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THE NUMBER AND SUM OF NEAR $m-$EXTREMES

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Abstract. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with common continuous distribution function F. In this article we discuss distributional and asymptotical properties of the point process $N_{n,m}(\cdot) = \sum_{i=1}^n \mathbf{1}(X_{n-m+1:n} - X_i \in \cdot)$ driven by the mth upper order statistic $X_{n-m+1:n}$. Further we derive some limiting results for related sums, which are of some interest in insurance applications.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with common continuous distribution function F. By $X_{1:n} < \cdots < X_{n:n}$ we denote the order statistics of $X_1, \ldots, X_n$. Define the number of near $m-$extremes $K_n(a, m), a > 0, m \in \mathbb{N}$ by

$$K_n(a,m) \quad := \quad \sum_{i=1}^n \mathbf{1}(X_{n-m+1:n} - a < X_i \leq X_{n-m+1:n}),$$

with $\mathbf{1}(\cdot)$ the indicator function. More generally, let

$$N_{n,m}(\cdot) \quad := \quad \sum_{i=1}^n \mathbf{1}(X_{n-m+1:n} - X_i \in \cdot) \tag{1}$$

denote the related point process defined on $[0, \infty)$ driven by the mth upper order statistics. The marginal random variable $N_{n,m}(T)$ with T a Borel set of $[0, \infty)$ is in other words the number of $m-$extremes falling into the Borel set T. Setting $T = [0, a)$ we have $N_{n,m}(T) = K_n(a, m)$.

The number of near $m-$extremes is dealt with in several papers. It was introduced and carefully examined by Pakes and Steutel (1997) and Khmaladze et al. (1997) (considering $m = 1$ only). Hashorva (2003, 2004) showed that studying $K_n(a, m)$ for dependent samples is of some relevance for insurance applications. Estimation of the tail coefficient based on the number of near $m-$extremes is further discussed in Hashorva and Hüsler (2004). Recent papers on the topic are Balakrishnan and Stepanov (2004, 2005), Dembinska et al. (2007), where new ideas and results in connection with near extremes have been presented.

Point process approach was considered by Hashorva and Hüsler (2000) deriving both distributional and asymptotical results for $N_{n,1}(\cdot)$.

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The main topic of this paper is the asymptotic behaviour of the near $m-$extreme point process $N_{n,m}(\cdot)$. Special attention is given to $K_n(a,m)$; we show weak convergence of $K_n(a_n,m), n \geq 1$ without supposing the iid assumption. Further we discuss some asymptotic properties of sums of the near $m-$extremes (dealt with initially in Pakes (2000)) defined by

$$S_n(a,m) \quad := \quad \sum_{i=1}^{n} X_i \mathbf{1}\Big(X_i \in (X_{n-m+1:n}-a, X_{n-m+1:n}]\Big). \tag{2}$$

It is well-known that asymptotic properties of the mth upper order statistic are in some sense invariant to m. For example $X_{n-m+1:n}$ converges almost surely for any $m > 1$ to the upper endpoint of the distribution function F. We show in this paper that similar invariance properties are demonstrated by the number of near $m-$extremes.

Brief outline of the paper: We continue next with some preliminary results followed by Section 3 where several asymptotical results for the iid setup are presented. In Section 4 we show that convergence in distribution for $K_n(a_n,m)$ with $a_n \to a \geq 0$ and $m \in \mathbb{N}$ holds under certain dependence assumptions on the random sequence $X_i, i \geq 1$.

2. Preliminaries

Write in the following l_F, u_F for the lower and the upper endpoint of the distribution function F, respectively. We state now the following obvious lemma:

Lemma 1. *Let $\{X_n, n \geq 1\}$ be independent random variables with common continuous distribution function F. The random variable $N_{n,m}(T) - 1(0 \in T)$ with T some Borel set of $[0,\infty)$ and $n > m \geq 1$ has a mixed binomial distribution $B(n-m, p(T,x))$ with mixing random variable $X_{n-m+1:n}$ where*

$$p(T,x) := \boldsymbol{P}\{x - W \in T | W \leq x\}, \quad l_F < x < u_F,$$

with W a random variable with distribution function F.

