= \frac{A_1 + Bu_2^2}{2C},\end{aligned}$$

where

$$\begin{aligned} A_1 &= \frac{-\alpha\beta}{1-\beta} + \frac{2\beta^2}{(1-\beta)^2}\left[\alpha-\gamma+\beta(1-\alpha)\right] \\ B &= \frac{-\alpha\beta}{1-\beta} + 2\left[(\alpha-\gamma)+\beta(1-\alpha)\right], \\ C &= (1-\alpha)(1-\beta)\left[\left(\frac{\beta}{1-\beta}\right)^2 + u_2^2\right] > 0. \end{aligned}$$

Now $Re\{\Psi(iu_2, v_1)\} \leq 0$ if $A_1 \leq 0$ and $B \leq 0$. For $A_1 \leq 0$, we find β as given by (4) with $0 < \alpha \leq \gamma \leq \frac{3}{2}\alpha < 1$ and $B \leq 0$ gives us $0 < \beta < 1$. This shows that condition (iii) of Lemma 1.1 holds. Applying Lemma 1.1, we see that

$$Re\{h_i(z)\} > 0; \quad i = 1, 2, \quad z \in E.$$

Consequently $h \in P_k$ and therefore $p \in P_k(\beta)$, where β is given by (4) This completes the proof. □

We now discuss some special cases.

Special Cases

(i) Let $\alpha = \gamma$ Then $\beta = \frac{2\alpha}{\alpha+\sqrt{\alpha(4-3\alpha)}}$. This improves a result proved in [6] for the case $k = 2$.

(ii) Let $\gamma = \frac{3}{2}\alpha$. Then we have $\beta = \frac{4\alpha}{\alpha+\sqrt{\alpha(16-15\alpha)}}$.

By taking $\alpha = \frac{1}{2}$, we get $\beta = \frac{4}{1+\sqrt{7}}$. If we take $\beta = 0$ and $\alpha \leq \gamma < 1$, then $A_1 = 0$, $B = 2(\alpha-\gamma) \leq 0$ and Lemma 1.1. is applicable. This gives a result proved in [6] for $k = 2$.

With similar technique used in Theorem 2.1, we can easily prove the following.

Theorem 2.2. Let, for $0 < \alpha < 1$, $f \in N_\alpha(k, \frac{1}{2})$. Then $f \in P'_k(\frac{1}{2})$, $z \in E$.

Theorem 2.3. For $0 \leq \alpha_2 < \alpha_1 < 1$, $N_{\alpha_1}(k, \frac{1}{2}) \subset N_{\alpha_2}(k, \frac{1}{2})$.

Proof. Since

$$(1-\alpha_2)f'(z) + \alpha_2(1 + \frac{zf''(z)}{f'(z)} = \left(1 - \frac{\alpha_2}{\alpha_1}\right) f'(z) + \frac{\alpha_2}{\alpha_1}\left[(1-\alpha_1)f'(z) + \alpha_1\left(1 + \frac{zf''(z)}{f'(z)}\right)\right],$$

the result follows by using Theorem 2.2 and the fact that $P_k(\frac{1}{2})$ is a convex set, see [3]. □

Theorem 2.4. Let, for $0 < \alpha < 1$, $0 \leq \beta < 1$, $f \in P'_k(\beta$. Then $f \in N_\alpha(k, \beta_1)$, for $|z| < r_0$, where

$$\beta_1 = \beta + \alpha(1-\beta),$$

r_0 is given as in Lemma 1.2 with

$$s = \frac{a\alpha}{(1-\alpha)(1-\beta)}, \quad \nu = \frac{\beta}{1-\beta}.$$

The value of r_0 is exact.

Proof. Let $f \in P'_k(\beta)$. Then

(10) $$f'(z) = (1-\beta)p(z) + \beta, \quad p \in P_k.$$

Now

$$J(\alpha, f) = (1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right).$$

Using (10), we have

$$\frac{1}{(1-\alpha)(1-\beta)}\left[J(\alpha, f) - \{\beta + \alpha(1-\beta)\}\right] = p(z) + \frac{\frac{\alpha}{(1-\alpha)(1-\beta)}zp'(z)}{p(z) + \frac{\beta}{1-\beta}}.$$

This gives us

$$\frac{1}{1-\beta_1}\left[J(\alpha, f) - \beta_1\right] = p(z) + \frac{szp'(z)}{p(z)+\nu},$$

where

$$\beta_1 = \beta + \alpha(1-\beta), \quad s = \frac{\alpha}{(1-\alpha)(1-\beta)}, \quad \nu = \frac{\beta}{1-\beta}.$$

Writing

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)p_2(z)$$

and using convolution technique as before, we can write

$$\begin{aligned} \frac{1}{1-\beta_1}\left[J(\alpha, f) - \beta_1\right] &= \left(\frac{k}{4} + \frac{1}{2}\right)\left[p_1(z) + \frac{szp_1'(z)}{p_1(z)+\nu}\right] \\ &\quad - \left(\frac{k}{4} - \frac{1}{2}\right)\left[p_2(z) + \frac{szp_1'(z)}{p_2(z)+\nu}\right], \quad p_i \in P, \quad i = 1, 2. \end{aligned} \tag{11}$$

We now apply Lemma 1.2 to have

$$Re\left\{p_i(z) + \frac{szp_i'(z)}{p_i(z)+\nu}\right\} > 0$$

for $|z| < r_0$, and using this in (11), we obtain the required result. □

As a special case, with $\beta = \frac{1}{2}$, $\alpha = \frac{1}{2}$, we note that $f \in P_k'(\frac{1}{2})$ implies that $f \in N_{\frac{1}{2}}(k, \frac{3}{4})$ for $|z| < r_0$, where

$$r_0 = \frac{2}{\sqrt{\frac{9}{2} + \sqrt{\frac{81}{4}}}} = \frac{2}{3}.$$

We can prove easily the following special case independently.

Theorem 2.5. Let $f \in P_k'(\frac{1}{2})$. Then $f \in N_{\frac{1}{2}}(k, \frac{1}{2})$ for $|z| < \frac{1}{2}$. The value $\frac{1}{2}$ is exact.

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