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BOUNDARY VALUE PROBLEMS FOR SECOND ORDER CONVEX AND NONCONVEX DIFFERENTIAL INCLUSIONS WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. We prove existence results for boundary value problems for second order convex and nonconvex differential inclusions with integral boundary conditions. The proofs use nonlinear alternatives of Leray-Schauder type and a selection theorem due to Bressan and Colombo.

1. Introduction

This paper is concerned with the boundary value problem for a second order ordinary differential inclusion with integral boundary conditions

$$x''(t) \in F(t, x(t)), \quad \text{a.e. } t \in J := [0,1], \tag{1.1}$$

$$x(0) - k_1 x'(0) = \int_0^1 h_1(x(s))ds, \qquad x(1) + k_2 x'(1) = \int_0^1 h_2(x(s))ds. \tag{1.2}$$

In problem (1.1)-(1.2), $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued function with nonempty compact values, $\mathcal{P}(\mathbb{R})$ is the class of all subsets of $\mathbb{R}$ and, for $i = 1, 2$, $h_i : \mathbb{R} \to \mathbb{R}$ are given functions and k_i are nonnegative constants.

Boundary value problems with integral boundary conditions constitute an important class of problems, because they include as special cases two, three, multi-point and nonlocal boundary value problems. Such problems for second order differential equations have been considered by many authors, for instance, see [3, 6, 8, 10, 12, 13] and the references therein. As far as we know, there are few authors who study the existence of solutions in the case of differential inclusions, among them we would like to cite Brykalov [2] and Halidias and Papageorgiou [7]. In [2], existence results for boundary value problems for differential inclusions with nonconvex right-hand sides and monotone nonlinear (integral) boundary conditions was studied. The technique of continuous selections of multivalued functions with decomposable values coupled with the method of monotone boundary conditions are used in these investigations. In [7], the authors use the method of upper and lower solutions with fixed point theorems to establish some existence results for second order differential inclusions with Sturm-Liouville and periodic boundary conditions.

Recently, Rahmat in [12] have used the method of upper and lower solutions with the method of generalized quasilinearization to study the existence of solutions of

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the boundary value problem for a second order differential equation with integral boundary conditions of the form (1.2)

$$x''(t) = f(t, x(t)), \quad \text{a.e. } t \in J := [0, 1], \tag{1.3}$$

$$x(0) - k_1 x'(0) = \int_0^1 h_1(x(s))ds, \qquad x(1) + k_2 x'(1) = \int_0^1 h_2(x(s))ds. \tag{1.4}$$

Motivated by this work, we consider problem (1.1)-(1.2) which is the multivalued form of problem (1.3)-(1.4). Our goal is to give some existence results for problem (1.1)-(1.2). Our method of study is to convert problem (1.1)-(1.2) into a fixed point problem. Then, we first apply the nonlinear alternative of Leray-Schauder type for multivalued functions [11] to prove an existence result when F has convex values. Next, we combine a continuous selection theorem [1] due to Bressan and Colombo with the nonlinear alternative of Leray-Schauder type for single valued functions [4] to prove the second existence result of this paper for F with nonconvex values. In both cases, the conditions established on the multivalued function F are common in the literature on differential equations and inclusions. In our main results, the only condition we require on the functions h_i, $i = 1, 2$, is continuity.

Let $C(J, \mathbb{R})$ and $L^1(J, \mathbb{R})$ denote the Banach spaces of continuous and Lebesgue integrable functions on J equipped with the normes $\|x\| = \max\{|x(t)| : t \in J\}$ and $\|x\|_{L^1} = \int_0^1 |x(t)|dt$, respectively. Consider $AC^1(J, \mathbb{R})$ the space of all continuous functions whose first derivatives exist and are absolutely continuous on J.

By a solution of (1.1)-(1.2) we mean a function $x \in AC^1(J, \mathbb{R})$ whose second derivative x'' exists and is a member of $L^1(J, \mathbb{R})$, that is, there exists a function $v \in L^1(J, \mathbb{R})$, $v(t) \in F(t, x(t))$ for almost every $t \in J$ such that $x''(t) = v(t)$ almost everywhere in J and x satisfies the conditions (1.2)

2. Preliminaries

In what follows we will enumerate some notions and results regarding single valued and multivalued functions. Although many of these are available in a more general framework, we will mention them only in the form we need in the present paper.