It is also easy to see that the joint conditional distribution of

$$N_{n,m}(T_1) - 1(0 \in T_1), \ldots, N_{n,m}(T_k) - 1(0 \in T_k), \quad k \geq 2,$$

with $T_1, \ldots, T_k$ Borel sets of $[0,\infty)$ given the mth upper order statistic is multinomial. This fact is crucial when dealing with both distributional and asymptotical propertied of the point process $N_{n,m}(\cdot)$.

The law of the point process $N_{n,m}(\cdot)$ can be described via Markov kernels (see Reiss (1993) for basic properties of Markov kernels). We have

$$L(N_{n,m}(\cdot) - \mathbf{1}(0 \in \cdot)) \quad = \quad \int_{\mathbb{R}} G_n(\cdot, x)\, dL(X_{n-m+1:n})(x), \tag{3}$$

where $G_{n,m}(\cdot,x) \stackrel{d}{=} B(n-m, p(\cdot,x))$, $L(\cdot)$ denotes the law of the corresponding random element, and $\stackrel{d}{=}$ stands for equality of distribution functions. Referring to Theorem 1.5.1 of Reiss (1989) the mth upper order statistic $X_{n-m+1:n}$ possess the F-density

$$\frac{n! F^{n-m}(x)(1-F(x))^{m-1}}{(n-m)!(m-1)!}. \tag{4}$$

Application of Fubini's Theorem for Markov kernels (see Reiss (1993)) yields further

$$\begin{aligned}
\boldsymbol{E}\{N_{n,m}(T)\} &= \mathbf{1}(0\in T)+\int_{\mathbb{R}}\left(\int_{\mathbb{R}} y\,dL(B(n-m,p(T,x)))(y)\right)dL(X_{n-m+1:n})(x)\\
&= \mathbf{1}(0\in T)+(n-m)\boldsymbol{E}\{p(T,X_{n-m+1:n})\}\\
&= \mathbf{1}(0\in T)+\frac{n!}{(n-m-1)!(m-1)!}\\
&\qquad\times\int_{\mathbb{R}}\boldsymbol{P}\{x-X_1\in T|X_1\le x\}F^{n-m}(x)(1-F(x))^{m-1}\,dF(x). \qquad (5)
\end{aligned}$$

Substituting we have

$$\begin{aligned}
\boldsymbol{E}\{K_n(a,m)\} &= 1+\frac{n!}{(n-m-1)!(m-1)!}\\
&\qquad\times\int_{\mathbb{R}}[F(x)-F(x-a)]F^{n-m-1}(x)(1-F(x))^{m-1}\,dF(x) \qquad (6)
\end{aligned}$$

for all $n>m\ge 1$.

Alternatively, moments of the random variable $N_{n,m}(T), T\subset[0,\infty)$ can be easily derived from the expression of probability generating function (p.g.f) given below.

Lemma 2. *Under the assumptions and the notation of Lemma 1 we have for any integer $m, n>m$ and $s\in(0,1)$*

$$\boldsymbol{E}\{s^{N_{n,m}(T)-\mathbf{1}(0\in T)}\}=\int_{\mathbb{R}}[1-(1-s)p(T,x)]^{n-m}\,dL(X_{n-m+1:n})(x), \qquad (7)$$

where the distribution of the mth largest order statistic has F-density as in (4).

Certain assumptions on the tail asymptotic behaviour of the distribution function F allow us to derive several limiting results. If F has upper endpoint $u_F=\infty$, and the following limit

$$\lim_{x\to\infty}\frac{1-F(x+a)}{1-F(x)} = l(a)\in[0,1] \qquad (8)$$

exists for any $a>0$, then as in Pakes and Steutel (1997) we call F a thin-tailed, a thick-tailed, or a medium-tailed distribution whenever $l(a), a>0$, is equal to 0, 1, or is strictly between 0 and 1 for all $a>0$, respectively. Note in passing that Balakrishnan and Stepanov (2005) mention that (8) holds for all $a>0$ if and only if (iff) it holds for two distinct constants a_1, a_2 such that a_1/a_2 is an irrational number.

