TORIC FIBRATIONS AND MIRROR SYMMETRY

ARTUR ELEZI

Abstract. The relation between the quantum $\mathcal{D}$-modules of a smooth variety $X$ and a toric bundle is studied here. We describe the relation completely when $X$ is a semi-ample complete intersection in a toric variety. In this case, we obtain all the relations in the small quantum cohomology ring of the bundle.

1. Introduction and Goals

For a smooth, projective variety $Y$ we denote by $Y_{k,\beta}$ the moduli stack of rational stable maps of class $\beta \in H_2(Y,\mathbb{Z})$ with $k$-markings (Fulton et al [8]) and $[Y_{k,\beta}]$ its virtual fundamental class (Behrend et al [3], Li et al [13]). Genus zero Gromov-Witten invariants are defined as appropriate integrals over $[Y_{k,\beta}]$. We let $e : Y_{1,\beta} \to Y$ be the evaluation map, $\psi$ - the first chern class of the cotangent line bundle on $Y_{1,\beta}$ and $f : Y_{1,\beta} \to Y_{0,\beta}$ - the forgetful morphism. The formal completion of an arbitrary ring $\mathcal{R}$ along the semigroup $MY$ of the rational curves of $Y$ is defined to be

\[
\mathcal{R}[[q^\beta]] := \{ \sum_{\beta \in MY} a_\beta q^\beta, \ a_\beta \in \mathcal{A}, \ \beta \ - \ effective \}\,.
\]

where $\beta \in H_2(Y,\mathbb{Z})$ is effective if it is a positive linear combination of rational curves. For each $\beta$, the set of $\alpha$ such that $\alpha$ and $\beta - \alpha$ are both effective is finite, hence $\mathcal{R}[[q^\beta]]$ behaves like a power series. Alternatively, we may define

\[
q^\beta := q_1^{d_1} \cdot ... \cdot q_k^{d_k} = \exp(t_1d_1 + ... + t_kd_k)
\]

where $\{d_1, d_2, ..., d_k\}$ are the coordinates of $\beta$ relative to the dual of a nef basis $\{p_1, ..., p_k\}$ of $H^2(Y,\mathbb{Q})$.

Let $*$ denote the small quantum product of $Y$. The small quantum cohomology ring

\[(QH_1^*;Y,*\)

is a deformation of the cohomology ring $(H^*(Y,\mathbb{Q}[q^\beta]), \cup)$. Its structural constants are three point Gromov-Witten invariants of genus zero. Let $h$ be a formal variable and

\[
J_\beta(Y) := e_* \left( \frac{[Y_{1,\beta}]}{h(h - \psi)} \right) = \sum_{k=0}^{\infty} \frac{1}{h^{2+k}} e_* (\psi^k \cap [Y_{1,\beta}]).
\]


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The sum is finite for dimension reasons. For \( t = (t_0, t_1, \ldots, t_k) \), let
\[
 tp := t_0 + \sum_{i=1}^{k} t_i p_i.
\]
The \( D \)-module for the quantum differential equation of \( Y \)
\[
1 \leq i \leq k, \quad \hbar \partial/\partial t_i = p_i^*,
\]
is generated by (Givental [10])
\[
 J(Y) = \exp \left( \frac{tp}{\hbar} \right) \sum_{\beta \in H_2(Y, \mathbb{Z})} q^{\beta} J_\beta(Y)
\]
where we use the convention \( J_0 = 1 \). The generator \( J(Y) \) encodes all of the genus zero, one marking Gromov-Witten invariants and gravitational descendants of \( Y \). The generator \( J(Y) \) is an element of the completion \( H^*(Y, \mathbb{Q})[[t]][[q^\beta]] \) that may be used to produce relations in \( QH^*_Y \) in the following way: let
\[
 P(h, \hbar \partial/\partial t_i, q_i)
\]
be a polynomial differential operator where \( q_i \) and \( \hbar \) act via multiplication and \( q_i = e^{t_i} \) are on the left of derivatives. If
\[
 P(h, \hbar \partial/\partial t_i, q_i) J(Y) = 0
\]
then
\[
 P(0, p_i, q_i) = 0
\]
is a relation in the small quantum cohomology ring \( QH^*_Y \).

