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OSCILLATION OF NONAUTONOMOUS SECOND ORDER NEUTRAL DELAY DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper, we establish some new oscillation criteria for nonautonomous second order neutral delay dynamic equation with several delays

$$(x(t)-r(t)x(\tau(t)))^{\triangle\triangle}+H(t,x(h(t)))+G(t,x(g(t)))=0,$$

on a time scale $\mathbb{T}$. The results not only can be applied on neutral differential equations when $\mathbb{T}=\mathbb{R}$, neutral delay difference equations when $\mathbb{T}=\mathbb{N}$ and for neutral delay $q-$ difference equations when $\mathbb{T}=q^{\mathbb{N}}$ for $q>1$, but also improved most previous results.

1. Introduction

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On any time scale $\mathbb{T}$, we defined the *forward* and *backward jump* operators by

$$\sigma(t):=\inf\{s\in\mathbb{T}:s>t\}\ \textit{and}\ \rho(t):=\sup\{s\in\mathbb{T}:s<t\}, \tag{1.1}$$

A point $t\in\mathbb{T},\ t>\inf\mathbb{T}$ is said to be *left-dense* if $\rho(t)=t$, *right-dense* if $t>\sup\mathbb{T}$ and $\sigma(t)=t$, *left-scattered* if $\rho(t)<t$ and *right-scattered* if $\sigma(t)>t$. The graininess function $\mu:\mathbb{T}\to[0,\infty)$, is defined by $\mu(t):=\sigma(t)-t$. For the function $f:\mathbb{T}\to\mathbb{R}$ the (*delta*) derivative is defined by

$$f^{\triangle}(t):=\frac{f(\sigma(t))-f(t)}{\sigma(t)-t}, \tag{1.2}$$

f is said to be differentiable if its derivative exists. A useful formula is

$$f^{\sigma}:=f(\sigma(t))=f(t)+\mu(t)f^{\triangle}(t), \tag{1.3}$$

If f,g are differentiable, then fg and the quotient $\frac{f}{g}$ (where $gg^{\sigma}\neq 0$) are differentiable with

$$(fg)^{\triangle}=f^{\triangle}g+f^{\sigma}g^{\triangle}=fg^{\triangle}+f^{\triangle}g^{\sigma}, \tag{1.4}$$

and

$$\left(\frac{f}{g}\right)^{\triangle}:=\frac{f^{\triangle}g-fg^{\triangle}}{gg^{\sigma}}. \tag{1.5}$$

If $f^{\triangle}(t)\geq 0$, then f is nondecreasing.

A function $f:[a,b]\to\mathbb{R}$ is said to be *right-dense continuous* if it right continuous at each right-dense point and there exists a finite left limit at all left-dense

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points. The set of all right-dense continuous functions is denoted by C_{rd} . A function $f : \mathbb{T} \to \mathbb{R}$ is called regressive, if $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}$. The set of all functions $f : \mathbb{T} \to \mathbb{R}$ which are regressive and $rd-$continuous will be denoted by $\mathcal{R}$. We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)f(t) \neq 0,\ t \in \mathbb{T}\}$. A function F with $F^{\triangle} = f$ is called an antiderivative of f and then we define

$$\int_a^b f(t)\triangle t = F(b) - F(a), \tag{1.6}$$

where $a, b \in \mathbb{T}$. It is well known that rd-continuous functions possess antiderivatives. A simple consequence of formula (2.3) is

$$\int_t^{\sigma(t)} f(s)\triangle s = \mu(t)f(t), \tag{1.7}$$

and infinite integrals are defined as

$$\int_a^{\infty} f(t)\triangle t = \lim_{b\to\infty} \int_a^b f(t)\triangle t. \tag{1.8}$$