Examples of thin-tailed distribution functions are the half Normal law or the Weibull one with parameter $\alpha>1$. Gamma family belongs to the medium-tailed class, whereas Pareto distribution is a thick-tailed one.

The above tail asymptotic behaviour of F is related to the max-domain of attraction of F (see Pakes and Steutel (1997)).

It is well known (see e.g. Galambos (1987), Resnick (1987), Reiss (1989), Falk et al. (2004), Kotz and Nadarajah (2005), de Haan and Ferreira (2006)) that the distribution function F belongs to the max-domain of attraction of an extreme value distribution function H (write $F\in \mathrm{MDA}(H)$) if

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}\Big|F^t(q(t)x+r(t))-H(x)\Big| = 0, \qquad (9)$$

with $q(t) > 0, r(t), t > 0$, two measurable functions. The univariate distribution functions H is either the Gumbel distribution $\Lambda(x) = \exp(-\exp(-x)), x \in \mathbb{R}$, the Weibull distribution $\Psi_\alpha(x) = \exp(-|x|^\alpha), x < 0, \alpha > 0$, or the Fréchet distribution $\Phi_\alpha(x) = \exp(-x^{-\alpha}), x > 0, \alpha > 0$.

3. Asymptotics in the IID-Setup

In this section we consider the iid-setup, i.e., $X_i, i \geq 1$ are independent with common continuous distribution function F. Convergence in distribution for both $K_n(a, m)$ and $S_n(a, m)$ can be shown utilising the explicit expression in the right hand side of (7), provided that F satisfies certain asymptotic conditions. For instance Pakes and Steutel (1997), Li (1999), Balakrishnan and Stepanov (2004, 2005) make extensive use of (8).

We split this section in three parts beginning with some equivalent conditions for (8). In the second part we discuss almost sure convergence and CLT for the sum of near $m-$extremes $S_n(a, m)$. In the last part we derive an approximation of the point process for the interesting cases that F is in the Gumbel or the Weibull max-domain of attraction.

3.1. **Condition** (8).

We give next a general result which provides several equivalent conditions to (8). Previous partial results can be found in Proposition 2.5.3 in Hashorva (1999), Theorem 1.1. of Li (1999), Theorem 2.1 in Balakrishnan and Stepanov (2005).

Proposition 3. *Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables with continuous distribution function F with upper endpoint $u_F = \infty$. Then the following four statements are equivalent:*
i) For any $a > 0$ the limit in (8) *exists and $l(a) \in (0, 1]$.*
ii) For any $a > 0$ and any integer m, the discrete random variable $K_n(a, m)$ converges in distribution to some random variable $K^*_{a,m}$, where $K^*_{a,m} - 1$ has a negative binomial distribution with p.g.f

$$\boldsymbol{E}\{s^{K^*_{a,m}-1}\} = \left(\frac{l(a)}{1 - s(1 - l(a))}\right)^m, \quad l(a) \in (0, 1]. \tag{10}$$

iii) For any $a > 0$ and any integer m

$$\lim_{n\to\infty} \boldsymbol{E}\{K_n(a, m)\} = m[1 - l(a)]/l(a) + 1, \quad l(a) \in (0, 1] \tag{11}$$

holds.
iv) For any $a > 0$ we have

$$\lim_{n\to\infty} \boldsymbol{P}\{K_n(a, 1) = 1\} = l(a) \in (0, 1]. \tag{12}$$

Moreover, for any $a > 0$ and any integer m we have $K_n(a, m) \xrightarrow{p} \infty$ iff $l(a) = 0$.

