

EXTRAGRADIENT METHOD FOR EQUILIBRIUM PROBLEMS AND VARIATIONAL INEQUALITIES

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ABSTRACT. In this paper, we suggest and analyze a new extragradient method for finding a common element of the set of solutions of an equilibrium problem, the set of fixed points of a nonexpansive mapping and the set of solutions of some variational inequality. Furthermore, we prove that the proposed iterative algorithm converges strongly to a common element of the above three sets. Our result includes the main result of Bnouhachem, Noor and Hao [A. Bnouhachem, M.A. Noor and Z. Hao, Some new extragradient methods for variational inequalities, *Nonlinear Analysis* (2008), doi:10.1016/j.na.2008.02.014] as a special case.

1. INTRODUCTION

Equilibrium problems, which were introduced by Blum and Oettli [21] and Noor and Oettli [22] in 1994, are being used as mathematical model for studying a wide class of problems arising in various branches of pure and applied sciences. It has been shown that equilibrium problems include variational inequalities, fixed point problems and Nash equilibrium problems as special cases. In recent years, several iterative methods including extragradient method and auxiliary technique have been developed for solving equilibrium problems and variational inequalities, see [16-25] and the references therein. Bnouhachem, Noor and Hao [15] has suggested and analyzed an extragradient type method for solving variational inequalities. Motivated and inspired by the on going research in this direction, we suggest and analyze a new extragradient type method for finding the common element of the set of solutions of the equilibrium problems, variational inequalities and fixed point problems of nonexpansive mapping. The proposed iterative method is quite general and include the recent methods as special cases. Our results can be viewed as a significant improvement of the recently obtained results.

Let C be a nonempty closed convex subset of a real Hilbert space H . Let $T : C \rightarrow H$ be a nonlinear mapping. The classical variational inequality, denoted by $VI(T, C)$, is to find $u^* \in C$ such that

$$\langle T(u^*), u - u^* \rangle \geq 0, \forall u \in C,$$

which was introduced by Stampacchia [1] in 1964. Since then, the variational inequality has been extensively studied in the literature. See, e.g., [2-11] and the

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references therein. Recall that a mapping T of C into H is called α -inverse-strongly monotone if there exists a positive real number α such that

$$\langle T(u) - T(v), u - v \rangle \geq \alpha \|T(u) - T(v)\|^2, \forall u, v \in C.$$

It is obvious that any α -inverse-strongly monotone mapping T is $\frac{1}{\alpha}$ Lipschitz continuous. A mapping $S : C \rightarrow H$ is said to be nonexpansive if

$$\|S(u) - S(v)\| \leq \|u - v\|, \quad \forall u, v \in C.$$

Denote the set of fixed points of S by $Fix(S)$.

For finding an element of $Fix(S) \cap VI(T, C)$ under the assumption that a set $C \subset H$ is closed and convex, a mapping S of C into itself is nonexpansive and a mapping T of C into H is α -inverse-strongly monotone, Takahashi and Toyoda [12] introduced the following iterative scheme:

$$(1) \quad u^{k+1} = \alpha_k u^k + (1 - \alpha_k) S(P_C[u^k - \rho_k T(u^k)]), \forall k \geq 0,$$

where P_C is the metric projection of H onto C , $u^0 = u \in C$, $\{\alpha_k\}$ is a sequence in $(0, 1)$, and $\{\rho_k\}$ is a sequence in $(0, 2\alpha)$. They showed that, if $Fix(S) \cap VI(T, C)$ is nonempty, then the sequence $\{u^k\}$ generated by (1) converges weakly to some $z \in Fix(S) \cap VI(T, C)$. Recently, Nadezhkina and Takahashi [13] introduced a so-called extragradient method motivated by the idea of Korpelevich [14] for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem. Zeng and Yao [11] introduced another extragradient method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem. Further, Bnouhachem, Noor and Hao [15] introduced the following extragradient iterative method:

$$(2) \quad \begin{cases} \tilde{u}^k = P_C[u^k - \rho_k T(u^k)], \\ u^{k+1} = \beta_k u^k + (1 - \beta_k) S(\alpha_k u + (1 - \alpha_k) P_C[u^k - \rho_k T(\tilde{u}^k)]). \end{cases}$$

Under mild assumptions, they proved a strong convergence theorem for finding a common element of the fixed points of a nonexpansive mapping S and the solution set of the variational inequality for an α -inverse strongly monotone mapping T in a Hilbert space.

Let F be an equilibrium bifunction of $C \times C$ into \mathbf{R} , where \mathbf{R} is the set of real numbers. The equilibrium problem for $F : C \times C \rightarrow \mathbf{R}$ is to find $u \in C$ such that

$$EP : \quad F(u, v) \geq 0 \text{ for all } v \in C.$$

The set of solutions of the equilibrium problem is denoted by $EP(F)$.

For solving the above equilibrium problem, some efforts have been made by many authors. For the more details, please refer to [15-18] and the references therein.

Motivated and inspired by the works in the literature, in this paper, we introduce an iterative algorithm based on extragradient method for finding a common element of the set of solutions of an equilibrium problem, the set of fixed points of a nonexpansive mapping and the set of solutions of some variational inequality. Furthermore, we prove that the proposed iterative algorithm converges strongly to a common element of the above three sets. Our result includes the main result of Bnouhachem, Noor and Hao [A. Bnouhachem, M.A. Noor and Z. Hao, Some

new extragradient methods for variational inequalities, *Nonlinear Analysis* (2008), doi:10.1016/j.na.2008.02.014] as a special case.