In the recent years, the theory of time scales has received a lot of attention which was introduced by Stefan Hilger in his Ph.D. thesis in 1988 in order to unify continuous and discrete analysis (see [10]). In fact there has been much activities concerning the oscillation and nonoscillation of solutions of dynamic equations on time scales (or measure chains). We refer the reader to recent papers [1-3, 7, 11, 13-18] and the references cited therein. A book on the subject of time scales, by Bohner and Peterson [5] summarizes and organizes much of time scales calculus, see also the book by Bohner and Peterson [4] for advances in dynamic equations on time scales. For oscillation of first-order neutral delay dynamic equations with a negative coefficient on the neutral term, Mathsen et. al. [14] considered the equation

$$(x(t) - r(t)x(\tau(t)))^{\triangle} + \alpha(t)x(h(t)) = 0. \tag{1.9}$$

and the authors posed the following question. What can be said about even order

equations

$$(x(t) - r(t)x(\tau(t)))^{\triangle^{2n}} + \alpha(t)x(h(t)) = 0$$

and various generalization?. Recently Saker in [16] considered the equation

$$(x(t) - r(t)x(\tau(t)))^{\triangle\triangle} + \alpha(t)x(h(t)) = 0. \tag{1.10}$$

Also, recently Liu et. al. [13] considered the equation

$$(x(t) - r(t)x(\tau(t)))^{\triangle} + H(t, x(h(t))) + G(t, x(g(t))) = 0 \tag{1.11}$$

on a time scale $\mathbb{T}$ and established some oscillation criteria, which in the special case when $\mathbb{T} = \mathbb{R}$ involve some oscillation criteria for neutral delay differential equations.

In this paper, we are concerned with the oscillation of the second-order nonlinear dynamic equation

$$(x(t)-r(t)x(\tau(t)))^{\triangle\triangle}+H(t,x(h(t)))+G(t,x(g(t)))=0 \tag{1.12}$$

on a time scale $\mathbb{T}$. Since we are interest in asymptotic behavior of solutions, we will suppose that the time scale $\mathbb{T}$ under consideration is not bounded above, i.e. it is a time scale interval of the form $[t_0,\infty)_{\mathbb{T}}=[t_0,\infty)\cap\mathbb{T}$. Through this paper, we assume that:

(H_1) $r\in C_{rd}(\mathbb{T},\mathbb{R}^+)$, h and $g\in C_{rd}(\mathbb{T},\mathbb{T}), \tau(t)<t, h(t)<t,\ g(t)<t$, $\lim\limits_{t\to\infty}\tau(t)=\infty$, $\lim\limits_{t\to\infty}h(t)=\infty$, $\lim\limits_{t\to\infty}g(t)=\infty$ and $0\le r(t)\le r<1$, $C_{rd}(\mathbb{T},\mathbb{S})$ denotes the set of all functions $f:\mathbb{T}\to\mathbb{S}$ -($\mathbb{S}$ is a time scale)- which are *right-dense continuous* on $\mathbb{T}$.

(H_2) $H(t,u),G(t,v)\in C(\mathbb{T}\times\mathbb{R},\mathbb{R})$ for each $t\in\mathbb{T}$ which are nondecreasing in u and $v, uH(t,u)>0$ for $u\neq0$ and $vG(t,v)>0$ for $v\neq0$.

(H_3) $|H(t,u)|\ge\alpha(t)\,|u|^{\lambda}$ and $|G(t,v)|\ge\beta(t)\,|v|^{\lambda}$, where $\alpha(t),\beta(t)\ge0$ and $0\le\lambda=\frac{p}{q}\le1$ with p,q are odd integers.

By a solution of equation (1.12), we mean a nontrivial real value function $x(t)$ which has the properties $(x(t)-r(t)x(\tau(t))\in C^2_{rd}[t_x,\infty), t_x>t_0$ and satisfying equation (1.12) for all $t>t_x$. Our attention is restricted to those solutions of equation (1.12) which exist on some half line $[t_x,\infty)$ and satisfy $\sup\{|x(t)|:t>t_1\}>0$ for any $t_1>t_x$.