Proof. First note that

$$\boldsymbol{P}\{K_n(a, 1) = 1\} = n \int_R F^{n-1}(x - a)\, dF(x), \quad \forall a > 0, n > 1.$$

Transforming the expression of the expectation in (5) we obtain

$$\begin{aligned}
\boldsymbol{E}\{K_n(a,m)\}-1 &= (n-m)\boldsymbol{E}\{[1-F(X_{n-m+1:n}-a)/F(X_{n-m+1:n})]\}\\
&= n-m-(n-m)\boldsymbol{E}\{F(X_{n-m+1:n}-a)/F(X_{n-m+1:n})\}\\
&= n\boldsymbol{E}\{[1-F(X_{n-m:n-1}-a)]\}-m\\
&= n\boldsymbol{E}\{F_{n-m:n-1}(X_1+a)\}-m. \qquad (13)
\end{aligned}$$

Statement $i)$ implies that F is in the Gumbel max-domain of attraction. Furthermore, the scaling function $q(t)$ can be choose to be constant in t. The proof follows now using (7), the Gumbel max-domain of attraction of F, and applying Lemma 4 below with the function $\chi(\cdot)$ constant and $\rho=1$. □

The following lemma follows immediately form Lemma 2.5.1 in Hashorva (1999) which is stated for multivariate distribution functions.

Lemma 4. *Let F,G be two continuous univariate distribution functions with upper endpoint ∞, and let $\rho\geq 1, C\in[0,\infty]$ be two given constants. If $\chi:[0,\infty)\to[0,\infty)$ is a measurable function such that $\lim_{x\to 0}\chi(tx)/\chi(x)=1, \forall t>0$, then the following two statements are equivalent:*
i) As $n\to\infty$ we have for any $c\in[0,\infty)$

$$n\int_{\mathbb{R}} G^{n-c}(x)\,dF(x) \quad = \quad (1+o(1))\frac{C}{n^{\rho-1}}\chi(1/n). \qquad (14)$$

ii) As $x\to\infty$

$$\frac{1-F(x)}{(1-G(x))^{\rho}\chi(1-G(x))} \quad = \quad (1+o(1))\frac{C}{\Gamma(\rho+1)}. \qquad (15)$$

Remark 1. a) Imposing an additional technical condition on the distribution function F Balakrishnan and Stepanov (2005) show in Theorem 2.2 therein that the almost sure convergence $K_n(a,m)\to 1$ is equivalent to $l(a)=1$ for all $a>0$. Note that if $l(a)=1$, then statement ii) in Proposition 3 means convergence in probability to 1.
b) Lemma 4 is motivated by Lemma 1.3 in Maller and Resnick (1984).

Our next result concerns the asymptotic approximation of the ratio of the number and sum of near $m-$extremes. It subsumes Theorem 7.1 in Pakes (2000).

Proposition 5. *Under the assumptions and the notation of Proposition 3*

$$S_n(a,m)/X_{n-m+1:n} \overset{d}{\to} 1+K^*_{a,m}, \quad n\to\infty \qquad (16)$$

holds for any $a>0$ and any integer m, iff either of the statements $i),ii),iii),iv)$ in Proposition 3 hold.
Furthermore, for any $a>0$ and any integer m the convergence in probability

$$S_n(a,m)/X_{n-m+1:n} \overset{p}{\to} \infty, \quad n\to\infty \qquad (17)$$

is valid iff $l(a)=0$, and

$$S_n(a,m)/X_{n-m+1:n} \overset{p}{\to} 1, \quad n\to\infty \qquad (18)$$

holds iff $l(a)=1$.