2. PRELIMINARIES

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and let C be a closed convex subset of H . It is well known that, for any $u \in H$, there exists unique $y_0 \in C$ such that

$$\|u - y_0\| = \inf\{\|u - y\| : y \in C\}.$$

We denote y_0 by $P_C[u]$, where P_C is called the metric projection of H onto C . The metric projection P_C of H onto C has the following basic properties:

- (i) $\|P_C[u] - P_C[v]\| \leq \|u - v\|$ for all $u, v \in H$,
- (ii) $\langle u - v, P_C[u] - P_C[v] \rangle \geq \|P_C[u] - P_C[v]\|^2$ for every $u, v \in H$,
- (iii) $\langle u - P_C[u], v - P_C[u] \rangle \leq 0$ for all $u \in H, v \in C$,
- (iv) $\|u - v\|^2 \geq \|u - P_C[u]\|^2 + \|v - P_C[u]\|^2$ for all $u \in H, v \in C$.

Let T be a monotone mapping of C into H . In the context of the variational inequality problem, it is easy to see from (iv) that

$$u \in VI(T, C) \Leftrightarrow u = P_C[u - \lambda T(u)], \quad \forall \lambda > 0.$$

A set-valued mapping $A : H \rightarrow 2^H$ is called monotone if, for all $u, v \in H, f \in Au$ and $g \in Av$ imply $\langle u - v, f - g \rangle \geq 0$. A monotone mapping $A : H \rightarrow 2^H$ is maximal if its graph $G(A)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping A is maximal if and only if, for $(u, f) \in H \times H, \langle u - v, f - g \rangle \geq 0$ for every $(v, g) \in G(A)$ implies $f \in Au$. Let T be a monotone mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$; i.e.,

$$N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C\}.$$

Define

$$Av = \begin{cases} T(v) + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then A is maximal monotone and $0 \in Av$ if and only if $v \in VI(T, C)$.

In this paper, for solving the equilibrium problems for an equilibrium bifunction $F : C \times C \rightarrow \mathbf{R}$, we assume that F satisfies the following conditions:

- (C1) $F(u, u) = 0$ for all $u \in C$;
- (C2) F is monotone, i.e., $F(u, v) + F(v, u) \leq 0$ for all $u, v \in C$;
- (C3) for each $u, v, w \in C, \lim_{t \downarrow 0} F(tw + (1-t)u, v) \leq F(u, v)$;
- (C4) for each $u \in C, v \mapsto F(u, v)$ is convex and lower semicontinuous.

If an equilibrium bifunction $F : C \times C \rightarrow \mathbf{R}$ satisfies conditions (C1)-(C4), then we have the following two important results. You can find them in [16].

Lemma 2.1 Let C be a nonempty closed convex subset of H and let F be an equilibrium bifunction of $C \times C$ into \mathbf{R} satisfies conditions (C1)-(C4). Let $r > 0$ and $u \in C$. Then, there exists $v \in C$ such that

$$F(v, w) + \frac{1}{r} \langle w - v, v - u \rangle \geq 0 \text{ for all } w \in C.$$

Lemma 2.2 Assume that F satisfies the same assumptions as Lemma 2.1. For $r > 0$ and $u \in C$, define a mapping $\Gamma_r : H \rightarrow C$ as follows:

$$\Gamma_r(u) = \{v \in C : F(v, w) + \frac{1}{r} \langle w - v, v - u \rangle \geq 0, \forall w \in C\}.$$

Then the following hold:

- (1) Γ_r is single-valued;
- (2) Γ_r is firmly nonexpansive, i.e., for any $u, v \in H$,

$$\|\Gamma_r u - \Gamma_r v\|^2 \leq \langle \Gamma_r u - \Gamma_r v, u - v \rangle;$$

- (3) $Fix(\Gamma_r) = EP(F)$;
- (4) $EP(F)$ is closed and convex.

We also need the following lemmas for proving our main results.

Lemma 2.3 ([19]) Let $\{u^k\}$ and $\{v^k\}$ be bounded sequences in a Banach space X and let $\{\beta_k\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{k \rightarrow \infty} \beta_k \leq \limsup_{k \rightarrow \infty} \beta_k < 1$. Suppose $u^{k+1} = (1 - \beta_k)v^k + \beta_k u^k$ for all integers $k \geq 0$ and

$$\limsup_{k \rightarrow \infty} (\|v^{k+1} - v^k\| - \|u^{k+1} - u^k\|) \leq 0.$$

Then, $\lim_{k \rightarrow \infty} \|v^k - u^k\| = 0$.

Lemma 2.4 ([20]) Assume $\{a^k\}$ is a sequence of nonnegative real numbers such that $a^{k+1} \leq (1 - \gamma_k)a^k + \delta^k$, where $\{\gamma_k\}$ is a sequence in $(0, 1)$ and $\{\delta^k\}$ is a sequence such that

- (1) $\sum_{k=1}^{\infty} \gamma_k = \infty$;
- (2) $\limsup_{k \rightarrow \infty} \delta^k / \gamma_k \leq 0$ or $\sum_{k=1}^{\infty} |\delta^k| < \infty$.

Then $\lim_{k \rightarrow \infty} a^k = 0$.

3. ITERATIVE ALGORITHMS

In this section, we suggest and analyze an iterative algorithm for finding a common element of the set of solutions of an equilibrium problem, the set of fixed points of a nonexpansive mapping and the set of solutions of some variational inequality. Let C be a nonempty closed convex subset of a real Hilbert space H . Let F be a bifunction from $C \times C \rightarrow \mathbf{R}$ satisfying (C1)-(C4). Let T be an α -inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $Fix(S) \cap VI(T, C) \cap EP(F) \neq \emptyset$.