We say that a subset A of $L^1(J, \mathbb{R})$ is decomposable if for all $u, v \in A$ and all $I \subset J$ measurable, the function $u\chi_I + v\chi_{J-I} \in A$, where χ_I stands for the characteristic function of I.

For X a Banach space, $\mathcal{P}(X)$ is the class of all subsets of X.

Let X_1 and X_2 be Banach spaces and $G : X_1 \to \mathcal{P}(X_2)$ be a multivalued function. G is said to be closed (resp. convex and compact) valued if $G(x)$ is closed (resp. convex and compact) subset of X_2 for each $x \in X_1$. We say that G is lower semi-continuous (in brief l.s.c.) if for every open subset A of X_2, the set $\{x \in X_1 : G(x) \cap A \neq \emptyset\}$ is open. We say that G is upper semi-continuous (in brief u.s.c.) if for every closed subset A of X_2, the set $\{x \in X_1 : G(x) \cap A \neq \emptyset\}$ is closed. G is called continuous when it is l.s.c and u.s.c.

A multivalued function $G : X_1 \to \mathcal{P}(X_2)$ (resp. A function $G : X_1 \to X_2$) is said to be completely continuous if $\overline{G(A)}$ is compact for all bounded subsets A of X_1. If $X_1 = X_2$, we say that G has a fixed point if there is $x \in X_1$ such that $x \in G(x)$ (resp. $x = G(x)$).

A multivalued function $G : J \to \mathcal{P}(\mathbb{R})$ with nonempty compact convex values is said to be measurable if for every $y \in \mathbb{R}$, the function $t \to d(y, G(t)) = \inf\{|y - x| : x \in G(t)\}$ is measurable.

A multivalued function $G : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ (resp. A function $G : J \times \mathbb{R} \to \mathbb{R}$) is said to satisfy Carathéodory's conditions if

(i) $t \mapsto G(t, x)$ is measurable for each $x \in \mathbb{R}$,
(ii) $x \mapsto G(t, x)$ is continuous almost everywhere in J.

Moreover, G is called L^1-Carathéodory, if, in addition,

(iii) for each real number $r > 0$, there exists a function $h_r \in L^1(J, \mathbb{R})$ such that

$$\|G(t, x)\| = \sup\{|v| : v \in G(t, x)\} \leq h_r(t) \text{ (resp. } |G(t, x)| \leq h_r(t)) \text{ a.e. } t \in J$$

for all $x \in \mathbb{R}$ with $|x| \leq r$.

For each $x \in C(J, \mathbb{R})$, define the set of selections of a multivalued function $G : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ that belong to $L^1(J, \mathbb{R})$ by

$$S^1_G(x) = \{v \in L^1(J, \mathbb{R}) : v(t) \in G(t, x(t)) \ a.e. \ t \in J\}. \tag{2.1}$$

Then we have the following lemma [9] due to Lasota and Opial.

Lemma 2.1. *Let $G : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be an L^1-Carathéodory multivalued function with nonempty compact convex values. Then $S^1_G(x) \neq \emptyset$ for each $x \in \mathbb{R}$.*

The following hypotheses on the multivalued function $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ and the functions h_i, $i = 1, 2$, in problem (1.1)-(1.2) will be used throughout this work:

(C1) F is Carathéodory.
(C2) Each function $h_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, is continuous.
(C3) There exists an L^1-Carathéodory function $\psi : J \times \mathbb{R}_+ \to \mathbb{R}_+$ such that
  (i) $|F(t, x)| \leq \psi(t, |x|)$, for almost all $t \in J$ and all $x \in \mathbb{R}$,
  (ii) $\psi(t, x)$ is nondecreasing in x for almost all $t \in J$,
  as well as a constant $M^* > 0$ such that
  (iii) $M^* > \sup\limits_{|u| \leq M^*} |h_1(u)| + \sup\limits_{|u| \leq M^*} |h_2(u)| + C_0 \|\psi(\cdot, M^*)\|_{L^1}$, where

$$C_0 := \frac{(1 + k_1)(1 + k_2)}{1 + k_1 + k_2}.$$

Remark 2.2. *From conditions (C1) and (C3)-(i) we deduce that the multivalued function F is L^1-Carathéodory.*

3. Main results

3.1. **Convex case.** In this section, we are concerned with the existence of solutions for the problem (1.1)-(1.2) when the right hand side has convex values. So, we suppose that $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ in (1.1) is a multivalued function with nonempty compact convex values.

We need the following result in the sequel.