Proof. For any positive a and any integer m

$$[X_{n-m+1:n} - a]K_n(a,m) \le S_n(a,m) \le X_{n-m+1:n}K_n(a,m)$$

holds almost surely. Furthermore the almost sure convergence

$$X_{n-m+1:n} \to \infty, \quad n \to \infty$$

implies

$$a/X_{n-m+1:n} \to 0, \quad n \to \infty,$$

thus the proof follows using Proposition 3. □

3.2. **Almost Sure Convergence and CLT.** Hashorva (1999) shows the convergence in probability of

$$K_n(a,m)/n \quad \stackrel{p}{\to} \quad 1 - F(u_F - a), \quad n \to \infty,$$

provided that the upper endpoint u_F of F is finite. Since

$$\lim_{n\to\infty} \boldsymbol{E}\{K_n(a,m)/n\} \quad = \quad 1 - F(u_F - a)$$

the convergence holds also in the rth ($r > 0$) mean. Almost sure convergence and CLT for $K_n(a,m)$ are stated in Hashorva and Hüsler (2004).
Next, we discuss some asymptotic properties for the sum of near $m-$extremes.

Proposition 6. *Let $\{X_n, n \ge 1\}$ be a positive sequence of independent random variables with continuous distribution function F. If both the lower and the upper endpoint l_F, u_F of the distribution function F are finite, then for any $a > 0$ and any integer m we have the almost sure convergence*

$$S(a,m)/n \quad \stackrel{a.s.}{\to} \quad \boldsymbol{E}\{X_1 \mathbf{1}(X_1 > u_F - a)\}, \quad n \to \infty. \tag{19}$$

Proof. Since for any $\varepsilon > 0$

$$X_i \mathbf{1}(u_F - X_{n-m+1:n} > \varepsilon) \le u_F \mathbf{1}(u_F - X_{n-m+1:n} > \varepsilon) \to 0, \quad n \to \infty$$

almost surely, we have for all large n

$$S(a,m)/n \quad \le \quad \sum_{i=1}^n X_i \mathbf{1}(X_i > u_F - a - \varepsilon)/n + u_F \mathbf{1}(u_F - X_{n-m+1:n} > \varepsilon)$$

and

$$S(a,m)/n \quad \ge \quad \sum_{i=1}^n X_i \mathbf{1}(X_i > u_F - a)/n - u_F(m-1)/n,$$

hence the proof follows by the Strong Law of Large Numbers. □

We show next the CLT for the sum of near $m-$extremes.

Proposition 7. *Let $\{X_n, n \ge 1\}$ be as in Proposition 6 and suppose further that the lower endpoint of F is non-negative and the upper endpoint u_F is finite. Assume that there exists a positive sequence $\{c_n, n \ge 1\}$ such that*

$$\lim_{n\to\infty} n \ln F(u_F - c_n) \quad = \quad -\infty, \tag{20}$$

and for some positive constant a we have

$$\lim_{n\to\infty} \sqrt{n}[F(u_F - a) - F(u_F - a - c_n)] \quad = \quad 0, \tag{21}$$

with $F(u_F - a) \in (0,1)$. *If further* $\sigma_a^2 := Var(X_1 \mathbf{1}(u_F - a \le X_1 \le u_F)) \in (0, \infty)$, *then the convergence in distribution*

$$[S_n(a,m) - n\boldsymbol{E}\{X_1 \mathbf{1}(X_1 > u_F - a)\}]/\sqrt{n} \quad \stackrel{d}{\to} \quad W, \quad n \to \infty \tag{22}$$

holds with W *a mean zero Gaussian random variable with variance* σ_a^2.

Proof. For any $m \ge 1$, condition (20) implies $\sqrt{n}\mathbf{1}(X_{n-m+1:n} < u_F - c_n) = o_p(1)$ as $n \to \infty$. Consequently (recall $u_F < \infty$)

$$S_n(a,m)/\sqrt{n} \le \sum_{i=1}^n X_i \mathbf{1}(u_F - X_i < a + c_n)/\sqrt{n} + o_p(1)$$

and

$$S_n(a,m)/\sqrt{n} \ge \sum_{i=1}^n X_i \mathbf{1}(u_F - X_i < a)/\sqrt{n} - u_F(m-1)/\sqrt{n}.$$

By (21)

$$\sum_{i=1}^n X_i \mathbf{1}(a \le u_F - X_i < a + c_n)/\sqrt{n} = o_p(1), \quad \text{as } n \to \infty,$$

thus the proof follows easily by applying the CLT for the random sequence $X_i \mathbf{1}(u_F - X_i < a), i \ge 1$ (see e.g. Kallenberg (1997)). □

3.3. **Asymptotics for the point process.** Finally we consider the asymptotic behaviour of the point process $N_{n,m}(\cdot)$. Since this point process is driven by the mth upper extreme order statistics, in order to deal with its asymptotic behaviour we consider the case when F is in a max-domain of attraction of a univariate distribution function H, assuming (9) holds with norming functions $q(t), r(t)$.