If \( Y \) is a complete intersection in a toric variety, \( J(Y) \) is related to an explicit hypergeometric series \( I(Y) \) via a change of variables (Givental [8], Lian et al [12],[13]). Furthermore, if \( Y \) is Fano then the change of variables is trivial, i.e.
\[
 J(Y) = I(Y).
\]
Since \( I(Y) \) is known explicitly, this yields two immediate benefits.

(1) The one point Gromov-Witten invariants and gravitational descendants of \( Y \) are determined completely.

(2) Differential operators that annihilate \( I(Y) \) are easy to find, hence producing relations in the small quantum cohomology ring of \( Y \).

In this paper we seek to relativize these results for Fano toric bundles, hence extending the results of the papers Elezi [6],[7].

2. Toric Bundles and Mirror Theorems

Toric varieties and bundles. We follow the approach and the terminology of Oda [15]. Let \( M \simeq \mathbb{Z}^m \) be a free abelian group of rank \( m \), \( N = \text{Hom}(M, \mathbb{Z}) \) its dual, and \( <,>: M \times N \rightarrow \mathbb{Z} \) the pairing between them. Let \( Y \) be an \( m \)-dimensional smooth, toric variety determined by a fan \( \Sigma \subset N \otimes \mathbb{R} \). Denote by
\[
 \Sigma(1) = \{ \rho_1, \ldots, \rho_m, \rho_{m+1}, \ldots, \rho_{r=m+k} \}
\]
the one dimensional cones of \( \Sigma \) and \( D_1, \ldots, D_r \) the corresponding toric divisors. Let \( v_i = (v_{i1}, \ldots, v_{im}) \) be the first lattice point along the ray \( \rho_i \). Let
\[
 \{a_1, a_2, \ldots, a_k\}
\]
with \( a_j := (a_{1j}, a_{2j}, ..., a_{mj}, a_{m+1j}, ..., a_{rj}) \) be a basis of the lattice of relations \( \Lambda \) between \( v_1, ..., v_r \). There is a short exact sequence

\[
0 \to \Lambda \to \mathbb{Z}^{\Sigma(1)} \overset{h}{\to} \mathbb{N} \to 0,
\]

where \( h(c_1, c_2, ..., c_r) = c_1 v_1 + ... + c_r v_r \). The lattice \( \Lambda \) may be identified with \( \text{Hom}(A_{m-1}(Y), \mathbb{Z}) \simeq H_2(Y, \mathbb{Z}) \). Under this isomorphism, \( a_{ij} \) is the intersection of \( a_j \), when interpreted as a two dimensional cycle, with the toric divisor \( D_i \). We choose \( a_j \) so that \( \{a_1, ..., a_k\} \) is a generating set for the Mori cone of classes of effective curves. Then \( a_{1i}, ..., a_{ki} \) are the coordinates of \( D_i \) with respect to the nef basis \( \{p_1, ..., p_k\} \) dual to \( \{a_1, ..., a_k\} \).

Assume that \( p_1, ..., p_m \) generate a maximal dimensional cone in \( \Sigma \). Since \( Y \) is smooth, \( \{v_1, v_2, ..., v_m\} \) forms a \( \mathbb{Z} \)-basis of \( \mathbb{N} \) and the absolute value of the matrix \((a_{ij}); \ i = m+1, ..., r; \ j = 1, 2, ..., k\) is 1.

The cohomology ring \( H^*(Y, \mathbb{Z}) \) is generated by the divisors \( D_1, ..., D_r \) subject to the following two types of relations:

**Type One:** Whenever \( \{\rho_{j_1}, ..., \rho_{j_s}\} \) do not generate a cone in \( \Sigma \), the intersection

\[
D_{j_1} \cdot ... \cdot D_{j_s} = 0.
\]

**Type Two:** For each \( 1 \leq i \leq m \),

\[
D_i = \sum_{j=1}^{k} a_{ij} p_j.
\]

