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DEGREE EVEN COVERINGS OF ELLIPTIC CURVES BY GENUS 2 CURVES

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Abstract. In this survey we study the genus 2 curves with (n, n)-split Jacobian for even n.

1. Introduction

Let C be a genus 2 curve defined over an algebraically closed field k, of characteristic zero. Let $\psi : C \to E$ be a degree n maximal covering (i.e. does not factor through an isogeny) to an elliptic curve E defined over k. We say that C has a *degree n elliptic subcover*. Degree n elliptic subcovers occur in pairs. Let (E, E') be such a pair. It is well known that there is an isogeny of degree n^2 between the Jacobian J_C of C and the product $E \times E'$. The locus of such C, denoted by $\mathcal{L}_n$, is a 2-dimensional algebraic subvariety of the moduli space $\mathcal{M}_2$ of genus two curves and has been the focus of many papers in the last decade; see [5, 7, 8, 9, 10, 1, 2].

The space $\mathcal{L}_2$ was studied in Shaska/Völklein [9]. The space $\mathcal{L}_3$ was studied in [5] were an algebraic description was given as sublocus of $\mathcal{M}_2$. Lately the space $\mathcal{L}_5$ has been studied in detail in [10]. The case of even degree has been less studied even though there have been some attempts lately to compute some of the cases for $n = 4$; see [4]. In this survey we study the genus 2 curves with (n, n)-split Jacobian for small n. While such curves have been studied by many authors, our approach is simply computational.

2. Curves of genus 2 with split Jacobians

Most of the results of this section can be found in [11]. Let C and E be curves of genus 2 and 1, respectively. Both are smooth, projective curves defined over k, $char(k) = 0$. Let $\psi : C \longrightarrow E$ be a covering of degree n. From the Riemann-Hurwitz formula, $\sum_{P \in C} (e_\psi(P) - 1) = 2$ where $e_\psi(P)$ is the ramification index of points $P \in C$, under ψ. Thus, we have two points of ramification index 2 or one point of ramification index 3. The two points of ramification index 2 can be in the same fiber or in different fibers. Therefore, we have the following cases of the covering ψ:

Case I: There are P_1, $P_2 \in C$, such that $e_\psi(P_1) = e_\psi(P_2) = 2$, $\psi(P_1) \neq \psi(P_2)$, and $\forall P \in C \setminus \{P_1, P_2\}$, $e_\psi(P) = 1$.

Case II: There are P_1, $P_2 \in C$, such that $e_\psi(P_1) = e_\psi(P_2) = 2$, $\psi(P_1) = \psi(P_2)$, and $\forall P \in C \setminus \{P_1, P_2\}$, $e_\psi(P) = 1$.

Case III: There is $P_1 \in C$ such that $e_\psi(P_1) = 3$, and $\forall P \in C \setminus \{P_1\}$, $e_\psi(P) = 1$.

In case I (resp. II, III) the cover ψ has 2 (resp. 1) branch points in E.

Denote the hyperelliptic involution of C by w. We choose $\mathcal{O}$ in E such that w restricted to E is the hyperelliptic involution on E. We denote the restriction of w on E by v, $v(P) = -P$. Thus, $\psi \circ w = v \circ \psi$. E[2] denotes the group of 2-torsion points of the elliptic curve E, which are the points fixed by v. The proof of the following two lemmas is straightforward and will be omitted.

Lemma 1. *a) If $Q \in E$, then $\forall P \in \psi^{-1}(Q)$, $w(P) \in \psi^{-1}(-Q)$.*
b) For all $P \in C$, $e_\psi(P) = e_\psi(w(P))$.

Let W be the set of points in C fixed by w. Every curve of genus 2 is given, up to isomorphism, by a binary sextic, so there are 6 points fixed by the hyperelliptic involution w, namely the Weierstrass points of C. The following lemma determines the distribution of the Weierstrass points in fibers of 2-torsion points.