We show first weak convergence of the scaled point process

$$N^*_{n,m}(\cdot) := \sum_{i=1}^n \mathbf{1}((X_{n-m+1:n} - X_i)/q_n \in \cdot),$$

with $q_n := q(n)$ the norming constant from (9). The asymptotics for $m = 1$ is dealt with in Hashorva (1999). As in that case, utilising the same arguments (see also Theorem 1.2 in Hashorva and Hüsler (2000)) the scaled point process $N^*_{n,m}$ can be approximated ($n \to \infty$) by a Cox process plus a point at 0. We have the following result:

Proposition 8. *Let* $\{X_n, n \ge 1\}$ *be a sequence of iid random variables with continuous distribution function* F *satisfying condition* (9). *If the univariate distribution function* H *is standard Gumbel or Weibull, then for all* $m \ge 1$

$$N^*_{n,m}(\cdot) \quad \stackrel{d}{\to} \quad N_m(\cdot) + \mathbf{1}(0 \in \cdot), \quad n \to \infty, \tag{23}$$

where $N_m(\cdot)$ *is a Cox process with stochastic intensity*

$$\nu([a,b), X_*^{(m)}) = \ln\left(\frac{H(X_*^{(m)} - a)}{H(X_*^{(m)} - b)}\right)$$

for $0 < a < b < \infty$. The mixing random variable $X_*^{(m)}$ has continuous distribution function H_m given by

$$H_m(x) \quad = \quad H(x) \sum_{r=0}^{m-1} \frac{(-\ln H(x))^m}{m!}, \quad \forall x \in \mathbb{R}.$$

The following corollary is immediate:

Corollary 9. *Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables with continuous distribution function F such that for all $a > 0$ condition (8) holds with $l(a) \in (0,1)$. Then (9) holds with $q(t) = q \in (0, \infty), r(t), t > 0$ and further for all $m \geq 1$*

$$N_{n,m}(\cdot) \quad \xrightarrow{d} \quad N_m(\cdot) + \mathbf{1}(0 \in \cdot), \quad n \to \infty, \tag{24}$$

where $N_m(\cdot)$ is a Cox process with stochastic intensity

$$\nu([a,b), X_*^{(m)}) = \ln\left(\frac{H(X_*^{(m)} - a/c)}{H(X_*^{(m)} - b/c)}\right), \quad 0 < a < b < \infty$$

and the mixing random variable $X_*^{(m)}$ as in Proposition 8.

Note in passing that weak convergence of the unscaled point process $N_{n,m}(\cdot)$ follows by the above result, since the scaling function is constant in Corollary 9.

4. Approximations in the case of dependence

Since we want to drop the iid assumption (which implies (7)), we follow here a different approach which we motivate below. If the distribution function F satisfies (9) with functions $q(t) > 0, r(t), t \in \mathbb{R}$, we have the joint convergence in distributions ($n \to \infty$)

$$\Big((X_{n:n} - r_n)/q_n, \ldots, (X_{n-l+1:n} - r_n)/q_n\Big) \quad \xrightarrow{d} \quad (X_*^{(1)}, \ldots, X_*^{(l)}), \tag{25}$$

where $q_n = q(n) > 0, r_n := r(n), n \geq 1$, and the random vector $(X_*^{(1)}, \ldots, X_*^{(l)})$ has the distribution function $\mathbf{H}_l$ with density function $\mathbf{h}_l$ given by