A solution $x(t)$ of (1.12) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

Note that if $\mathbb{T}=\mathbb{R}$, we have $\sigma(t)=\rho(t)=t$, $f^{\Delta}(t)=f^{'}(t)$, and (1.10), (1.12) become respectively, the second-order neutral delay differential equations

$$[x(t)-r(t)x(\tau(t)]^{''}+\alpha(t)x(h(t))=0 \tag{1.13}$$

and

$$[x(t)-r(t)x(\tau(t)]^{''}+H(t,x(h(t)))+G(t,x(g(t)))=0. \tag{1.14}$$

For oscillation of equation (1.13) Graef et.al. [8] proved that , if $\alpha>0, 0\le r(t)<1$ and

$$\int_{t_0}^{\infty}\alpha(s)ds=\infty \tag{1.15}$$

then every unbounded solution of (1.13) oscillates. Note that condition (1.15) can not be applied for the second order neutral equation

$$[x(t)-r(t)x(\tau(t)]^{''}+\frac{\gamma}{(t-h)^2}x(t-h)=0,$$

where $\gamma>0, 0\le r(t)<1$. Also, Dzurina and Mihalikova in [6] considered the equation (1.5) when $r(t)=r$ where r is constant and gave the following oscillation criteria. If

$$\int_{t_0}^{\infty}(\alpha(s)h(s)\frac{1-r^{n+1}}{1-r}-\frac{1}{4h(s)})ds=\infty, \tag{1.16}$$

then, every solution of equation (1.13) oscillates.

If $\mathbb{T}=\mathbb{Z}$, we have $\sigma(t)=t+1$, $\mu(t)=1$, $f^{\Delta}=\Delta f$, and (1.12) becomes the second-order neutral delay difference equation

$$\Delta^2\left[x(t)-r(t)x(\tau(t))\right]+H(t,x(h(t)))+G(t,x(g(t)))=0. \tag{1.17}$$

If $\mathbb{T}=h\mathbb{Z}$, $h>0$, we have $\sigma(t)=t+h$, $\mu(t)=h$, $f^{\Delta}=\Delta_h f=\frac{f(t+h)-f(t)}{h}$ and (1.4) becomes the second-order neutral delay difference equation

$$\Delta_h^2\left[x(t)-r(t)x(\tau(t))\right]+H(t,x(h(t)))+G(t,x(g(t)))=0. \tag{1.18}$$

If $\mathbb{T}=q^{\mathbb{N}}=\{t:t=q^n,\ n\in\mathbb{N},\ q>1\}$, we have $\sigma(t)=qt$, $\mu(t)=(q-1)t$, $x_q^{\Delta}(t)=\frac{x(qt)-x(t)}{(q-1)t}$, and (1.3) becomes the second order $q-$neutral delay difference equation

$$\Delta_q^2\left[x(t)-r(t)x(\tau(t))\right]+H(t,x(h(t)))+G(t,x(g(t)))=0. \tag{1.19}$$

This paper is organized as follows: In Section 2, we establish some new sufficient conditions for oscillation of (1.12). In Section 3, we present some illustrative examples to show that our results are not only new but also improved many previous results.

2. Main results

In this section, we establish some sufficient conditions for the oscillation of equation (1.12). For the remainder of the paper we assume that $\delta^{-1}(t)$ is the inverse of the function $\delta(t)$ exists and satisfies $\delta^{-(n+1)}(t)=t+n\delta$

Theorem 2.1. *Assume that H_1-H_3 hold. Then every solution of (1.12) oscillates, if*

$$\int_{t_5}^{\infty}\{\alpha(s)(r(h(s))\tau(h(s)))^{\lambda}+\beta(s)(r(g(s))\tau(g(s)))^{\lambda}\}\triangle s=\infty. \tag{2.1}$$

Proof. Suppose to the contrary that equation (1.12) has a nonoscillatory solution $x(t)$. We may assume without loss of generality that there exists $t_1\geq t_0$ such that $x(t)>0$, $x(\tau(t))>0$ and $x(\delta(t))>0$ where $\delta=\min\{h,g\}$ for all $t>t_1$. Set

$$y(t)=x(t)-r(t)x(\tau(t)), \tag{2.2}$$

Then, it follows from equation (1.12) we have

$$y^{\triangle\triangle}(t)=-H(t,x(h(t)))-G(t,x(g(t)))\ for\ all\ t>t_1. \tag{2.3}$$