Algorithm 3.1 For fixed $u \in C$ and given $u^0 \in C$ arbitrarily, find the approximate solution $\{u^{k+1}\}$ by the iterative schemes:

$$(3) \quad \begin{cases} F(v^k, w) + \frac{1}{r_k} \langle w - v^k, v^k - u^k \rangle \geq 0, \forall w \in C, \\ \tilde{u}^k = P_C[v^k - \rho_k T(v^k)], \\ u^{k+1} = \beta_k u^k + (1 - \beta_k) S(\alpha_k u + (1 - \alpha_k) P_C[v^k - \rho_k T(\tilde{u}^k)]), \end{cases}$$

where $\{\alpha_k\}$ and $\{\beta_k\}$ are two sequences in $(0, 1)$, $\{\rho_k\}$ is a sequence in $[0, 2\alpha]$ and $\{r_k\}$ is a sequence in $(0, \infty)$.

If we put $F(u, v) \equiv 0$ for all $u, v \in C$ and $r_k = 1$ for all $k \geq 0$ in Algorithm 3.1, then we have $v^k = P_C[u^k] = u^k$. Then we obtain the following iterative algorithm

Algorithm 3.2 For fixed $u \in C$ and given $u^0 \in C$ arbitrarily, find the approximate solution $\{u^{k+1}\}$ by the iterative schemes:

$$\begin{cases} \tilde{u}^k = P_C[u^k - \rho_k T(u^k)], \\ u^{k+1} = \beta_k u^k + (1 - \beta_k)S(\alpha_k u + (1 - \alpha_k)P_C[u^k - \rho_k T(\tilde{u}^k)]), \end{cases}$$

where $\{\alpha_k\}$ and $\{\beta_k\}$ are two sequences in $(0, 1)$, $\{\rho_k\}$ is a sequence in $[0, 2\alpha]$ and $\{r_k\}$ is a sequence in $(0, \infty)$.

If we put $S \equiv I$ the identity operator in Algorithm 3.2. Then we obtain the following iterative algorithm

Algorithm 3.3 For fixed $u \in C$ and given $u^0 \in C$ arbitrarily, find the approximate solution $\{u^{k+1}\}$ by the iterative schemes:

$$\begin{cases} \tilde{u}^k = P_C[u^k - \rho_k T(u^k)], \\ u^{k+1} = \beta_k u^k + (1 - \beta_k)(\alpha_k u + (1 - \alpha_k)P_C[u^k - \rho_k T(\tilde{u}^k)]), \end{cases}$$

where $\{\alpha_k\}$ and $\{\beta_k\}$ are two sequences in $(0, 1)$, $\{\rho_k\}$ is a sequence in $[0, 2\alpha]$ and $\{r_k\}$ is a sequence in $(0, \infty)$.

Let $\{u^k\}$ be a sequence defined by (3). In the sequence, we will assume that the algorithm parameters satisfy the following restrictions:

- (R1) $\lim_{k \rightarrow \infty} \alpha_k = 0$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$;
- (R2) $0 < \liminf_{k \rightarrow \infty} \beta_k \leq \limsup_{k \rightarrow \infty} \beta_k < 1$;
- (R3) $\lim_{k \rightarrow \infty} \rho_k = 0$;
- (R4) $\liminf_{k \rightarrow \infty} r_k > 0$ and $\lim_{k \rightarrow \infty} (r_{k+1} - r_k) = 0$.

In order to prove the strong convergence of Algorithm 3.1, we first prove the following lemmas.

Lemma 3.1 The sequence $\{u^k\}$ is bounded.

Proof. Let $u^* \in \text{Fix}(S) \cap \text{VI}(T, C) \cap \text{EP}(F)$. Then, it is clear that $u^* = P_C[u^* - \rho_k T(u^*)] = \Gamma_{r_k} u^*$. First, we note that $I - \rho_k T$ is nonexpansive for all $\rho_k \in [0, 2\alpha]$. Indeed, by the α -inverse-strongly monotonicity of T , we have

$$\begin{aligned} \|(I - \rho_k T)u - (I - \rho_k T)v\|^2 &= \|u - v\|^2 - 2\rho_k \langle T(u) - T(v), u - v \rangle \\ &\quad + \rho_k^2 \|T(u) - T(v)\|^2 \\ &\leq \|u - v\|^2 + \rho_k(\rho_k - 2\alpha) \|T(u) - T(v)\|^2 \\ &\leq \|u - v\|^2, \end{aligned}$$

which implies that $I - \rho_k T$ is nonexpansive. Set $w^k = P_C[v^k - \rho_k T(\tilde{u}^k)]$ for all $k \geq 0$. From the property (iv) of P_C , we have

$$\begin{aligned} \|w^k - u^*\|^2 &\leq \|v^k - \rho_k T(\tilde{u}^k) - u^*\|^2 - \|v^k - \rho_k T(\tilde{u}^k) - w^k\|^2 \\ &= \|v^k - u^*\|^2 - 2\rho_k \langle T(\tilde{u}^k), v^k - u^* \rangle + \rho_k^2 \|T(\tilde{u}^k)\|^2 \\ &\quad - \|v^k - w^k\|^2 + 2\rho_k \langle T(\tilde{u}^k), v^k - w^k \rangle - \rho_k^2 \|T(\tilde{u}^k)\|^2 \\ (4) \quad &= \|v^k - u^*\|^2 - \|v^k - w^k\|^2 + 2\rho_k \langle T(\tilde{u}^k), u^* - w^k \rangle \\ &= \|v^k - u^*\|^2 - \|v^k - w^k\|^2 + 2\rho_k \langle T(\tilde{u}^k) - T(u^*), u^* - \tilde{u}^k \rangle \\ &\quad + 2\rho_k \langle T(u^*), u^* - \tilde{u}^k \rangle + 2\rho_k \langle T(\tilde{u}^k), \tilde{u}^k - w^k \rangle. \end{aligned}$$

Using the fact that T is monotonic and u^* is a solution of the variational inequality problem $\text{VI}(T, C)$, we have $\langle T(\tilde{u}^k) - T(u^*), u^* - \tilde{u}^k \rangle \leq 0$ and $\langle T(u^*), u^* - \tilde{u}^k \rangle \leq 0$.