$$\mathbf{h}_l(\mathbf{x}) \quad = \quad H(x_l) \prod_{i=1}^{l} \frac{H'(x_i)}{H(x_i)}, \quad \text{with } x_l < \cdots < x_2 < x_1 \tag{26}$$

and x_1, x_l are such that $H(x_1), H(x_l) \in (0,1)$ (see e.g. Reiss (1989)). Hence for $0 < k \leq n - m - 1$, $a_n = c_a q_n(1 + o(1)), n \geq 1$, with $c_a > 0$ some constant, we have

$$\begin{aligned}
&\boldsymbol{P}\{K_n(a_n, m) > k\} \\
&\quad = \quad \boldsymbol{P}\{X_{n-m+1:n} - X_{n-m+1-k:n} < a_n\} \\
&\quad = \quad \boldsymbol{P}\{(X_{n-m+1:n} - r_n)/q_n - (X_{n-m+1-k:n} - r_n)/q_n < c_a(1 + o(1))\} \\
&\qquad\qquad [\text{ since the convergence holds locally uniformly, we get }] \\
&\quad \to \quad \boldsymbol{P}\{X_*^{(m)} - X_*^{(m+k)} \leq c_a\} =: 1 - \mathbf{H}^{(m)}(c_a, k), \quad n \to \infty. \qquad (27)
\end{aligned}$$

In deriving (27) we only needed (25), consequently at this point dropping the iid assumption is possible. We need however to introduce mixing type conditions D and D' as in Leadbetter et al. (1983), see Corollary 3.2.

The next result generalises Theorem 2 and Theorem 3 of Li and Pakes (1998b). We denote in the sequel by α_H the lower endpoint of the distribution function H.

Proposition 10. *Assume that condition (25) holds for the sequence of random variables $\{X_n, n \geq 1\}$ with continuous distribution function F, with constants $q_n > 0$, r_n and $\mathbf{H}_m$ as in (26). If $a_n \sim c_a q_n$, $c_a > 0$, then for $m, l \geq 1$ and any $x \in \mathbb{R}$ such that $H(x) \in (0,1)$ we have*

$$\begin{aligned} &\lim_{n\to\infty} \boldsymbol{P}\{K_n(a_n, m) \leq k, (X_{n-l+1:n} - r_n)/q_n \leq x\} \\ &= \boldsymbol{P}\{X_*^{(m)} - X_*^{(m+k)} > c_a, X_*^{(l)} \leq x\} \\ &=: \mathbf{H}^{(m)}(c_a, x, k, l), \quad k \in \mathbb{N}. \end{aligned} \tag{28}$$

The random sequence $K_n(a_n, m)$ converges weakly to a positive non-degenerate random variable $K^*_{a,m}$ with distribution function $\mathbf{H}^{(m)}(c_a, k)$ iff $\alpha_H = -\infty$.
Further if for some $j > m - 1$

$$(X_{j-m+1:n} - r_n)/q_n \xrightarrow{p} x_0, \quad n \to \infty \tag{29}$$

then we have

$$\lim_{n\to\infty} \boldsymbol{P}\{K_n(a_n, m) > n - j\} = H(c_a + x_0) \sum_{r=0}^{m-1} (-\ln H(c_a + x_0))^r / r!. \tag{30}$$

Proof. By the assumptions we have

$$\begin{aligned} &\lim_{n\to\infty} \boldsymbol{P}\{K_n(a_n, m) \leq k, (X_{n-l+1:n} - r_n)/q_n \leq x\} \\ &= \boldsymbol{P}\{X_*^{(m)} - X_*^{(m+k)} > c_a, X_*^{(l)} \leq x\}, \end{aligned}$$

hence the weak convergence of $K_n(a_n, m)$ to a non-degenerated random variable follows if we show further that

$$\lim_{k\to\infty} \boldsymbol{P}\{X_*^{(m)} - X_*^{(m+k)} > c_a\} = 1.$$

Since H is continuous, in light of (26) the limit distribution of the ith largest order statistic is

$$\boldsymbol{P}\{X_*^{(i)} \leq x\} = H(x) \sum_{r=0}^{i-1} \frac{(-\ln H(x))^r}{r!} =: H_i(x), \quad x : H(x) > 0,$$

hence we get for all $x \in \mathbb{R}$ such that $H(x) > 0$

$$\lim_{i\to\infty} \boldsymbol{P}\{X_*^{(i)} \leq x\} = \lim_{i\to\infty} H(x) \sum_{r=0}^{i-1} \frac{(-\ln H(x))^r}{r!} = H(x)\exp(-\ln H(x)) = 1.$$