Lemma 2. *The following hold:*

(1) $\psi(W) \subset E[2]$
(2) *If n is an odd number then*
  i) $\psi(W) = E[2]$
  ii) If $Q \in E[2]$ then $\#(\psi^{-1}(Q) \cap W) = 1 \mod (2)$
(3) *If n is an even number then for all $Q \in E[2]$,* $\#(\psi^{-1}(Q) \cap W) = 0 \mod (2)$

Let $\pi_C : C \longrightarrow \mathbb{P}^1$ and $\pi_E : E \longrightarrow \mathbb{P}^1$ be the natural degree 2 projections. The hyperelliptic involution permutes the points in the fibers of π_C and π_E. The ramified points of π_C, π_E are respectively points in W and $E[2]$ and their ramification index is 2. There is $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ such that the diagram commutes.

$$\begin{array}{ccc} C & \xrightarrow{\pi_C} & \mathbb{P}^1 \\ \psi \downarrow & & \downarrow \phi \\ E & \xrightarrow{\pi_E} & \mathbb{P}^1 \end{array} \tag{1}$$

Next, we will determine the ramification of induced coverings $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$. First we fix some notation. For a given branch point we will denote the ramification of points in its fiber as follows. Any point P of ramification index m is denoted by (m). If there are k such points then we write $(m)^k$. We omit writing symbols for unramified points, in other words $(1)^k$ will not be written. Ramification data between two branch points will be separated by commas. We denote by $\pi_E(E[2]) = \{q_1, \ldots, q_4\}$ and $\pi_C(W) = \{w_1, \ldots, w_6\}$.

2.0.1. *The Case When n is Even.* Let us assume now that $deg(\psi) = n$ is an even number. The following theorem classifies the induced coverings in this case.

Theorem 1. *If n is an even number then the generic case for $\psi : C \longrightarrow E$ induce the following three cases for $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$:*

I: $\left((2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n}{2}}, (2)\right)$

II: $\left((2)^{\frac{n-4}{2}},(2)^{\frac{n-2}{2}},(2)^{\frac{n}{2}},(2)^{\frac{n}{2}},(2)\right)$
III: $\left((2)^{\frac{n-6}{2}},(2)^{\frac{n}{2}},(2)^{\frac{n}{2}},(2)^{\frac{n}{2}},(2)\right)$

Each of the above cases has the following degenerations (two of the branch points collapse to one)

I: (1) $\left((2)^{\frac{n}{2}},(2)^{\frac{n-2}{2}},(2)^{\frac{n-2}{2}},(2)^{\frac{n}{2}}\right)$
(2) $\left((2)^{\frac{n-2}{2}},(2)^{\frac{n-2}{2}},(4)(2)^{\frac{n-6}{2}},(2)^{\frac{n}{2}}\right)$
(3) $\left((2)^{\frac{n-2}{2}},(2)^{\frac{n-2}{2}},(2)^{\frac{n-2}{2}},(4)(2)^{\frac{n-4}{2}}\right)$
(4) $\left((3)(2)^{\frac{n-4}{2}},(2)^{\frac{n-2}{2}},(2)^{\frac{n-2}{2}},(2)^{\frac{n}{2}}\right)$

II: (1) $\left((2)^{\frac{n-2}{2}},(2)^{\frac{n-2}{2}},(2)^{\frac{n}{2}},(2)^{\frac{n}{2}}\right)$
(2) $\left((2)^{\frac{n-4}{2}},(2)^{\frac{n}{2}},(2)^{\frac{n}{2}},(2)^{\frac{n}{2}}\right)$
(3) $\left((4)(2)^{\frac{n-8}{2}},(2)^{\frac{n-2}{2}},(2)^{\frac{n}{2}},(2)^{\frac{n}{2}}\right)$
(4) $\left((2)^{\frac{n-4}{2}},(4)(2)^{\frac{n-6}{2}},(2)^{\frac{n}{2}},(2)^{\frac{n}{2}}\right)$
(5) $\left((2)^{\frac{n-4}{2}},(2)^{\frac{n-2}{2}},(2)^{\frac{n-4}{2}},(2)^{\frac{n}{2}}\right)$
(6) $\left((3)(2)^{\frac{n-6}{2}},(2)^{\frac{n-2}{2}},(4)(2)^{\frac{n}{2}},(2)^{\frac{n}{2}}\right)$
(7) $\left((2)^{\frac{n-4}{2}},(3)(2)^{\frac{n-4}{2}},(2)^{\frac{n}{2}},(2)^{\frac{n}{2}}\right)$