Lemma 3.1. [9] *Let $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be an L^1-Carathéodory multivalued function with nonempty compact convex values, and $\mathcal{K} : L^1(J, \mathbb{R}) \to C(J, \mathbb{R})$ be a linear continuous function. Then the operator, with nonempty compact convex values, $\mathcal{K} \circ S^1_F : C(J, \mathbb{R}) \to \mathcal{P}(C(J, \mathbb{R}))$ has a closed graph in $C(J, \mathbb{R}) \times C(J, \mathbb{R})$.*

Our main existence theorem in this section (i.e. Theorem 3.3) is obtained via the following nonlinear alternative of Leray-Schauder type for multivalued functions [11].

Theorem 3.2. *Let X be a Banach space, U an open and bounded subset of X with $0 \in U$ and $\Gamma : \overline{U} \to \mathcal{P}(X)$ a multivalued function. Suppose that*

(i) Γx *is nonempty, convex and closed for each* $x \in \overline{U}$,
(ii) Γ *has closed graph,*
(iii) Γ *is completely continuous.*

Then, either

(A1) Γ *has a fixed point in* $\overline{U}$, *or*
(A2) *there exists* $x \in \partial U$ *(the boundary of* U*) and* $\lambda \in (0,1)$ *with* $x \in \lambda \Gamma x$.

Now, we are able to state and prove our main theorem.

Theorem 3.3. *Suppose that conditions (C1)-(C3) are satisfied. Then problem (1.1)-(1.2) has a solution on* J.

Proof. To establish our result, we will apply Theorem 3.2 to the operator $\Gamma : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ defined, for any $x \in C(J,\mathbb{R})$, by Γx the set of functions $y \in C(J,\mathbb{R})$ such that

$$y(t) = P(t) + \int_0^1 G(t,s)v(s)ds, \quad t \in J, \quad v \in S_F(x),$$

where the function $P : J \to \mathbb{R}$ is defined, for any $t \in J$, by

$$P(t) = \frac{1}{1+k_1+k_2}\left[(1-t+k_2)\int_0^1 h_1(x(s))ds + (k_1+t)\int_0^1 h_2(x(s))ds\right] \tag{3.1}$$

and $G : J \times J \to \mathbb{R}$, the Green function associated with problem (1.1)-(1.2), is given by

$$G(t,s) = \frac{-1}{1+k_1+k_2}\begin{cases}(k_1+t)(1-s+k_2), & 0 \le t < s \le 1,\\ (k_1+s)(1-t+k_2), & 0 \le s < t \le 1.\end{cases} \tag{3.2}$$

Note that $|G(t,s)| \le C_0$ on $J \times J$, where C_0 is given in condition (C3)-(iii).

It is clear that Γ is well defined. By standard argument one can check that fixed points of Γ are solutions to problem (1.1)-(1.2). It remains to show that Γ satisfies all the conditions of Theorem 3.2.

Claim 1: Γx *is nonempty and convex for each* $x \in C(J,\mathbb{R})$. This is an immediate consequence of the fact that $S_F(x)$ is nonempty (see Lemma 2.1) and $F(x)$ is convex, respectively.

Claim 2: Γ *has closed graph.* So, let $(x_n)_n$ be a sequence in $C(J,\mathbb{R})$ and $x \in C(J,\mathbb{R})$ such that $x_n \to x$. Let $y_n \in \Gamma x_n$ such that $y_n \to y$. We will show that $y \in \Gamma x$.

Define the operator $\mathcal{K} : L^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ by

$$(\mathcal{K}v)(t) = \int_0^1 G(t,s)v(s)ds, \quad v \in L^1(J,\mathbb{R}), \quad t \in J.$$

We can easily see that $\mathcal{K}$ is well defined, linear and continuous. Let $n \in \mathbb{N}$ and $v_n \in S_F(x_n)$ such that

$$y_n(t) = P_n(t) + \int_0^1 G(t,s)v_n(s)ds, \quad t \in J.$$

We have $y_n - P_n \in \mathcal{K} \circ S_F(x_n)$ and $y_n - P_n \to y - P$. By Lemma 3.1, $\mathcal{K} \circ S_F$ has a closed graph, so that $y - P \in \mathcal{K} \circ S_F(x)$, that is,

$$y(t) = P(t) + \int_0^1 G(t,s)v(s)ds$$

for some $v \in S_F(x)$, which proves that $y \in \Gamma x$.