Consequently, the monotonicity of the random sequence $\{X_*^{(i)}, i \geq 1\}$ implies

$$X_*^{(i)} \overset{a.s.}{\to} \alpha_H, \quad \text{as } i \to \infty$$

and thus

$$\lim_{k\to\infty} \boldsymbol{P}\{X_*^{(m)} - X_*^{(m+k)} > c_a\} = 1 - \boldsymbol{P}\{X_*^{(m)} \leq c_a + \alpha_H\} < 1$$

if and only if we have $\alpha_H > -\infty$. Finally by (29) for any j fixed

$$\begin{aligned} &\boldsymbol{P}\{K_n(a_n, m) > n - j\} \\ &= \boldsymbol{P}\{X_{n-m+1:n} - X_{j-m+1:n} < a_n\} \\ &= \boldsymbol{P}\{(X_{n-m+1:n} - r_n)/q_n - (X_{j-m+1:n} - r_n)/q_n < c_a(1 + o(1))\} \\ &\to \boldsymbol{P}\{X_*^{(m)} \leq c_a + x_0\}, \quad n \to \infty, \end{aligned}$$

hence the proof is complete. □

In the above theorem we do not assume the independence of X_i, but only condition (25). Next, we focus attention to stationary random sequences which satisfy certain mixing type conditions. For the extreme value theory the distributional mixing conditions $D_3(\boldsymbol{u}_n)$ for the long range dependence and $D'(u_n)$ for the local dependence are sufficient to assume (see Leadbetter et al. (1983)). Here $\boldsymbol{u}_n = (u_{ni}) \in \mathbb{R}^3$ with $u_{ni} = q_n x_i + r_n$ and any $x_i \in \mathbb{R}, i \le 3$, where q_n and r_n are from (5). We mention these conditions which imply (25).

Condition $D_3(\boldsymbol{u}_n)$: For any fixed p, q and integers $1 \le i_1 < i_2 < \cdots < i_p < j_1 < j_2 < \cdots < j_q \le n$ with $j_1 - i_p > \ell$, and any $k_h, k'_{h'} \in \{1,2,3\}$ for $h \le p, h' \le q$, assume that

$$\Big| \boldsymbol{P}\Big\{X_{i_h} \le u_{n,k_h}, h \le p, X_{j_{h'}} \le u_{n,k'_{h'}}, h' \le q\Big\} - \boldsymbol{P}\Big\{X_{i_h} \le u_{n,k_h}, h \le p\Big\} \times \boldsymbol{P}\Big\{X_{j_{h'}} \le u_{n,k'_{h'}}, h' \le q\Big\}\Big| \le \alpha_{n,\ell},$$

where $\alpha_{n,\ell} \to 0$ for some sequence $\ell = \ell(n) = o(n)$.

Condition $D'(u_n)$: Assume that for $k \to \infty$

$$\limsup_{n\to\infty} \sum_{1<i\le n/k} P\{X_1 > u_n, X_i > u_n\} \to 0.$$

Proposition 11. *Let $\{X_n, n \ge 1\}$ be a stationary random sequence with distribution function F so that (9) holds. If also the conditions $D_3(\boldsymbol{u}_n)$ and $D'(u_n)$ are satisfied for any x with $H(x) \in (0,1)$ and $u_n = u_n(x) = q_n x + r_n$, then (28) holds.*

Proof. The proof follows immediately from Theorem 5.5.1 of Leadbetter et al. (1983). □

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