From the short exact sequence (2) we obtain

\[
0 \to \mathbb{T}^k \overset{\alpha}{\to} \mathbb{T}^r \overset{\beta}{\to} \mathbb{T}^m \to 0,
\]

where the maps are defined as follows:

\[
\alpha(t_1, t_2, ..., t_k) = (k \prod_{i=1}^{k} t_i^{a_{1i}}, ..., k \prod_{i=1}^{k} t_i^{a_{ri}}), \quad \beta(t_1, ..., t_r) = (r \prod_{i=1}^{r} t_i^{v_{1i}}, ..., r \prod_{i=1}^{r} t_i^{v_{rim}}).
\]

Let \( Z(\Sigma) \subset \mathbb{C}^r \) be the variety whose ideal is generated by the products of those variables which do not generate a cone in \( \Sigma \). The toric variety \( Y \) is the geometric quotient (Cox [5])

\[
\mathbb{C}^r - Z(\Sigma)//\mathbb{T}^k
\]

where the torus acts as follows

\[
t \cdot x = \left( k \prod_{i=1}^{k} t_i^{a_{1i}} x_1, ..., k \prod_{i=1}^{k} t_i^{a_{ri}} x_r \right).
\]

The short exact sequence (5) yields an action of the quotient \( T := \mathbb{T}^m \) on \( Y \).

The first chern class of the tangent bundle to \( Y \) is equal to

\[
\sum_{i=1}^{r} D_i = \sum_{i=1}^{k} n_i p_i.
\]

The toric variety \( Y \) is Fano iff \( n_i > 0 \) for all \( i \).
We relativize the previous construction as follows. Consider the principal \( T \)-bundle 
\[ E := \bigoplus_{i=1}^{m} (L_i - \{0\}) \to X, \]
where \( L_i \) are line bundles over a smooth, projective variety \( X \). Let \( T \) act fibrewisely on \( E \) and the diagonally on the first \( m \)-homogeneous coordinates of \( Y \). The quotient space 
\[ Y(E) := E \times_T Y \]
is a toric bundles over \( X \) with fiber isomorphic to \( Y \). The bundle \( Y(E) \) inherits a \( T \)-action. There is a projection map \( \pi : Y(E) \to Y \). The maximal cone generated by \( \{\rho_1, \rho_2, ..., \rho_m\} \) determines a \( T \) fixed point \( q \) in \( Y \) whose homogeneous coordinates are \((0, 0, ..., 0, 1, 1, ...1)\). In the relativized setting, the \( T \)-equivariant inclusion 
\[ q \hookrightarrow Y \]
yields a map 
\[ q(E) \simeq X \xrightarrow{\pi} Y(E) \]
which is a section of \( \pi \). This is also a fixed point component for the action of \( T \) on \( Y(E) \). The other \( T \)-fixed points of \( Y \) yield sections of \( \pi \) and these are all the fixed point components.

Toric divisors lift to divisors in \( Y(E) \); these liftings will be denoted by the same letter in this paper. It was shown in Sankaran and Uma [17] that the two types of relations (3) and (4) lift in a natural way in \( H^*(Y(E), \mathbb{Z}) \); namely 
\[ D_{j_1} \cdot ... \cdot D_{j_s} = 0 \]
whenever \( \{\rho_{j_1}, ..., \rho_{j_s}\} \) do not generate a cone in \( \Sigma \), and
\[ D_i = \sum_{j=1}^{k} a_{ij} p_j + c_1(L_i) \]
for each \( 1 \leq i \leq m \), where as in the case of \( H^*(Y, \mathbb{Z}) \) the divisors
\[ p_1, ..., p_k \]
generate freely \( H^*(Y(E), \mathbb{Z}) \). In fact, there is a simple relation between the \( T \)-equivariant cohomology of \( Y \) and the cohomology of \( Y(E) \) which will be used throughout this paper. Recall, that the rational cohomology of the classifying space \( B\mathbb{T} \) is \( \mathbb{Q}[\lambda_1, ..., \lambda_m] \) where \( \lambda_i \) is the first chern class of the equivariant line bundle corresponding to the character
\[ \nu_i : \mathbb{T} \to \mathbb{C}^* \quad \nu_i(t_1, ..., t_m) = t_i. \]
A relation in the equivariant cohomology ring of \( Y \) becomes a relation in \( H^*(Y(E)) \) after substituting \( c_1(L_i) \) for \( \lambda_i \).