Claim 3: Γx *is closed for each* $x \in C(J,\mathbb{R})$. This assertion follows from Claim 2 by setting $x_n \equiv x$.

Claim 4: Γ *is completely continuous on* $C(J,\mathbb{R})$. To show this, we first show that Γ maps bounded sets into bounded sets. Let B be a bounded subset of $C(J,\mathbb{R})$. Then there exists a constant $r > 0$ such that $\|x\| \le r$ for all $x \in B$. Let $x \in B$, $y \in \Gamma x$ and $v \in S_F(x)$ such that, for $t \in J$,

$$y(t) = P(t) + \int_0^1 G(t,s)v(s)ds.$$

Conditions (C2) and (C3) yield

$$\begin{aligned} |y(t)| &\le |P(t)| + C_0 \int_0^1 |F(s,x(s))|ds \\ &\le \sup_{|u|\le r} |h_1(u)| + \sup_{|u|\le r} |h_2(u)| + C_0\|\psi(\cdot,r)\|_{L^1} := \eta, \end{aligned}$$

which implies that y is uniformly bounded with a uniform bound η. This finish to prove that ΓB is bounded.

Next we show that Γ maps bounded sets into equicontinuous sets. Let B be a bounded subset of $C(J,\mathbb{R})$ as above. Let $x \in B$, $y \in \Gamma x$ and $t, \tau \in J$. Then

$$\begin{aligned} |y(t) - y(\tau)| &\le |P(t) - P(\tau)| + \int_0^1 |G(t,s) - G(\tau,s)||F(s,x(s))|ds \\ &\le |P(t) - P(\tau)| + \int_0^1 |G(t,s) - G(\tau,s)|\psi(s,q)ds \\ &\le |P(t) - P(\tau)| + \|G(t,\cdot) - G(\tau,\cdot)\|_{L^1}\|\psi(\cdot,q)\|_{L^1}. \end{aligned}$$

In view of the continuity of P and G, and by use of the Lebesgue's convergence theorem, the right hand side tends to zero as $\tau \to t$. So ΓB is equicontinuous.

The results above, together with the Arzelá-Ascoli Theorem, allow us to conclude that, for any bounded subset B of $C(J,\mathbb{R})$, ΓB is relatively compact. Hence, Γ is completely continuous.

Now take M^* as in condition (C3)-(iii), set

$$U = \{x \in C(J,\mathbb{R}) : \|x\| < M^*\}$$

and consider the operator $\Gamma : \overline{U} \to \mathcal{P}(C(J,\mathbb{R}))$. From Theorem 3.2 it follows that either the operator inclusion $x \in \Gamma x$ has a solution (i.e. problem (1.1)-(1.2) has a solution) or there exists $x \in \partial U$ and $\lambda \in (0,1)$ such that $x \in \lambda\Gamma x$.

Claim 5: *The second alternative above does not occur.* Let x be a solution of $x \in \lambda\Gamma x$ with $\lambda \in (0,1)$ and suppose that $\|x\| = M^*$. Then, for $t \in J$ and some

$v \in S_F(x)$,

$$\begin{aligned}|x(t)| &= \left|\lambda\left(P(t) + \int_0^1 G(t,s)v(s)ds\right)\right| \\ &\leq \int_0^1 |h_1(x(s))|ds + \int_0^1 |h_2(x(s))|ds + C_0\int_0^1 \psi(s,|x(s)|)ds \\ &\leq \sup_{|u|\leq M^*}|h_1(u)| + \sup_{|u|\leq M^*}|h_2(u)| + C_0\|\psi(\cdot,M^*)\|_{L^1}.\end{aligned}$$

Consequently

$$M^* \leq \sup_{|u|\leq M^*}|h_1(u)| + \sup_{|u|\leq M^*}|h_2(u)| + C_0\|\psi(\cdot,M^*)\|_{L^1},$$

which contradicts condition (C3)-(iii). The conclusion of our theorem is straightforward from Theorem 3.2. □

Hereafter, we discuss a special case to illustrate how condition (C3)-(iii) can be satisfied.

Let us suppose that, for each $i = 1, 2$, the function h_i is continuous and there exist $\alpha_i, \beta_i, x_i \geq 0$, with $\alpha_1 + \alpha_2 < 1$, such that $|h_i(x)| \leq \alpha_i x + \beta_i$, for all $x \geq x_i$. Also, suppose that there exist a nondecreasing continuous eventually α_3-sublinear function $\overline{\psi} : \mathbb{R}_+ \to \mathbb{R}_+$, with $\alpha_3 < \frac{1-(\alpha_1+\alpha_2)}{C_0\|p\|_{L^1}}$, and a function $p \in L^1(J, \mathbb{R}_+)$ such that $\psi(t,x) = p(t)\overline{\psi}(x)$, for all $x \geq 0$.