This together with (4) implies that

$$\begin{aligned}
\|w^k - u^*\|^2 &\leq \|v^k - u^*\|^2 - \|v^k - w^k\|^2 + 2\rho_k \langle T(\tilde{u}^k), \tilde{u}^k - w^k \rangle \\
&= \|v^k - u^*\|^2 - \|v^k - \tilde{u}^k\|^2 - 2\langle v^k - \tilde{u}^k, \tilde{u}^k - w^k \rangle \\
(5) \quad &\quad - \|\tilde{u}^k - w^k\|^2 + 2\rho_k \langle T(\tilde{u}^k), \tilde{u}^k - w^k \rangle \\
&= \|v^k - u^*\|^2 - \|v^k - \tilde{u}^k\|^2 + 2\langle v^k - \rho_k T(\tilde{u}^k) - \tilde{u}^k, w^k - \tilde{u}^k \rangle \\
&\quad - \|\tilde{u}^k - w^k\|^2.
\end{aligned}$$

By using the property (iii) of P_C , we have $\langle v^k - \rho_k T(v^k) - \tilde{u}^k, w^k - \tilde{u}^k \rangle \leq 0$. Therefore, we get

$$\begin{aligned}
\langle v^k - \rho_k T(\tilde{u}^k) - \tilde{u}^k, w^k - \tilde{u}^k \rangle &= \langle v^k - \rho_k T(v^k) - \tilde{u}^k, w^k - \tilde{u}^k \rangle \\
&\quad + \rho_k \langle T(v^k) - T(\tilde{u}^k), w^k - \tilde{u}^k \rangle \\
(6) \quad &\leq \rho_k \langle T(v^k) - T(\tilde{u}^k), w^k - \tilde{u}^k \rangle \\
&\leq \rho_k \|T(v^k) - T(\tilde{u}^k)\| \|w^k - \tilde{u}^k\| \\
&\leq \frac{\rho_k}{\alpha} \|v^k - \tilde{u}^k\| \|w^k - \tilde{u}^k\|.
\end{aligned}$$

Combining (5) and (6), we obtain

$$\begin{aligned}
\|w^k - u^*\|^2 &\leq \|v^k - u^*\|^2 - \|v^k - \tilde{u}^k\|^2 - \|\tilde{u}^k - w^k\|^2 \\
&\quad + 2\frac{\rho_k}{\alpha} \|v^k - \tilde{u}^k\| \|w^k - \tilde{u}^k\| \\
(7) \quad &\leq \|v^k - u^*\|^2 - \|v^k - \tilde{u}^k\|^2 - \|\tilde{u}^k - w^k\|^2 \\
&\quad + \frac{\rho_k^2}{\alpha^2} \|v^k - \tilde{u}^k\|^2 + \|w^k - \tilde{u}^k\|^2 \\
&= \|v^k - u^*\|^2 + \left(\frac{\rho_k^2}{\alpha^2} - 1\right) \|v^k - \tilde{u}^k\|^2.
\end{aligned}$$

Note that $\lim_{k \rightarrow \infty} \rho_k = 0$, we may assume without loss of generality that $\rho_k < \alpha$. Hence, from (7), we have

$$\|w^k - u^*\|^2 \leq \|v^k - u^*\|^2 = \|\Gamma_{r_k} u^k - \Gamma_{r_k} u^*\|^2 \leq \|u^k - u^*\|^2.$$

From (3), we deduce that

$$\begin{aligned}
\|u^{k+1} - u^*\| &= \|\beta_k(u^k - u^*) + (1 - \beta_k)(S(\alpha_k u + (1 - \alpha_k)w^k) - u^*)\| \\
(8) \quad &\leq \beta_k \|u^k - u^*\| + (1 - \beta_k) \|\alpha_k(u - u^*) + (1 - \alpha_k)(w^k - u^*)\| \\
&\leq \beta_k \|u^k - u^*\| + (1 - \beta_k)(\alpha_k \|u - u^*\| + (1 - \alpha_k) \|w^k - u^*\|) \\
&\leq (1 - \beta_k) \alpha_k \|u - u^*\| + (1 - (1 - \beta_k) \alpha_k) \|u^k - u^*\|.
\end{aligned}$$

It follows from (8) induction that

$$\|u^k - u^*\| \leq \max\{\|u - u^*\|, \|u^0 - u^*\|\}, \quad k \geq 0.$$

Therefore $\{u^k\}$ is bounded. It is easy to prove that $\{\tilde{u}^k\}$, $\{v^k\}$ and $\{w^k\}$ are all bounded.

Lemma 3.2 $\lim_{k \rightarrow \infty} \|u^{k+1} - u^k\| = 0$.