We may assume that \( L_i = O_X, i > m \) without loss of generality. This is due to the fact that \( \rho_1, ..., \rho_m \) generate a maximal cone in \( \Sigma \).

**The quantum \( D \)-module structure of a toric bundle.** The generator \( J \) of a quantum \( D \)-structure is weighted by the lattice points of the Mori cone. Hence we first study the relation between the Mori cones of \( Y \) and \( Y(E) \).
Lemma 1. If $L_i^*$ are generated by global sections, then the liftings of the nef divisors $p_1, \ldots, p_k$ in $Y(E)$ are also nef. Furthermore, the Mori cone of $Y(E)$ is a direct sum of the Mori cone of $X$, embedded via the section $s$, and the Mori cone of the fiber $Y$.

Proof. In toric varieties, every nef divisor $p$ is generated by global sections (Oda [14]). Let $x_1, x_2, \ldots, x_r$ be homogeneous coordinates in $Y$. The vector space of global sections $H^0(O(p))$ has a monomial basis

$$\prod_{i=1}^{r} x_i^{m_i}.$$ 

Let \{\phi_{ij}\} be a collection of generating sections for the line bundles $L_i^*$. The “monomials”

$$\prod_{i=1}^{r} (x_i \phi_{ij})^{m_i}$$

are generating sections the line bundle

$$\prod_{i=1}^{r} (O(D_i) \otimes (L_i^*))^{m_i}$$

which is isomorphic to $O(p)$ in $Y(E)$. Thus $p$ lifts to a nef divisor in $Y(E)$.

This shows that the addition of $p_1, \ldots, p_k$ to a nef basis \{p_{k+1}, \ldots, p_l\} of $X$ yields a nef basis

$$\{p_1, \ldots, p_l\}$$

of $Y(E)$. Now for a curve $C \subset Y(E)$ we have

$$\pi_*(\langle [C] - s_*(\pi_*(\langle C \rangle)) \rangle) = 0.$$ 

Notice that the restrictions of the divisors $p_1, p_2, \ldots, p_k$ in the section $q(E)$ are all zero since they may be written as $\mathbb{Z}$-linear combinations of $D_{m+1}, \ldots, D_{m+k}$. Hence $\forall i = 1, 2, \ldots, k$, $p_i \cdot (\langle [C] - s_*(\pi_*(\langle C \rangle)) \rangle) \geq 0$ and we have a unique decomposition

$$[C] = s_*(\pi_*(\langle C \rangle)) + [C'],$$

where $[C']$ and $\pi_*(\langle C \rangle)$ are curve classes respectively in the fiber of $\pi$ and $X$. 

We introduce a “mixed” $I(Y(E))$ that admits contributions from both $J(X)$ and an $E$-twisted $J(Y)$. Let $(\nu, d)$ denote a curve class in the Mori cone of $Y(E)$, with $\nu$ a curve class in the fiber of $\pi$ and $d$ a curve class in $X$.

Define

$$I(Y(E)) := \exp\left(\frac{1p}{h}\right) \sum_{(d, \nu)} q_1^{\nu} q_2^{d} \prod_{i=1}^{m} \prod_{m=0}^{\infty} \frac{(D_i + mh)}{(D_i + mh)} \pi^*(J_d(X)).$$

If $X$ is a point then $Y(E) = Y$. Furthermore, as mentioned in the introduction $J(Y) = I(Y)$ if $Y$ is a Fano toric variety. In this paper we show that the same holds for the relativized $Y(E)$.

Proposition 1. If $X$ is a semi-ample complete intersection in a toric variety, and both $Y$ and $Y(E)$ are Fano, then $J(Y(E)) = I(Y(E))$. 

Proposition 1 will follow as a corollary of another statement which we now formulate and prove. Let \( Z \) be a toric variety, \( \tilde{L}_i, \ i = 0, 1, ..., n \) toric line bundles over \( Z \) and \( \tilde{E} = \bigoplus_{i=0}^n \tilde{L}_i \). The bundle

\[ \pi : Y(\tilde{E}) \to Z \]

is also a toric variety (Oda [15]). The edges of the fan for \( Y(\tilde{E}) \) corresponds to the liftings \( B_1, ..., B_r \) to \( Y(E) \) of the toric base divisors \( b_1, ..., b_r \) and the divisors \( D_i \) from \( Y \).