III: (1) $\left((2)^{\frac{n-4}{2}},(2)^{\frac{n}{2}},(2)^{\frac{n}{2}},(4)(2)^{\frac{n}{2}}\right)$
(2) $\left((2)^{\frac{n-6}{2}},(4)(2)^{\frac{n-4}{2}},(2)^{\frac{n}{2}},(2)^{\frac{n}{2}}\right)$
(3) $\left((2)^{\frac{n}{2}},(2)^{\frac{n}{2}},(2)^{\frac{n}{2}},(4)(2)^{\frac{n-10}{2}}\right)$
(4) $\left((3)(2)^{\frac{n-8}{2}},(2)^{\frac{n}{2}},(2)^{\frac{n}{2}},(2)^{\frac{n}{2}}\right)$

Proof. We skip the details of the proof. □

Remark 1. *The case $n = 8$ is the first true generic case when all the subcases occur.*

2.1. **Maximal coverings $\psi : C \longrightarrow E$.** Let $\psi_1 : C \longrightarrow E_1$ be a covering of degree n from a curve of genus 2 to an elliptic curve. The covering $\psi_1 : C \longrightarrow E_1$ is called a **maximal covering** if it does not factor through a nontrivial isogeny. A map of algebraic curves $f : X \to Y$ induces maps between their Jacobians $f^* : J_Y \to J_X$ and $f_* : J_X \to J_Y$. When f is maximal then f^* is injective and $ker(f_*)$ is connected, see [8] for details.

Let $\psi_1 : C \longrightarrow E_1$ be a covering as above which is maximal. Then ${\psi^*}_1 : E_1 \to J_C$ is injective and the kernel of $\psi_{1,*} : J_C \to E_1$ is an elliptic curve which we denote by E_2; see [2]. For a fixed Weierstrass point $P \in C$, we can embed C to its Jacobian via

$$\begin{aligned} i_P : C &\longrightarrow J_C \\ x &\to [(x) - (P)] \end{aligned} \tag{2}$$

Let $g : E_2 \to J_C$ be the natural embedding of E_2 in J_C, then there exists $g_* : J_C \to E_2$. Define $\psi_2 = g_* \circ i_P : C \to E_2$. So we have the following exact sequence

$$0 \to E_2 \xrightarrow{g} J_C \xrightarrow{\psi_{1,*}} E_1 \to 0$$

The dual sequence is also exact

$$0 \to E_1 \xrightarrow{\psi_1^*} J_C \xrightarrow{g_*} E_2 \to 0$$

If $deg(\psi_1)$ is an odd number then the maximal covering $\psi_2 : C \to E_2$ is unique up to isomorphism of elliptic curves. If the cover $\psi_1 : C \longrightarrow E_1$ is given, and therefore ϕ_1, we want to determine $\psi_2 : C \longrightarrow E_2$ and ϕ_2. The study of the relation between the ramification structures of ϕ_1 and ϕ_2 provides information in this direction. The following lemma (see [2, pg. 160]) answers this question for the set of Weierstrass points $W = \{P_1, \dots, P_6\}$ of C when the degree of the cover is odd.