Proof. First, we estimate $\|w^{k+1} - w^k\|$. Noting that P_C and $I - \rho_k T$ is nonexpansive, we have

$$\begin{aligned}
(9) \quad \|w^{k+1} - w^k\| &= \|P_C[v^{k+1} - \rho_{k+1}T(\tilde{u}^{k+1})] - P_C[v^k - \rho_k T(\tilde{u}^k)]\| \\
&\leq \|(v^{k+1} - \rho_{k+1}T(\tilde{u}^{k+1})) - (v^k - \rho_k T(\tilde{u}^k))\| \\
&= \|(v^{k+1} - \rho_{k+1}T(v^{k+1})) - (v^k - \rho_{k+1}T(v^k)) \\
&\quad + \rho_{k+1}(T(v^{k+1}) - T(\tilde{u}^{k+1}) - T(v^k)) + \rho_k T(\tilde{u}^k)\| \\
&\leq \|(v^{k+1} - \rho_{k+1}T(v^{k+1})) - (v^k - \rho_{k+1}T(v^k))\| \\
&\quad + (\rho_{k+1} + \rho_k)M_1 \\
&\leq \|v^{k+1} - v^k\| + (\rho_{k+1} + \rho_k)M_1,
\end{aligned}$$

where M_1 is some constant such that

$$\sup\{\|T(v^{k+1}) - T(\tilde{u}^{k+1}) - T(v^k)\| + \|T(\tilde{u}^k)\|, \quad k \geq 0\} \leq M_1.$$

On the other hand, from $v^k = \Gamma_{r_k} u^k$ and $v^{k+1} = \Gamma_{r_{k+1}} u^{k+1}$, we have

$$(10) \quad F(v^k, w) + \frac{1}{r_k} \langle w - v^k, v^k - u^k \rangle \geq 0, \quad \forall w \in C$$

and

$$(11) \quad F(v^{k+1}, w) + \frac{1}{r_{k+1}} \langle w - v^{k+1}, v^{k+1} - u^{k+1} \rangle \geq 0, \quad \forall w \in C.$$

Putting $w = v^{k+1}$ in (10) and $w = v^k$ in (11), we have

$$(12) \quad F(v^k, v^{k+1}) + \frac{1}{r_k} \langle v^{k+1} - v^k, v^k - u^k \rangle \geq 0,$$

and

$$(13) \quad F(v^{k+1}, v^k) + \frac{1}{r_{k+1}} \langle v^k - v^{k+1}, v^{k+1} - u^{k+1} \rangle \geq 0.$$

From the monotonicity of F , we have

$$F(v^k, v^{k+1}) + F(v^{k+1}, v^k) \leq 0.$$

So, from (12) and (13), we can conclude that

$$\langle v^{k+1} - v^k, \frac{v^k - u^k}{r_k} - \frac{v^{k+1} - u^{k+1}}{r_{k+1}} \rangle \geq 0$$

and hence

$$\langle v^{k+1} - v^k, v^k - v^{k+1} + v^{k+1} - u^k - \frac{r_k}{r_{k+1}}(v^{k+1} - u^{k+1}) \rangle \geq 0.$$

Since $\liminf_{k \rightarrow \infty} r_k > 0$, without loss of generality, we may assume that there exists a real number b such that $r_k > b > 0$ for all $k \in N$. Then, we have

$$\begin{aligned}
\|v^{k+1} - v^k\|^2 &\leq \langle v^{k+1} - v^k, u^{k+1} - u^k + (1 - \frac{r_k}{r_{k+1}})(v^{k+1} - u^{k+1}) \rangle \\
&\leq \|v^{k+1} - v^k\| \{ \|u^{k+1} - u^k\| + |1 - \frac{r_k}{r_{k+1}}| \|v^{k+1} - u^{k+1}\| \}
\end{aligned}$$

and hence

$$(14) \quad \|v^{k+1} - v^k\| \leq \|u^{k+1} - u^k\| + \frac{M_2}{b} |r_{k+1} - r_k|,$$

where M_2 is a constant such that $\sup\{\|v^{k+1} - u^{k+1}\|, k \geq 0\} \leq M_2$. Substituting (14) into (9), we have

$$(15) \quad \begin{aligned} \|w^{k+1} - w^k\| &\leq \|u^{k+1} - u^k\| + (\rho_{k+1} + \rho_k)M_1 \\ &\quad + \frac{M_2}{b}|r_{k+1} - r_k|. \end{aligned}$$

Define $u^{k+1} = \beta_k u^k + (1 - \beta_k)x^k, \forall k \geq 0$. It follows that

$$(16) \quad \begin{aligned} x^{k+1} - x^k &= \frac{u^{k+2} - \beta_{k+1}u^{k+1}}{1 - \beta_{k+1}} - \frac{u^{k+1} - \beta_k u^k}{1 - \beta_k} \\ &= S(\alpha_{k+1}u + (1 - \alpha_{k+1})w^{k+1}) - S(\alpha_k u + (1 - \alpha_k)w^k). \end{aligned}$$

It follows from (15) and (16) that

$$\begin{aligned} &\|x^{k+1} - x^k\| - \|u^{k+1} - u^k\| \\ &\leq \|(\alpha_{k+1}u + (1 - \alpha_{k+1})w^{k+1}) - (\alpha_k u + (1 - \alpha_k)w^k)\| \\ &\quad - \|u^{k+1} - u^k\| \\ &\leq \alpha_{k+1}(\|u\| + \|w^{k+1}\|) + \alpha_k(\|u\| + \|w^k\|) \\ &\quad + (\rho_{k+1} + \rho_k)M_1 + \frac{M_2}{b}|r_{k+1} - r_k|, \end{aligned}$$

which implies that $\limsup_{k \rightarrow \infty} (\|x^{k+1} - x^k\| - \|u^{k+1} - u^k\|) \leq 0$. This together with Lemma 2.3 implies that $\lim_{k \rightarrow \infty} \|x^k - u^k\| = 0$. Consequently $\lim_{k \rightarrow \infty} \|u^{k+1} - u^k\| = \lim_{k \rightarrow \infty} (1 - \beta_k)\|x^k - u^k\| = 0$.