Let $\varepsilon > 0$ be such that

$$\alpha_3 < \frac{1-(\alpha_1+\alpha_2)-\varepsilon}{C_0\|p\|_{L^1}}$$

and set

$$q(x) = \beta_1 + \beta_2 + (\alpha_1+\alpha_2-1)x + C_0\|p\|_{L^1}\overline{\psi}(x), \quad x \geq 0.$$

As $\overline{\psi}$ is eventually α_3-sublinear, there exists $x_3 > 0$ such that, for all $x \geq x_3$, $\overline{\psi}(x) \leq \alpha_3 x$ and then

$$\begin{aligned}q(x) &\leq \beta_1 + \beta_2 + (\alpha_1+\alpha_2-1)x + C_0\|p\|_{L^1}\alpha_3 x \\ &\leq \beta_1 + \beta_2 + (\alpha_1+\alpha_2-1)x + C_0\|p\|_{L^1}\frac{1-(\alpha_1+\alpha_2)-\varepsilon}{C_0\|p\|_{L^1}}x \\ &= \beta_1 + \beta_2 - \varepsilon x.\end{aligned}$$

Thus, for $x > \max\{x_3, \frac{\beta_1+\beta_2}{\varepsilon}\}$, we see that $q(x) < 0$, i.e.

$$q(x) = \beta_1 + \beta_2 + (\alpha_1+\alpha_2-1)x + C_0\|p\|_{L^1}\overline{\psi}(x) < 0$$

or

$$x > \beta_1 + \beta_2 + (\alpha_1+\alpha_2)x + C_0\|p\|_{L^1}\overline{\psi}(x).$$

This implies that

$$x > \sup_{|u|\leq x}|h_1(u)| + \sup_{|u|\leq x}|h_2(u)| + C_0\|\psi(\cdot,x)\|_{L^1},$$

whenever

$$x > \max\left\{x_1, x_2, x_3, \frac{\beta_1+\beta_2}{\varepsilon}\right\}.$$

Therefore, if h_i, $i = 1, 2$ and ψ are as above, we can always find a constant $M^* > 0$ satisfying condition (C3)-(iii). Hence, we have the following corollary.

Corollary 3.4. *Assume that (C1) holds. In addition, assume that the following conditions (C2') and (C3') are satisfied.*

(C2') *Each function* $h_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, *is continuous and there exist constants* α_i, β_i, $x_i \geq 0$, *with* $\alpha_1 + \alpha_2 < 1$, *such that* $|h_i(x)| \leq \alpha_i x + \beta_i$, *for all* $x \geq x_i$.

(C3') *There exist a continuous nondecreasing function* $\overline{\psi} : \mathbb{R}_+ \to \mathbb{R}_+$ *which is eventually* α_3*-sublinear, with* $\alpha_3 < \frac{1-(\alpha_1+\alpha_2)}{C_0\|p\|_{L^1}}$, *and a function* $p \in L^1(J, \mathbb{R}_+)$ *such that*

$$|F(t, x)| \leq p(t)\overline{\psi}(|x|), \text{ for almost all } t \in J \text{ and all } x \in \mathbb{R}.$$

Then problem (1.1)-(1.2) has a solution on J.

3.2. **Nonconvex case.** Suppose that the multivalued function $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ in (1.1) has nonempty compact (nonconvex) values. We assign to F the multivalued operator $\mathcal{F} : C(J, \mathbb{R}) \to \mathcal{P}(L^1(J, \mathbb{R}))$ defined by $\mathcal{F}(x) = S^1_F(x)$, where $S^1_F(x)$ is given by (2.1). We say that F is of lower semi-continuous type (in brief l.s.c. type) if the operator $\mathcal{F}$ has property (BC), that is,

1) $\mathcal{F}$ is l.s.c.,
2) $\mathcal{F}$ has nonempty closed and decomposable values.

The following selection result [1] due to Bressan and Colombo and Lemma 3.6 below are of great importance in the proof of Theorem 3.8.