Let \( \mathcal{L}_a : a = 1, 2, ..., l \) be globally generated line bundles over \( Z \) and \( X \) the zero locus of a generic section \( s \) of

\[ V = \bigoplus_{a=1}^l \mathcal{L}_a. \]

Such an \( X \) will be called a semi-ample complete intersection. Denote by \( E_i \) and \( E \) the restrictions of \( \tilde{L}_i \) and \( \tilde{E} \) to \( X \). The total space of \( Y(E) \) is easily seen to be the zero locus of the section \( \pi^*(s) \) of the pull back bundle \( \pi^*(V) \).

Assume that the line bundles \( \tilde{L}_i^* \) are globally generated and \( -K_Z - \sum_{a=1}^l c_1(\mathcal{L}_a) + \sum_{i=0}^n c_1(\tilde{L}_i) \) is ample. (This will ensure that the conditions of Proposition 1 for the bundle \( Y(E) \) over \( X \) are satisfied.)

Let \( V_d \) be the bundle on \( Z_{1,d} \) whose fiber over the moduli point \( (C, x_1, f) \) is \( \oplus_a H^0(f^*(\mathcal{L}_a)) \). Denote by \( s_V \) its canonical section induced by \( s \), i.e.

\[ s_V((C, x_1, f)) = f^*(s). \]

The stack theoretic zero section of \( s_V \) is the disjoint union

\[ Z(s_V) = \coprod_{i, \langle \beta \rangle = d} X_{1, \beta}. \]

The map \( i_* : H_2 X \to H_2 Z \) is not injective in general, hence the zero locus \( Z(s_V) \) may have more than one connected component. An example is the quadric surface in \( \mathbb{P}^3 \). The sum of the virtual fundamental classes \( [X_{1, \beta}] \) is the refined top Chern class of \( V_d \) with respect to \( s_V \).

Let \( \tilde{V}_{v,d} \) and \( \tilde{s}_V \) be the pull backs of \( V_d \) and \( s_V \) via the stack morphism

\[ Y(\tilde{E})_{1,(v,d)} \to Z_{1,d}. \]

The zero section of \( \tilde{s}_V \) is the disjoint union

\[ z(\tilde{s}_V) = \coprod_{i, \langle \beta \rangle = d} Y(\mathcal{E})_{1,(v,\beta)}. \]

It follows that

\[ \sum_{i, \langle \beta \rangle = d} [Y(\mathcal{E})_{1,(v,\beta)}] = c_{\text{top}}(\tilde{V}_{v,d}) \cap [Y(\tilde{E})_{1,(v,d)}]. \]

Recall that the nef basis \( \{p_1, p_2, ..., p_k, p_{k+1}, ..., p_l\} \) of \( Y(E) \) is obtained by completing a nef basis \( \{p_{k+1}, ..., p_l\} \) of \( X \). We will use \( tp \) to denote both \( \sum_{i=1}^l t_i p_i \) and \( \sum_{i=k+1}^l t_i p_i \). The difference will be clear from the context. Consider the following generating functions

\[ J^v(Y(\mathcal{E})) = \exp \left( \frac{tp}{\hbar} \right) \sum_{\langle v,d \rangle} q_1 v q_2^d e^* \left( \frac{c_{\text{top}}(\tilde{V}_{v,d}) \cap [Y(\tilde{E})_{1,(v,d)}]}{\hbar(\hbar - c)} \right). \]
and

\[ \tilde{I}^V(Y(\tilde{E})) = \exp\left(\frac{tp}{\hbar}\right) \sum_{(\nu,d)} q_1^\nu q_2^d \Omega_{\nu,d} \pi^* e_\nu \left( \frac{c_{\text{top}}(V_d) \cap [Z_{1,d}]}{\hbar(h-c)} \right), \]

where

\[ \Omega_{\nu,d} = \prod_{i=1}^{\infty} \frac{\prod_{m=0}^{\infty} (D_i + mh)}{\prod_{m=0}^{\infty} (D_i + mh)}. \]