Now (H_2) with $x(\delta(t))>0$ implies that $y^{\Delta\Delta}(t)<0$. Thus $y^{\Delta}(t)$ is strictly decreasing. Now, we prove that $y^{\Delta}(t)>0$ on the interval $[t_1,\infty)_{\mathbb{T}}$. Assume not. Then there exists $t_2\geq t_1$ such that $y^{\Delta}(t_2)=C<0$. Then, since $y^{\Delta\Delta}(t)<0$, we have

$$y^{\Delta}(t)\leq y^{\Delta}(t_2)=C,\quad \text{for}\ \ t\geq t_2, \tag{2.4}$$

and therefore

$$y^{\Delta}(t)\leq C\quad for\ all\ \ t\geq t_2. \tag{2.5}$$

Integrating the last inequality from t_2 to t, we obtain

$$y(t) = y(t_2) + \int_{t_2}^{t} y^{\Delta}(s)\Delta s \leq y(t_2) + C(t - t_2), \tag{2.6}$$

and consequently $y(t) \to -\infty$ as $t \to \infty$ which implies that there exists $c > 0$ and $t_3 \geq t_2$ such that $y(t) < -c$ for $t \geq t_3$. Then, we have from (3.2) that

$$x(t) < -c + r(t)x(\tau(t)) \leq -c + rx(\tau(t)), \ for \ t \geq t_3, \tag{2.7}$$

which implies that $x(\delta^{-1}(t_3)) < -c + rx(t_3)$. Thus

$$x(\delta^{-(n+1)}(t_3)) \leq -c\sum_{i=0}^{n} r^i + r^{n+1}x(t_3) \leq -c + r^{n+1}x(t_3), \tag{2.8}$$

and so $x(\delta^{-(n+1)}(t_3)) < 0$ for large n, which contradicts the fact that $x(t) > 0$ for all $t \geq t_1$. Hence $y^{\Delta}(t) > 0$ and this implies that $y(t)$ is strictly increasing on $[t_1, \infty)$. We prove now that $y(t) > 0$ for $t \geq t_2$ where t_2 is large enough. Suppose not. Then there exists a $t_3 \geq t_1$ with $y(t_3) < 0$. Now, since $y(t)$ is strictly increasing then $y(t) > 0$ for $t \geq t_3$ (for if there exists a $t_4 > t_3$ with $y(t_4) > 0$, then $y(t) > 0$ for $t \geq t_4$, but we are assuming that $y(t) > 0$ for t large enough is not true). Then from (2.2) that $x(t) < rx(\tau(t))$, for $t \geq t_3$. Thus $x(\tau^{-1}(t)) \leq rx(t)$ and this implies after iteration that $x(\delta^{-(n+1)}(t)) \leq r^{n+1}x(t) \to 0$ for large n, since $0 < r < 1$ and so $x(\delta^{-(n+1)}(t)) < 0$ again, which contradicts the fact that $x(t) > 0$ for all $t \geq t_1$. Then, we have

$$y(t) > 0, \ y^{\Delta}(t) > 0, \ y^{\Delta\Delta}(t) < 0 \ for \ t \geq t_1. \tag{2.9}$$

Since $y^{\Delta\Delta}(t) < 0$ and $y(t) > 0$, then

$$\begin{aligned} y(t) &= y(t_4) + \int_{t_4}^{t} y^{\triangle}(s)\triangle s > (t - t_4)y^{\triangle}(t) > kty^{\triangle}(t) \ \text{ for } t > \frac{t_4}{(1-k)} := t_5, \\ 0 &< k < 1. \end{aligned}$$

Now $y(t) > 0$ implies that $y(t) < x(t)$ and $x(t) > r(t)x(\tau(t))$.Since $H(t, x)$ and $G(t, x)$ are nondecreasing in x , we get

$$\begin{aligned} 0 &= y^{\Delta\Delta}(t) + H(t, x(h(t))) + G(t, x(g(t))) \\ &\geq y^{\Delta\Delta}(t) + H(t, r(h(t))x(\tau(h(t)))) + G(t, r(g(t))x(\tau(g(t)))) \\ &\geq y^{\Delta\Delta}(t) + H(t, r(h(t))y(\tau(h(t)))) + G(t, r(g(t))y(\tau(g(t)))) \\ &\geq y^{\Delta\Delta}(t) + \alpha(t)(r(h(t))y(\tau(h(t))))^{\lambda} + \beta(t)(r(g(t))y(\tau(g(t))))^{\lambda} \\ &\geq y^{\Delta\Delta}(t) + \alpha(t)(kr(h(t))\tau(h(t))y^{\triangle}(\tau(h(t))))^{\lambda} \\ &\quad +\beta(t)(kr(g(t))\tau(g(t))y^{\triangle}(\tau(g(t))))^{\lambda} \end{aligned}$$