Lemma 3.3 $\lim_{k \rightarrow \infty} \|S(\tilde{u}^k) - \tilde{u}^k\| = 0$.

Proof. Since $u^{k+1} = \beta_k u^k + (1 - \beta_k)S(\alpha_k u + (1 - \alpha_k)w^k)$, we have

$$\begin{aligned} \|u^k - S(w^k)\| &\leq \|u^k - u^{k+1}\| + \|u^{k+1} - S(w^k)\| \\ &\leq \|u^k - u^{k+1}\| + \beta_k \|u^k - S(w^k)\| + (1 - \beta_k)\alpha_k \|u - w^k\|, \end{aligned}$$

that is

$$\|u^k - S(w^k)\| \leq \frac{1}{1 - \beta_k} \|u^k - u^{k+1}\| + \alpha_k \|u - w^k\|.$$

It follows that

$$(17) \quad \lim_{n \rightarrow \infty} \|u^k - S(w^k)\| = 0.$$

Since Γ_{r_k} is firmly nonexpansive, we have

$$\begin{aligned} \|v^k - u^*\|^2 &= \|\Gamma_{r_k} u^k - \Gamma_{r_k} u^*\|^2 \\ &\leq \langle \Gamma_{r_k} u^k - \Gamma_{r_k} u^*, u^k - u^* \rangle \\ &= \langle v^k - u^*, u^k - u^* \rangle \\ &= \frac{1}{2} (\|v^k - u^*\|^2 + \|u^k - u^*\|^2 - \|u^k - v^k\|^2) \end{aligned}$$

and hence

$$(18) \quad \|v^k - u^*\|^2 \leq \|u^k - u^*\|^2 - \|u^k - v^k\|^2.$$

By (3), we have

$$\begin{aligned}
\|u^{k+1} - u^*\|^2 &= \|\beta_k(u^k - u^*) + (1 - \beta_k)[S(\alpha_k u + (1 - \alpha_k)w^k) - u^*]\|^2 \\
&\leq \beta_k\|u^k - u^*\|^2 + (1 - \beta_k)\|\alpha_k u + (1 - \alpha_k)w^k - u^*\|^2 \\
(19) \quad &\leq \beta_k\|u^k - u^*\|^2 + (1 - \beta_k)(\alpha_k\|u - u^*\|^2 \\
&\quad + (1 - \alpha_k)\|w^k - u^*\|^2) \\
&= \beta_k\|u^k - u^*\|^2 + (1 - \beta_k)\alpha_k\|u - u^*\|^2 \\
&\quad + (1 - \beta_k)(1 - \alpha_k)\|w^k - u^*\|^2.
\end{aligned}$$

From (7) and (19), we have

$$\begin{aligned}
\|u^{k+1} - u^*\|^2 &\leq \beta_k\|u^k - u^*\|^2 + (1 - \beta_k)\alpha_k\|u - u^*\|^2 \\
&\quad + (1 - \beta_k)(1 - \alpha_k)\|v^k - u^*\|^2 \\
&\quad + (1 - \beta_k)(1 - \alpha_k)\left(\frac{\rho_k^2}{\alpha^2} - 1\right)\|v^k - \tilde{u}^k\|^2 \\
&\leq \beta_k\|u^k - u^*\|^2 + (1 - \beta_k)\alpha_k\|u - u^*\|^2 \\
&\quad + (1 - \beta_k)(1 - \alpha_k)\|u^k - u^*\|^2 \\
&\quad + (1 - \beta_k)(1 - \alpha_k)\left(\frac{\rho_k^2}{\alpha^2} - 1\right)\|v^k - \tilde{u}^k\|^2.
\end{aligned}$$

Then we derive

$$\begin{aligned}
(1 - \beta_k)(1 - \alpha_k)\left(1 - \frac{\rho_k^2}{\alpha^2}\right)\|v^k - \tilde{u}^k\|^2 \\
\leq \beta_k\|u^k - u^*\|^2 + (1 - \beta_k)\alpha_k\|u - u^*\|^2 \\
(20) \quad + (1 - \beta_k)(1 - \alpha_k)\|u^k - u^*\|^2 - \|u^{k+1} - u^*\|^2 \\
\leq (1 - \beta_k)\alpha_k\|u - u^*\|^2 + \|u^k - u^*\|^2 - \|u^{k+1} - u^*\|^2 \\
\leq (1 - \beta_k)\alpha_k\|u - u^*\|^2 + (\|u^k - u^*\| + \|u^{k+1} - u^*\|)\|u^k - u^{k+1}\|.
\end{aligned}$$

It is clear that $\liminf_{k \rightarrow \infty} (1 - \beta_k)(1 - \alpha_k)\left(1 - \frac{\rho_k^2}{\alpha^2}\right) > 0$. So, from (R1) and (20), we have

$$(21) \quad \lim_{k \rightarrow \infty} \|v^k - \tilde{u}^k\| = 0.$$

From (18) and (19), we have

$$\begin{aligned}
\|u^{k+1} - u^*\|^2 &\leq \beta_k\|u^k - u^*\|^2 + (1 - \beta_k)\alpha_k\|u - u^*\|^2 \\
&\quad + (1 - \beta_k)(1 - \alpha_k)\|v^k - u^*\|^2 \\
&\leq \beta_k\|u^k - u^*\|^2 + (1 - \beta_k)\alpha_k\|u - u^*\|^2 \\
&\quad + (1 - \beta_k)(1 - \alpha_k)(\|u^k - u^*\|^2 - \|u^k - v^k\|^2) \\
&\leq \|u^k - u^*\|^2 + (1 - \beta_k)\alpha_k\|u - u^*\|^2 \\
&\quad - (1 - \beta_k)(1 - \alpha_k)\|u^k - v^k\|^2,
\end{aligned}$$