Lemma 3. *Let $\psi_1 : C \longrightarrow E_1$, be maximal of degree n. Then, the map $\psi_2 : C \to E_2$ is a maximal covering of degree n. Moreover,*

i) *if n is odd and $\mathcal{O}_i \in E_i[2]$, $i = 1, 2$ are the places such that $\#(\psi_i^{-1}(\mathcal{O}_i) \cap W) = 3$, then $\psi_1^{-1}(\mathcal{O}_1) \cap W$ and $\psi_2^{-1}(\mathcal{O}_2) \cap W$ form a disjoint union of W.*

ii) *if n is even and $Q \in E[2]$, then $\#\left(\psi^{-1}(Q)\right) \cap W = 0$ or 2.*

The above lemma says that if ψ is maximal of even degree then the corresponding induced covering can have only type **I** ramification, see Theorem 1.

Example 1. Let $\psi : C \to E$ be a degree $n = 8$ maximal covering of the elliptic curve E by a genus 2 curve C. Then, we have Type I covering as in previous theorem. Hence, the ramification is

$$\left((2)^3, (2)^3, (2)^3, (2)^4, (2)\right)$$

This case is the first case which has all its subcases with ramifications as follows:

i) $\left((2)^4, (2)^3, (2)^3, (2)^4\right)$

ii) $\left((2)^3, (2)^3, (4)(2), (2)^4\right)$

iii) $\left((2)^3, (2)^3, (2)^3, (4)(2)^2\right)$

iv) $\left((3)(2)^2, (2)^3, (2)^3, (2)^4\right)$

The locus of genus 2 curves in the generic case is a 2-dimensional subvariety of the moduli space $\mathcal{M}_2$. It would be interesting to explicitly compute such subvariety since it is the first case which could give some clues to what happens in the general case for even degree.

3. The locus of genus two curves with (n, n) split Jacobians

In this section we will discuss the Hurwitz spaces of coverings with ramification as in the previous section and the Humbert spaces of discriminant n^2.

3.1. **Hurwitz spaces of covers $\phi : \mathbb{P}^1 \to \mathbb{P}^1$.** Two covers $f : X \to \mathbb{P}^1$ and $f' : X' \to \mathbb{P}^1$ are called **weakly equivalent** if there is a homeomorphism $h : X \to X'$ and an analytic automorphism g of $\mathbb{P}^1$ (i.e., a Moebius transformation) such that $g \circ f = f' \circ h$. The covers f and f' are called **equivalent** if the above holds with $g = 1$.

Consider a cover $f : X \to \mathbb{P}^1$ of degree n, with branch points $p_1, ..., p_r \in \mathbb{P}^1$. Pick $p \in \mathbb{P}^1 \setminus \{p_1, ..., p_r\}$, and choose loops γ_i around p_i such that $\gamma_1, ..., \gamma_r$ is a standard generating system of the fundamental group $\Gamma := \pi_1(\mathbb{P}^1 \setminus \{p_1, ..., p_r\}, p)$, in particular, we have $\gamma_1 \cdots \gamma_r = 1$. Such a system $\gamma_1, ..., \gamma_r$ is called a homotopy basis of $\mathbb{P}^1 \setminus \{p_1, ..., p_r\}$. The group Γ acts on the fiber $f^{-1}(p)$ by path lifting, inducing a transitive subgroup G of the symmetric group S_n (determined by f up to conjugacy in S_n). It is called the **monodromy group** of f. The images of $\gamma_1, ..., \gamma_r$ in S_n form a tuple of permutations $\sigma = (\sigma_1, ..., \sigma_r)$ called a tuple of **branch cycles** of f.

We say a cover $f : X \to \mathbb{P}^1$ of degree n is of type σ if it has σ as tuple of branch cycles relative to some homotopy basis of $\mathbb{P}^1$ minus the branch points of f. Let $\mathcal{H}_\sigma$ be the set of weak equivalence classes of covers of type σ. The **Hurwitz space** $\mathcal{H}_\sigma$ carries a natural structure of an quasiprojective variety.