From nondecreasing property of $\tau(t)$, we have $\tau(h(t)) < \tau(t) < t$ and nonincreasing of $y^{\triangle}(t)$ implies that

$$y^{\triangle}(\tau(h(t))) \geq y^{\triangle}(\tau(t)) \geq y^{\triangle}(t).$$

and

$$y^{\triangle}(\tau(h(t))) \geq y^{\triangle}(\tau(t)) \geq y^{\triangle}(t).$$

Hence,

$$
\begin{aligned}
&0 \geq y^{\Delta\Delta}(t) + \alpha(t)(kr(h(t))\tau(h(t))y^{\triangle}(\tau(h(t))))^{\lambda} \\
(2.10) \qquad &+ \beta(t)(kr(g(t))\tau(g(t))y^{\triangle}(\tau(g(t))))^{\lambda} \\
&\geq y^{\Delta\Delta}(t) + [\alpha(t)(kr(h(t))\tau(h(t)))^{\lambda} + \beta(t)(kr(g(t))\tau(g(t)))^{\lambda}](y^{\triangle}(t))^{\lambda}.
\end{aligned}
$$

Then

$$
\alpha(t)(kr(h(t))\tau(h(t)))^{\lambda} + \beta(t)(kr(g(t))\tau(g(t)))^{\lambda} \leq -\frac{y^{\triangle\triangle}(t)}{(y^{\triangle}(t))^{\lambda}}.
$$

Integrating the above inequality from t_5 to ∞, we get

$$
\begin{aligned}
&\int_{t_5}^{\infty} \{\alpha(s)(r(h(s))\tau(h(s)))^{\lambda} + \beta(s)(r(g(s))\tau(g(s)))^{\lambda}\}\triangle s \\
\leq & \ -\int_{t_5}^{\infty} \frac{y^{\triangle\triangle}(s)}{(y^{\triangle}(s))^{\lambda}}\triangle s. \\
= & \ \lim_{t\to\infty} \int_{y^{\triangle}(t_5)}^{y^{\triangle}(t)} \frac{\triangle s}{s^{\lambda}} \\
= & \ \int_{y^{\triangle}(t_5)}^{0} \frac{\triangle s}{s^{\lambda}} < \infty.
\end{aligned}
$$

But

$$
\int_{t_5}^{\infty} \{\alpha(s)(r(h(s))\tau(h(s)))^{\lambda} + \beta(s)(r(g(s))\tau(g(s)))^{\lambda}\}\triangle s = \infty,
$$

so, equation (1.12) has no eventually positive solution. Similarly, we can prove that equation (1.12) has no eventually negative solution. Thus equation (1.12) is oscillatory.

Theorem 2.2. *Assume that $H_1 - H_3$ hold. Then every solution of (1.12) oscillates, if the inequality*

$$
(2.11) \qquad z^{\triangle}(t) + [\alpha(t)(kr(h(t))\tau(h(t)))^{\lambda} + \beta(t)(kr(g(t))\tau(g(t)))^{\lambda}]z^{\lambda}(\tau(t)) \leq 0,
$$

has no eventually positive solution.

Proof. Assume to the contrary that equation (1.12) has a nonoscillatory solution $x(t)$. Following the same steps used in the proof of Theorem 2.1, until to get (1.10). Putting $z(t) = y^{\triangle}(t)$ in (2.10) we get (2.11) which have a positive solution. Consequently if (2.11) has no eventually positive solution, then all solutions of (1.12) are oscillatory. This completes the proof of the theorem.