**Proposition 2.** If \(-K_Y - \sum_{a=1}^{\nu} c_1(L_a) - \sum_{i=0}^{n} c_1(L_i)\) is ample then

\[ J^V((\tilde{E})) = \tilde{I}^V(Y(\tilde{E})). \]

**Proof.** Let

\[ I_d^V(Z) = \prod_a \frac{\prod_{m=-\infty}^{\infty} (L_a + mh)}{\prod_{m=-\infty}^{\infty} (L_a + mh)} \prod_i \frac{\prod_{m=-\infty}^{0} (B_i + mh)}{\prod_{m=-\infty}^{0} (B_i + mh)}. \]

From Givental [9], Lian et al [12], Lian et al [13] we know that \(J^V(Y(\tilde{E}))\) is related via a mirror transformation to

\[ I_d^V(Z) = \exp\left(\frac{tp}{\hbar}\right) \sum_{(\nu,d)} q_1^\nu q_2^d \Omega_{\nu,d} I_d^V(Z). \]

Likewise

\[ J^V(Z) = \exp\left(\frac{tp}{\hbar}\right) \sum q_2^d \Omega_{d} \left( \frac{c_{\text{top}}(V_d) \cap [Z_{1,d}]}{\hbar(h-c)} \right), \]

is related to

\[ I_d^V(Z) = \exp\left(\frac{tp}{\hbar}\right) \sum q_2^d I_d^V(Z). \]

Since \(-K_{Y(\tilde{E})} - \sum a c_1(L_a)\) and \(-K_Z - \sum a c_1(L_a)\) are ample, the mirror transformations are particularly simple. Indeed, both series can be written as power series of \(\hbar^{-1}\) as follows:

\[ I_d^V(Y(\tilde{E})) = 1 + \frac{P_1(q_1, q_2)}{\hbar} + o(\hbar^{-1}), \quad I_d^V(Z) = 1 + \frac{P_2(q_2)}{\hbar} + o(\hbar^{-1}), \]

where \(P_1(q_1, q_2), P_2(q_2)\) are both polynomials supported respectively in

\[ \Lambda_1 := \{ (\nu, d) \mid (-K_{Y(\tilde{E})} - \sum a c_1(L_a)) = 1; D_j \geq 0, \forall j; B_i \geq 0, \forall i \}, \]

and

\[ \Lambda_2 := \{ d \mid (-K_Z - \sum a c_1(L_a)) = 1; B_i \geq 0 \forall i \}. \]

Then

\[ J_d^V(Y(\tilde{E})) = \exp\left(\frac{-P_1(q_1, q_2)}{\hbar}\right) I_d^V(Y(\tilde{E})), \]

and

\[ J_d^V(Z) = \exp\left(\frac{-P_2(q_2)}{\hbar}\right) I_d^V(Z). \]

Simple algebraic manipulations show that

- \(c_1(L_j) \cdot d = 0, \forall d \in \Lambda_2, \forall j = 1, 2, ..., n\)
- \(\Lambda_1 = \{ (0, d) \mid d \in \Lambda_2 \}. \)
It follows that $\Omega_{0,d} = 1, \forall d \in \Lambda_2$ hence $P_1(q_1, q_2) = P_2(q_2)$. Notice also that if we expand
\[
\exp \left( -\frac{P_2(q_2)}{\hbar} \right) = \sum_{\alpha} c_\alpha q_2^\alpha
\]
then
\[
c_1(L_j) \cdot \alpha = 0, \forall j = 1, 2, ..., n.
\]
Hence for each $(\nu, d) \in M_P(\tilde{V})$ we have $\Omega_{\nu,d} = \Omega_{\nu,d+\alpha}$. Now the proposition follows easily.