Lemma 3.5. *Let* $\mathcal{F} : C(J, \mathbb{R}) \to \mathcal{P}(L^1(J, \mathbb{R}))$ *be a multivalued operator which has property (BC). Then* $\mathcal{F}$ *has a continuous selection, that is, there exists a continuous function (single valued)* $f_0 : C(J, \mathbb{R}) \to L^1(J, \mathbb{R})$ *such that* $f_0(x) \in \mathcal{F}(x)$ *for all* $x \in C(J, \mathbb{R})$.

Lemma 3.6. [5] *Let* $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *be a multivalued function with nonempty compact values. Assume (C1) and (C3)-(i) hold. Then* F *is of l.s.c. type.*

For the proof of Theorem 3.8, we rely on the well-known Leray-Schauder nonlinear alternative for single valued functions [4].

Theorem 3.7. *Let* X *be a Banach space and* U *an open and bounded subset of* X *with* $0 \in U$. *Suppose that* $\Gamma : \overline{U} \to X$ *is a continuous and completely continuous operator. Then, either*

(i) Γ *has a fixed point in* $\overline{U}$, *or*
(ii) *there exists a* $x \in \partial U$ *(the boundary of* U*) and a* $\lambda \in (0, 1)$ *with* $x = \lambda\Gamma x$.

Now, our main result of this section reads as follows.

Theorem 3.8. *Assume that conditions (C1), (C2) and (C3) hold. Then problem (1.1)-(1.2) has a solution on* J.

Proof. By Lemma 3.6 together with Lemma 3.5, the multivalued operator $\mathcal{F}$ defined above has a continuous selection $f_0 : C(J, \mathbb{R}) \to L^1(J, \mathbb{R})$ such that $f_0(x) \in \mathcal{F}(x)$ for all $x \in C(J, \mathbb{R})$. By analogy with the single valued case, we denote $f(\cdot, x(\cdot)) = f_0(x)(\cdot)$, for any $x \in C(J, \mathbb{R})$.

Consider then the problem

$$x''(t) = f(t, x(t)), \quad \text{a.e. } t \in J, \tag{3.3}$$

$$x(0) - k_1 x'(0) = \int_0^1 h_1(x(s))ds, \qquad x(1) + k_2 x'(1) = \int_0^1 h_2(x(s))ds. \tag{3.4}$$

It is clear that if $x \in AC^1(J,\mathbb{R})$ is a solution of (3.3)-(3.4), then x is a solution to the problem (1.1)-(1.2).

Integrating (3.3) on $[0,t]$ for $t \in J$, problem (3.3)-(3.4) becomes equivalent to the integral equation $x(t) = (\Gamma x)(t)$ where the operator $\Gamma : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is given by

$$(\Gamma x)(t) = P(t) + \int_0^1 G(t,s) f(s,x(s))ds, \quad x \in C(J,\mathbb{R}), \quad t \in J,$$

where the functions P and G are as in (3.1) and (3.2), respectively.

We will prove that Γ fulfills the hypotheses of Theorem 3.7.

We first show that Γ is continuous. To this end, let $\{x_n\}$ with $x_n \to x$ in $C(J,\mathbb{R})$. After some standard calculations we obtain, for $t \in J$,

$$\begin{aligned} |(\Gamma x_n)(t) - (\Gamma x)(t)| \leq \|h_1(x_n(\cdot)) - h_1(x(\cdot))\|_{L^1} + \|h_2(x_n(\cdot)) - h_2(x(\cdot))\|_{L^1} \\ +C_0\|f(\cdot,x_n(\cdot)) - f(\cdot,x(\cdot))\|_{L^1}. \end{aligned} \tag{3.5}$$

Let $B = \{u \in C(J,\mathbb{R}) : \|u\| \leq r\}$ for some $r > 0$ such that $\|x_n\|, \|x\| \leq r$, for all $n \in \mathbb{N}$. Since, by (C3)-(i),

$$|f(s,x_n(s)) - f(s,x(s))| \leq 2\psi(s,r), \text{ a.e. on } J,$$

then by the continuity of h_1, h_2 and f in its second variable and the Lebesgue's convergence theorem, from (3.5) we deduce that $\Gamma x_n \to \Gamma x$; which completes the proof that Γ is continuous.

Now, as the proofs that Γ is completely continuous and that the second alternative in Theorem 3.7 is deactivate follow the same lines as in the proof that the operator Γ in the proof of Theorem 3.3 possesses the same property and that the second alternative in Theorem 3.2 does not occur, they are omitted.

The conclusion of our theorem follows immediately by Theorem 3.7. □

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