Theorem 2.2 reduces the question of oscillation of (1.12) to that of the absence of eventually positive solutions of the dynamic inequality (2.11).

Theorem 2.3 *Assume that $H_1 - H_2$ hold and $|H(t,u)| \geq \alpha(t)\,|u|$ and $|G(t,v)| \geq \beta(t)\,|v|$,where $\alpha(t), \beta(t) \geq 0$. If*

$$
(2.12) \qquad \int_{t_5}^{\infty} \{\alpha(s)(r(h(s))\tau(h(s))) + \beta(s)(r(g(s))\tau(g(s)))\}\triangle s = \infty.
$$

Then , every solution of

$$(x(t) - r(t)x(\tau(t)))^{\triangle\triangle} + \alpha(t)x(h(t)) + \beta(t)x(g(t)) = 0 \tag{2.13}$$

is oscillatory.

Proof. The proof follows directly from the Theorem 2.1. So we omitted it.

3. Examples

In this section, we give some examples to illustrate our main results.

Example 3.1. Consider the following second order neutral delay dynamic equation

$$(x(t) - \frac{1}{c}x(\gamma_1 t))^{\triangle\triangle} + \frac{(2+\sin t)}{t^{\alpha}}x^{\lambda}(\gamma_2 t) + \frac{(3+\cos t)}{t^{\beta}}x^{\lambda}(\gamma_3 t) = 0, t \in \mathbb{T}, \tag{3.1}$$

where $\mathbb{T}$ is a time scale, with $c > 1, 0 \leq \lambda = \frac{p}{q} \leq 1$, p, q are odd integers, $\alpha_1, \alpha_2 \in [\lambda, \lambda+1]$ and $\gamma_1, \gamma_2, \gamma_3 \in (0,1)$. In equation (1.12) $r(t) = \frac{1}{c}, \tau(t) = \gamma_1 t, h(t) = \gamma_2 t, g(t) = \gamma_2 t, H(t, x(h(t)) = \frac{(2+\sin t)}{t^{\alpha_1}}x^{\lambda}(h(t))$ and $G(t, x(g(t)) = \frac{(3+\cos t)}{t^{\alpha_2}}x^{\lambda}(g(t))$. (i.e. $\alpha(t) = \frac{(2+\sin t)}{t^{\alpha_1}}$ and $\beta = \frac{(3+\cos t)}{t^{\alpha_2}}$). Then we have

$$\begin{aligned}
&\int_{t_5}^{\infty} \{\alpha(s)(r(h(s))\tau(h(s)))^{\lambda} + \beta(s)(r(g(s))\tau(g(s)))^{\lambda}\}\triangle s \\
= &\int_{t_5}^{\infty} \{\frac{(2+\sin s)}{s^{\alpha_1}}((\frac{1}{c})(\gamma_1\gamma_2 s))^{\lambda} + \frac{(3+\cos s)}{s^{\alpha_2}}((\frac{1}{c})(\gamma_1\gamma_3 s))^{\lambda}\}\triangle s \\
\geq &\int_{t_5}^{\infty} \{\frac{1}{s^{\alpha_1}}((\frac{1}{c})(\gamma_1\gamma_2 s))^{\lambda} + \frac{1}{s^{\alpha 2}}((\frac{1}{c})(\gamma_1\gamma_3 s))^{\lambda}\}\triangle s \\
= &(\frac{\gamma_1\gamma_2}{c})^{\lambda}\int_{t_5}^{\infty} \frac{\triangle s}{s^{\alpha_1 - \lambda}} + (\frac{\gamma_1\gamma_3}{c})^{\lambda}\int_{t_5}^{\infty} \frac{\triangle s}{s^{\alpha_2-\lambda}} = \infty \text{ for } \alpha_1, \alpha_2 \in [\lambda, \lambda+1].
\end{aligned}$$

Hence, by Theorem (2.1) every solution of equation (3.1) oscillates.