**Proof. of Proposition 1.** We know return to the proof of Proposition 1. Recall that the map
\[(8) \quad i_* : H_2(X) \to H_2(Z)\]
is not necessarily injective in general. If it is, then
\[
[X_{1,\beta}] = c_{\text{top}}(V_{i,\beta}) \cap [Y_{1,i,\beta}]
\]
and
\[
[Y(E)_{1,(\nu,\beta)}] = c_{\text{top}}(\tilde{V}_{i,\beta}) \cap [Y(\tilde{E})_{1,(\nu,\beta)}].
\]
In this case one can easily show that
\[
i_*(J_{\nu,\beta}(Y(E))) = J_{\nu,i,\beta}(Y(\tilde{E}))
\]
and
\[
i_*(I_{\nu,\beta}(Y(E))) = I_{\nu,i,\beta}(Y(\tilde{E})).
\]
Proposition 2 shows that Proposition 1 holds for complete intersection in toric varieties for which the map (8) is injective.

\[\square\]

3. LIFTING THE QUANTUM COHOMOLOGY STRUCTURE

In this section we use Proposition 1 to study small quantum cohomology ring of $Y(E)$. As explained in the introduction, some of the relations in the small quantum cohomology ring come from differential operators.

**Proposition 3.** Whenever Proposition 1 holds, quantum differential operators of $X$ may be lifted in $Y(E)$, while the quantum differential operators of the fiber $Y$ may be extended to $Y(E)$. Both types of operators produce relations in the quantum cohomology $QH^*_Y(Y(E))$.

**Proof.** Recall that $D_i = \sum a_{ij} p_j$. Let
\[
c_1(L_i) = \sum_{j=k+1}^l c_{ij} p_j, \quad i = 0, 1, ..., n.
\]
Recall that the nef basis $\{p_1, p_2, ..., p_k, p_{k+1}, ..., p_l\}$ of $H^2(Y(E), \mathbb{Z})$ is obtained by completing a nef basis $\{p_{k+1}, ..., p_l\}$ of $X$. Let
\[
P(h, \hbar \partial/\partial t_{k+1}, ..., \hbar \partial/\partial t_l, q_2) = \sum_{\alpha \in \Lambda} q_2^\alpha P_\alpha
\]
be a polynomial differential operator with $\Lambda$ a finite subset of the Mori cone of $X$. Suppose that

$$0 = \mathcal{P}J(X) = \sum_{\alpha \in \Lambda} q_2^\alpha \sum_\beta \mathcal{P}_\alpha \left( \exp(\frac{pl}{h}) q_2^\beta \right) J_\beta(X)$$

$$= \sum_{\alpha \in \Lambda} q_2^\alpha \sum_\beta c_{\alpha, \beta} \exp(\frac{pl}{h}) q_2^\beta J_\beta(X) = \exp(\frac{pl}{h}) \sum_{\alpha \in \Lambda, \beta} q_2^{\alpha+\beta} c_{\alpha, \beta} J_\beta(X).$$

Let

$$\delta_\alpha = \prod_{i=1}^n -L_i, a^{-1} \prod_{r_j=0}^k (\sum_{j=1}^{l} a_{ij} \hbar \frac{\partial}{\partial t_j} + \sum_{j=k+1}^{l} c_{ij} \hbar \frac{\partial}{\partial t_j} - r_i \hbar), \quad \hat{\mathcal{P}} = \sum_{\alpha \in \Lambda} q_2^\alpha \delta_\alpha \mathcal{P}_\alpha,$$

with the convention that if

$$L_i(\alpha) = 0,$$

the factors of $\delta_\alpha$ corresponding to $L_i$ are missing. Notice that

$$L_{n+1}(\alpha) = \ldots = L_m(\alpha) = 0$$

since we have chosen $L_i$ to be trivial for $i > n$. We compute

$$\hat{\mathcal{P}} J(Y(E)) = \sum_{\alpha \in \Lambda} q_2^\alpha \delta_\alpha \sum_\nu \mathcal{P}_\alpha \left( q_2^\beta \exp(\frac{pl}{h}) \right) q_1^\nu \Omega_{\nu, \beta} J_\beta =$$

$$= \sum_{\alpha \in \Lambda} q_2^\alpha \delta_\alpha \sum_\nu c_{\alpha, \beta} \exp(\frac{pl}{h}) q_1^\nu q_2^\beta \Omega_{\nu, \beta} J_\beta.$$

One can easily show that

$$\delta_\alpha \left( \exp(\frac{pl}{h}) q_1^\nu q_2^\beta \Omega_{\nu, \beta} \right) = \exp(\frac{pl}{h}) q_1^\nu q_2^\beta \Omega_{\nu, \alpha+\beta}.$$