We have $\mathcal{H}_\sigma = \mathcal{H}_\tau$ if and only if the tuples σ, τ are in the same **braid orbit** $\mathcal{O}_\tau = \mathcal{O}_\sigma$. In the case of the covers $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ from above, the corresponding braid orbit consists of all tuples in S_n whose cycle type matches the ramification structure of ϕ.

This and the genus of $\mathcal{H}_\sigma$ in the degenerate cases (see the following table) has been computed in GAP by the BRAID PACKAGE written by K. Magaard.

deg	Case	cycle type of σ	$\#(\mathcal{O}_\sigma)$	G	$\dim \mathcal{H}_\sigma$	genus of $\mathcal{H}_\sigma$
8		$(2^3, 2^3, 2^3, 2^4, 2)$	224	S_8	2	–
	1	$(2^4, 2^3, 2^3, 2^4)$	4	16	1	0
	2	$(2^3, 2^3, (4)(2), 2^4)$	48	S_8	1	4
	3	$(2^3, 2^3, 2^3, (4)(2)^2)$	96	S_8	1	16
	4	$((3)2^2, 2^3, 2^3, 2^4)$	36	S_8	1	4

TABLE 1. The length of braid orbits, the order of the group, and the genus of 1-dimensional subspaces for even degree maximal coverings.

As the reader can imagine even such computations are not easy for higher n. It is unclear what are the monodromy groups that appear in all the subcases and the formulas for the lengths of the braid orbits.

3.2. **Humbert surfaces.** Let $\mathcal{A}_2$ denote the moduli space of principally polarized abelian surfaces. It is well known that $\mathcal{A}_2$ is the quotient of the Siegel upper half space $\mathfrak{H}_2$ of symmetric complex 2×2 matrices with positive definite imaginary part by the action of the symplectic group $Sp_4(\mathbb{Z})$.

Let Δ be a fixed positive integer and N_Δ be the set of matrices $\tau = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathfrak{H}_2$ such that there exist nonzero integers a, b, c, d, e with the following properties:

$$\begin{aligned} & az_1 + bz_2 + cz_3 + d(z_2^2 - z_1 z_3) + e = 0 \\ & \Delta = b^2 - 4ac - 4de \end{aligned} \tag{3}$$

The *Humbert surface* $\mathcal{H}_\Delta$ of discriminant Δ is called the image of N_Δ under the canonical map

$$\mathfrak{H}_2 \to \mathcal{A}_2 := Sp_4(\mathbb{Z}) \setminus \mathfrak{H}_2.$$

It is known that $\mathcal{H}_\Delta \neq \emptyset$ if and only if $\Delta > 0$ and $\Delta \equiv 0$ *or* $1 \mod 4$. Humbert (1900) studied the zero loci in Eq. (3) and discovered certain relations between points in these spaces and certain plane configurations of six lines.

For a genus 2 curve C defined over $\mathbb{C}$, $[C]$ belongs too $\mathcal{L}_n$ if and only if the isomorphism class $[J_C] \in \mathcal{A}_2$ of its (principally polarized) Jacobian J_C belongs to the Humbert surface $\mathcal{H}_{n^2}$, viewed as a subset of the moduli space $\mathcal{A}_2$ of principally polarized abelian surfaces. There is a one to one correspondence between the points in $\mathcal{L}_n$ and points in $\mathcal{H}_{n^2}$. Thus, we have the map:

$$\begin{aligned} \mathcal{H}_\sigma \ &\longrightarrow\ \mathcal{L}_n \ \longrightarrow\ \mathcal{H}_{n^2} \\ ([f], (p_1, \dots, p_r) &\to [\mathcal{X}] \to [J_{\mathcal{X}}] \end{aligned} \tag{4}$$

In particular, every point in $\mathcal{H}_{n^2}$ can be represented by an element of $\mathfrak{H}_2$ of the form

$$\tau = \begin{pmatrix} z_1 & \frac{1}{n} \\ \frac{1}{n} & z_2 \end{pmatrix}, \qquad z_1,\, z_2 \in \mathfrak{H}.$$

There have been many attempts to explicitly describe these Humbert surfaces. For some small discriminant this has been done by several authors; see [9], [5]. Geometric characterizations of such spaces for $\Delta = 4, 8, 9$, and 12 were given by Humbert (1900) in [3] and for $\Delta = 13, 16, 17, 20, 21$ by Birkenhake/Wilhelm (2003).