that is

$$\begin{aligned} & (1 - \beta_k)(1 - \alpha_k)\|u^k - v^k\|^2 \\ & \leq (1 - \beta_k)\alpha_k\|u - u^*\|^2 + \|u^k - u^*\|^2 - \|u^{k+1} - u^*\|^2 \\ & \leq (1 - \beta_k)\alpha_k\|u - u^*\|^2 + (\|u^k - u^*\| + \|u^{k+1} - u^*\|) \times \|u^{k+1} - u^k\|, \end{aligned}$$

which implies that

$$(22) \quad \lim_{k \rightarrow \infty} \|u^k - v^k\| = 0.$$

Since

$$\begin{aligned} \|S(\tilde{u}^k) - \tilde{u}^k\| & \leq \|S(\tilde{u}^k) - S(w^k)\| + \|S(w^k) - u^k\| + \|u^k - v^k\| + \|v^k - \tilde{u}^k\| \\ & \leq \|\tilde{u}^k - w^k\| + \|S(w^k) - u^k\| + \|u^k - v^k\| + \|v^k - \tilde{u}^k\| \\ & = \|P_C[v^k - \rho_k T(v^k)] - P_C[v^k - \rho_k T(\tilde{u}^k)]\| + \|S(w^k) - u^k\| \\ & \quad + \|u^k - v^k\| + \|v^k - \tilde{u}^k\| \\ & \leq \rho_k \|T(v^k) - T(\tilde{u}^k)\| + \|S(w^k) - u^k\| \\ & \quad + \|u^k - v^k\| + \|v^k - \tilde{u}^k\|. \end{aligned}$$

This together with (R3), (17), (21) and (22) implies that $\lim_{k \rightarrow \infty} \|S(\tilde{u}^k) - \tilde{u}^k\| = 0$.

Lemma 3.4 $\limsup_{k \rightarrow \infty} \langle u - z^0, w^k - z^0 \rangle \leq 0$, where $z^0 = P_\Omega(u)$ and $\Omega = \text{Fix}(S) \cap \text{VI}(T, C) \cap \text{EP}(F)$.

Proof. First, we show that $\limsup_{k \rightarrow \infty} \langle u - z^0, S(\tilde{u}^k) - z^0 \rangle \leq 0$. To show this inequality, we can choose a subsequence $\{\tilde{u}^{k_j}\}$ of $\{\tilde{u}^k\}$ such that

$$\lim_{j \rightarrow \infty} \langle u - z^0, S(\tilde{u}^{k_j}) - z^0 \rangle = \limsup_{k \rightarrow \infty} \langle u - z^0, S(\tilde{u}^k) - z^0 \rangle.$$

Since $\{\tilde{u}^{k_j}\}$ is bounded, there exists a subsequence $\{\tilde{u}^{k_{j_i}}\}$ of $\{\tilde{u}^{k_j}\}$ which converges weakly to z . Without loss of generality, we can assume that $\tilde{u}^{k_j} \rightarrow z$ weakly. From $\|S(\tilde{u}^k) - \tilde{u}^k\| \rightarrow 0$, we obtain $S(\tilde{u}^{k_j}) \rightarrow z$ weakly.

First we show $z \in \text{EP}(F)$. By $v^k = \Gamma_{r_k} u^k$, we have

$$F(v^k, w) + \frac{1}{r_k} \langle w - v^k, v^k - u^k \rangle \geq 0, \quad \forall w \in C.$$

From the monotonicity of F , we have $\frac{1}{r_k} \langle w - v^k, v^k - u^k \rangle \geq -F(v^k, w) \geq F(w, v^k)$ and hence $\langle w - v^{k_j}, \frac{v^{k_j} - u^{k_j}}{r_{k_j}} \rangle \geq F(w, v^{k_j})$. Since $\frac{v^{k_j} - u^{k_j}}{r_{k_j}} \rightarrow 0$ and $v^{k_j} \rightarrow z$ weakly, from the lower semi-continuity of $F(u, v)$ on the second variable v , we have

$$F(w, z) \leq 0, \forall w \in C.$$

For t with $0 < t \leq 1$ and $w \in C$, let $w_t = tw + (1 - t)z$. Since $w \in C$ and $z \in C$, we have $w_t \in C$ and hence $F(w_t, z) \leq 0$. So, from the convexity of equilibrium bifunction $F(u, v)$ on the second variable v , we have

$$0 = F(w_t, w_t) \leq tF(w_t, w) + (1 - t)F(w_t, z) \leq tF(w_t, w).$$

Hence $F(w_t, w) \geq 0$. Then, we have $F(z, w) \geq 0, \forall w \in C$. This indicates that $z \in \text{EP}(F)$.

Second, we show that $z \in VI(T, C)$. Set

$$Av = \begin{cases} T(v) + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then A is maximal monotone. Let $(v, u) \in G(A)$. Since $u - T(v) \in N_C v$ and $\tilde{u}^k \in C$, we have

$$\langle v - \tilde{u}^k, u - T(v) \rangle \geq 0.$$

On the other hand, from $\tilde{u}^k = P_C[v^k - \rho_k T(v^k)]$, we have

$$\langle v - \tilde{u}^k, \tilde{u}^k - (v^k - \rho_k T(v^k)) \rangle \geq 0$$

and hence

$$\langle v - \tilde{u}^k, \frac{\tilde{u}^k - v^k}{\rho_k} + T(v^k) \rangle \geq 0.$$