Example 3.2. Consider the following second order neutral delay dynamic equation

$$(x(t) - e^{-\frac{1}{\lambda}(t-\tau)}x(t-\tau))^{\triangle\triangle} + x^{\lambda}(t-h_1) + \frac{1}{e^{-t}+1}x^{\lambda}(t-h_2) = 0, t \in \mathbb{T}, \tag{3.2}$$

where $\mathbb{T}$ is a time scale, where $0 \leq \lambda = \frac{p}{q} \leq 1$, p, q are odd integers, $\tau, h_1, h_2 > 0$, $r(t) = e^{-\frac{1}{\lambda}(t-\tau)}, \tau(t) = t - \tau, h(t) = t - h_1, g(t) = t - h_2, H(t, x(h(t)) = x^{\lambda}(h(t))$ and $G(t, x(g(t)) = \frac{1}{e^{-t}+1}x^{\lambda}(g(t))$. (i.e. $\alpha(t) = 1$ and $\beta = \frac{1}{e^{-t}+1}$). Then we have

$$\begin{aligned}
&\int_{t_5}^{\infty} \{\alpha(s)(r(h(s))\tau(h(s)))^{\lambda} + \beta(s)(r(g(s))\tau(g(s)))^{\lambda}\}\triangle s \\
= &\int_{t_5}^{\infty} \{e^{-\frac{1}{\lambda}(t-\tau-h_1)}(t-\tau-h_1)\}^{\lambda}\triangle s \\
&+ \int_{t_5}^{\infty} \frac{1}{e^{-s}+1}(e^{-\frac{1}{\lambda}(t-\tau-h_2)}(t-\tau-h_2))^{\lambda}\}\triangle s \\
\geq &\frac{3}{2}(\frac{\lambda}{e})^{\lambda}\int_{t_5}^{\infty} \triangle s = \infty.
\end{aligned}$$

Therefore, by Theorem (2.1), equation (3.2) is oscillatory.

Example 3.3. Consider the following specific second order neutral delay dynamic equation

$$(x(t)-\frac{1}{2}x(t-\tau))^{\triangle\triangle}+\frac{\gamma}{(t-h)^2}x^{\lambda}(t-h)=0, t\in\mathbb{T}, \tag{3.3}$$

where $\mathbb{T}$ is a time scale, where $\tau, h>0$, $r(t)=\frac{1}{2}, \tau(t)=t-\tau, h(t)=t-h, H(t,x(h(t))=x(h(t))$ and $G(t,x(g(t))=0$.(i.e. $\alpha(t)=\frac{\gamma}{(t-h)^2}, \gamma>0$ and $\beta(t)=0$). Then we have

$$\begin{aligned}&\int_{t_5}^{\infty}\{\alpha(s)(r(h(s))\tau(h(s)))+\beta(s)(r(g(s))\tau(g(s)))\}\triangle s\\ =&\int_{t_5}^{\infty}\frac{\gamma(s-\tau-h)}{2(s-h)^2}\triangle s\\ =&\frac{\gamma}{2}\int_{t_5}^{\infty}\frac{1}{s-h}(1-\frac{\tau}{s-h})\triangle s=\infty.\end{aligned}$$

Hence, by Theorem 2.3, every solution of equation (3.3) is oscillatory. This example shows that the results by Dzurina and Mihalikova [6] in the case when $\mathbb{T}=\mathbb{R}$, is not sharp, since by choosing $n=\infty$, we have

$$\begin{aligned}\int_{t_0}^{\infty}\left[\alpha(s)h(s)\frac{1-r^{n+1}}{1-r}-\frac{1}{4h(s)}\right]ds &= \int_{t_0}^{\infty}\left[\frac{\gamma}{s-h}(\frac{1}{1-\frac{1}{2}})-\frac{1}{4(s-h)}\right]ds\\ &= \int_{t_0}^{\infty}\left[\frac{2\gamma-\frac{1}{4}}{(s-h)}\right]ds=\infty, \text{ if } \gamma>\frac{1}{2}.\end{aligned}$$

Also, the result by Saker not sharp for equation (3.3). For, in his results [Example 2.2, 16], it was proved that this equation is oscillatory if $\gamma>\frac{1}{4}$ and Graef et. al. [8] condition (1.15) can not be applied. Therefore our results are not only new but also improve some previous results.

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