4. Computing the locus $\mathcal{L}_n$ in $\mathcal{M}_2$

We take the most general case for maximal coverings of even degree, namely n, Type I. The ramification structure of $\phi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ is

$$\left((2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n}{2}}, (2) \right)$$

We denote the branch points respectively $q_1, \dots, q_5$. Let $q_1 = 0, q_2 = 1, q_3 = \infty$. The red places in $\mathbb{P}^1_x$ denote the unramified places and the black places all have ramification index 2. We pick the coordinate x such that it is $x = 0, x = 1, x = \infty$ in the unramified places of $\mathbb{P}^1_z$ and respectively in the fibers of $0, 1, \infty$ as in the picture.

There are exactly $d = \frac{n-2}{2}$ places of index 2 in $\phi^{-1}(0)$. Let $P(x)$ denote the polynomial whose roots are exactly these places. Similarly denote by $R(x), Q(x)$ such polynomials for fibers of 1 and ∞. The other unramified places in the fibers of 0,1, ∞ we denote by w_4, w_5, w_6 respectively.

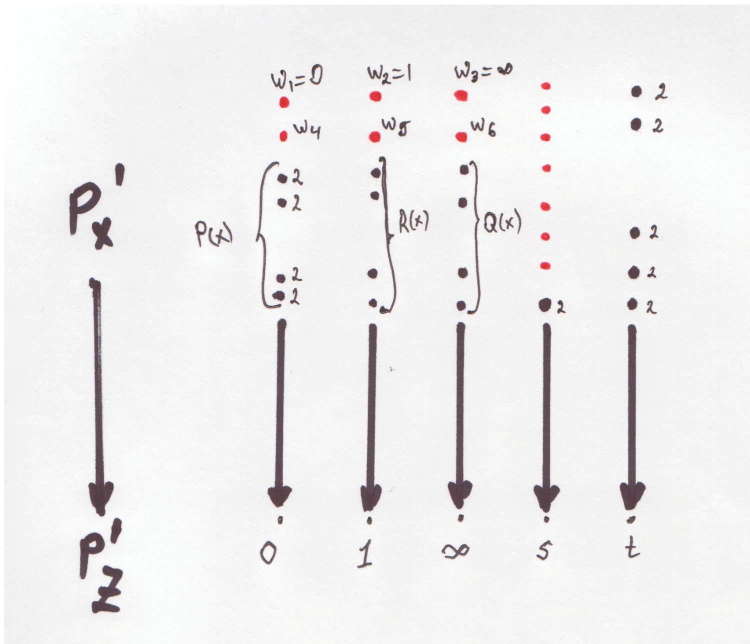
Then, we have

$$z = \lambda \cdot x \frac{x - w_4}{x - w_6} \cdot \frac{P^2(x)}{Q^2(x)}$$

for some $\lambda \cdot \in \mathbb{C}$, $\lambda \neq 0$. Furthermore,

$$z - 1 = \lambda \cdot (x - 1) \cdot \frac{x - w_5}{x - w_6} \cdot \frac{R^2(x)}{Q^2(x)}$$

where $P(x), Q(x), R(x)$ are monic polynomials of degree $d = \frac{n-2}{2}$ with no multiple roots and no common roots.