It follows that

$$\begin{aligned} \langle v - \tilde{u}^{k_j}, u \rangle &\geq \langle v - \tilde{u}^{k_j}, T(v) \rangle \\ &\quad - \langle v - \tilde{u}^{k_j}, \frac{\tilde{u}^{k_j} - v^{k_j}}{\rho_{k_j}} + T(v^{k_j}) \rangle \\ &= \langle v - \tilde{u}^{k_j}, T(v) - \frac{\tilde{u}^{k_j} - v^{k_j}}{\rho_{k_j}} - T(v^{k_j}) \rangle \\ &= \langle v - \tilde{u}^{k_j}, T(v) - T(\tilde{u}^{k_j}) \rangle + \langle v - \tilde{u}^{k_j}, T(\tilde{u}^{k_j}) - T(v^{k_j}) \rangle \\ &\quad - \langle v - \tilde{u}^{k_j}, \frac{\tilde{u}^{k_j} - v^{k_j}}{\rho_{k_j}} \rangle \\ &\geq \langle v - \tilde{u}^{k_j}, T(\tilde{u}^{k_j}) - T(v^{k_j}) \rangle - \langle v - \tilde{u}^{k_j}, \frac{\tilde{u}^{k_j} - v^{k_j}}{\rho_{k_j}} \rangle \end{aligned}$$

which implies that $\langle v - z, u \rangle \geq 0$. We have $z \in A^{-1}(0)$ and hence $z \in VI(T, C)$.

Thirdly, we prove that $z \in Fix(S)$. Assume that $z \notin Fix(S)$. Since $\tilde{u}^{k_j} \rightarrow z$ and $z \neq S(z)$, by Opial's condition we have

$$\begin{aligned} \liminf_{j \rightarrow \infty} \|\tilde{u}^{k_j} - z\| &< \liminf_{j \rightarrow \infty} \|\tilde{u}^{k_j} - S(z)\| \\ &\leq \liminf_{j \rightarrow \infty} (\|\tilde{u}^{k_j} - S(\tilde{u}^{k_j})\| + \|S(\tilde{u}^{k_j}) - S(z)\|) \\ &\leq \liminf_{j \rightarrow \infty} \|\tilde{u}^{k_j} - z\|, \end{aligned}$$

which is a contradiction. Then we get $z \in Fix(S)$. Hence, we deduce that $z \in Fix(S) \cap VI(T, C) \cap EP(F)$. Therefore, from the property (iii) of P_C , we have

$$\begin{aligned} \limsup_{k \rightarrow \infty} \langle u - z^0, w^k - z^0 \rangle &= \limsup_{k \rightarrow \infty} \langle u - z^0, S(\tilde{u}^k) - z^0 \rangle \\ (23) \quad &= \lim_{j \rightarrow \infty} \langle u - z^0, S(\tilde{u}^{k_j}) - z^0 \rangle \\ &= \langle u - z^0, z - z^0 \rangle \leq 0. \end{aligned}$$

4. STRONG CONVERGENCE

Now we prove the strong convergence of Algorithm 3.1.

Theorem 4.1 The sequence $\{u^k\}$ defined by (3) converges strongly to $z^0 = P_\Omega(u)$.

Proof. From (3), we have

$$\begin{aligned}
 \|u^{k+1} - z^0\|^2 &\leq \beta_k \|u^k - z^0\|^2 + (1 - \beta_k) \|S(\alpha_k u + (1 - \alpha_k)w^k) - z^0\|^2 \\
 &\leq \beta_k \|u^k - z^0\|^2 + (1 - \beta_k) \|\alpha_k(u - z^0) + (1 - \alpha_k)(w^k - z^0)\|^2 \\
 &\leq \beta_k \|u^k - z^0\|^2 + (1 - \beta_k) [(1 - \alpha_k) \|w^k - z^0\|^2 \\
 &\quad + 2\alpha_k \langle u - z^0, \alpha_k(u - z^0) + (1 - \alpha_k)(w^k - z^0) \rangle] \\
 (24) \quad &\leq \beta_k \|u^k - z^0\|^2 + (1 - \beta_k) [(1 - \alpha_k) \|u^k - z^0\|^2 \\
 &\quad + 2\alpha_k \langle u - z^0, \alpha_k(u - z^0) + (1 - \alpha_k)(w^k - z^0) \rangle] \\
 &= [1 - (1 - \beta_k)\alpha_k] \|u^k - z^0\|^2 + 2(1 - \beta_k)\alpha_k^2 \|u - z^0\|^2 \\
 &\quad + 2(1 - \beta_k)\alpha_k(1 - \alpha_k) \langle u - z^0, w^k - z^0 \rangle \\
 &= [1 - (1 - \beta_k)\alpha_k] \|u^k - z^0\|^2 + (1 - \beta_k)\alpha_k \left\{ 2\alpha_k \|u - z^0\|^2 \right. \\
 &\quad \left. + 2(1 - \alpha_k) \langle u - z^0, w^k - z^0 \rangle \right\}.
 \end{aligned}$$

Note that $\limsup_{k \rightarrow \infty} \left\{ 2\alpha_k \|u - z^0\|^2 + 2(1 - \alpha_k) \langle u - z^0, w^k - z^0 \rangle \right\} \leq 0$. Hence, by Lemma 2.4 and (24), we conclude that the sequence $\{u^k\}$ converges strongly to z^0 . This completes the proof.

It is clear that the following conclusion holds.

Theorem 4.2 Let C be a nonempty closed convex subset of a real Hilbert space H . Let T be an α -inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $Fix(S) \cap VI(T, C) \neq \emptyset$. Let $\{u^k\}$ be the sequence defined by Algorithm 3.2. If the algorithm parameters satisfy conditions (R1)-(R3), then the sequence $\{u^k\}$ converge strongly to $P_{Fix(S) \cap VI(T, C)}(u)$. \square

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