Substituting for z we get a degree n equation

$$\lambda x(x-w_4)P^2(x)-(x-w_6)Q^2(x)-\lambda\cdot(x-1)(x-w_5)R^2(x)=0$$

By equaling coefficients of this polynomial with zero we get a nonlinear system of $n+1$ equations. In the same way we get the corresponding equations from the fibers of the other two branch points s and t. Solving such system would determine also w_4, w_5, w_6. The equation of the genus 2 curve C is given by

$$y^2=x(x-1)(x-w_4)(x-w_5)(x-w_6)$$

4.1. **Degree 4 covers.** In this section we focus on the case $\deg(\phi)=4$ (not necessarily maximal). The goal is to determine all ramifications σ and explicitly compute $\mathcal{L}_4(\sigma)$. There is one generic case and one degenerate case in which the ramification of $\deg(\phi)=4$ applies, as given by the above possible ramification structures.

i) $(2,2,2,2^2,2)$ (generic)
ii) $(2,2,2,4)$ (degenerate)

4.2. **Degenerate Case.** In this case one of the Weierstrass points has ramification index 3, so the cover is totally ramified at this point.

Let the branch points be 0, 1, λ, and ∞, where ∞ corresponds to the element of index 4. Then, above the fibers of 0, 1, λ lie two Weierstrass points. The two Weierstrass points above 0 can be written as the roots of a quadratic polynomial x^2+ax+b; above 1, they are the roots of x^2+px+q; and above λ, they are the roots of x^2+sx+t. This gives us an equation for the genus 2 curve C:

$$C: y^2=(x^2+ax+b)(x^2+px+q)(x^2+sx+t).$$

The four branch points of the cover ϕ are the 2-torsion points $E[2]$ of the elliptic curve E, allowing us to write the elliptic subcover as

$$E: y^2=x(x-1)(x-\lambda).$$

We have the following theorem:

Theorem 2. *Let C be a genus 2 curve with a degree 4 degenerate elliptic subcover. Then C is isomorphic to the curve given by*

$$
\begin{aligned}
C: y^2 &= \left(\frac{1-b}{3} + \frac{2}{3}(1-b)x + x^2\right)\left(\frac{1}{12}(b-4)b + \frac{1}{3}(b-4)x + x^2\right) \\
&\quad \left(b - \frac{2}{3}(b+2)x + x^2\right) \\
E: v^2 &= u(u-1)\left(u - \frac{b^3(4-b)}{16(b-1)}\right)
\end{aligned}
\tag{5}
$$

where the corresponding discriminants of the right sides must be non-zero. Hence,

$$\Delta_C := b(b-4)(b-2)(b-1)(2+b) \neq 0 \tag{6}$$

$$\Delta_E := \frac{(b-4)^2(b-2)^6 b^6 (b+2)^2}{65536(b-1)^4} \neq 0. \tag{7}$$

and its invariants satisfy

$$
\begin{aligned}
&1541086152812576000\, {J_2}^2 {J_4}^2 - 22835312232360960000\, J_2\, J_4\, J_6 + 5009676947631\, {J_2}^6 \\
&- 8782271900467200000\, {J_6}^2 + 1176812184652746480\, {J_2}^4 J_4 + 12448207102988800000\, {J_4}^3 \\
&- 3715799948429529600\, {J_2}^3 J_6 = 0 \\
&186626560000\, {J_2}^2 {J_4}^4 + 138962144767343358744576000000\, {J_{10}}^2 + \frac{282429536481}{10^4}\, {J_2}^{10} \\
&+ 619923800736\, {J_2}^6 {J_4}^2 - 25600000000\, {J_4}^5 - \frac{28249152375924}{100}\, {J_2}^8 J_4 \\
&+ 2665762699498787920\, {J_2}^5 J_{10} - 510202022400\, {J_2}^4 {J_4}^3 \\
&+ 6930676241452032000000\, J_2\, {J_4}^2 J_{10} + 1763516708182388736000\, {J_2}^3 J_4\, J_{10} = 0.
\end{aligned}
\tag{8}
$$

Proof. See [4]. □

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