It follows that

$$\hat{\mathcal{P}} J(Y(E)) = \exp(\frac{pl}{h}) \sum_\nu q_1^\nu \sum_{\alpha \in \Lambda, \beta} c_{\alpha, \beta} q_2^{\alpha+\beta} \Omega_{\nu, \alpha+\beta} J_\beta(X) = 0.$$

Hence the relation $\mathcal{P}(0, p_{k+1}, \ldots, p_l, q_2) = 0$ in $QH^*_X$ lifts into the relation

$$\mathcal{P}(0, p_{k+1}, \ldots, p_l, q_2 \prod_{i=1}^n D_i) = 0$$

in $QH^*_Y(E)$, where

$$\left( \prod_{i=1}^n D_i \right)^\alpha := \prod_{i=1}^n D_i^{-L_i(\alpha)} \forall \alpha \in MX.$$

For a curve class $\nu$ in the fiber of $\pi$, consider the following differential operator

$$\Delta_{\nu}(\hbar \frac{\partial}{\partial t_1}, \ldots, \hbar \frac{\partial}{\partial t_l} q_j) := \prod_{i:D_i(\nu) > 0} D_i^{D_i(\nu) - 1} \prod_{m=0}^{D_i(\nu) - 1} \left( \sum_{j=1}^{k} a_{ij} \hbar \frac{\partial}{\partial t_j} - \sum_{j=k+1}^{l} c_{ij} \hbar \frac{\partial}{\partial t_j} + m \hbar \right)$$

$$-q^\nu \prod_{i:D_i(\nu) < 0} D_i^{-D_i(\nu) - 1} \prod_{m=0}^{D_i(\nu) - 1} \left( \sum_{j=1}^{k} a_{ij} \hbar \frac{\partial}{\partial t_j} - \sum_{j=k+1}^{l} c_{ij} \hbar \frac{\partial}{\partial t_j} + m \hbar \right).$$
It is easy to show that it satisfies
\[ \Delta_\nu J(Y(E)) = 0. \]

It follows that
\[ \Delta_\nu (p_1, \ldots, p_l, q_j) = 0 \]
in \( QH^*_Y(E) \), i.e.
\[ \prod_{i=1}^r D_i^{D_i(\nu)} = q^\nu. \]

These are precisely the extensions to \( Y(E) \) of the small quantum cohomology relations of the fiber \( Y \).

\[ \square \]

Sometimes all the relations in \( QH^*_Y \) come from quantum differential operators, hence \( QH^*_Y \) pulls back to \( QH^*_Y(E) \). This is the case when \( X \) is a Fano toric variety. The results of this section yield a complete description of \( QH^*_Y(E) \) which generalizes previous results of Costa et al [4] and Qin et al [15] and Givental [9].

4. The General (Nontoric) Case

We believe that Proposition 1 holds for any \( X \). On one end, the equality of the \( d = 0 \) terms in \( J(Y(E)) = I(Y(E)) \) is easy to establish. Indeed, the relative Gromov-Witten theory of the \( Y \)-bundle over \( BT \) associated with the universal bundle \( ET \rightarrow BT \) is precisely the \( \mathbb{T} \)-equivariant GW theory of \( Y \) (Astashkevich and Sadov [1]). The latter pulls back under the classifying map \( X \rightarrow BT \) to the relative GW theory of \( Y(E) \) over \( X \). It follows that the restriction of \( J(Y(E)) \) to \( \nu = 0 \) is obtained by substituting \( c_1(L_i) \) for \( \lambda_i \) in \( J^T(Y) \). Since \( Y \) is assumed to be Fano, the generator \( J^T(Y) \) is known (see for example [8]) and the substitution \( c_1(L_i) \mapsto \lambda_i \) is easily seen to yield the desired equality. At the other end, the \( \nu = 0 \) equality follows as an application of the equivariant quantum Lefshetz principle for the action of a torus on the fibers of \( Y(E) \). The fixed point component relevant for the equivariant and localization considerations ([12]) consists of the maps that land in the section \( s(X) \). The top chern class of the virtual normal bundle for this component is that of \( \mathbb{H}_1 \)-bundle for \( \bigoplus_{i=1}^m L_i \). Calculations are easy to carry out (see for example Elezi [7]).

References

[4] L. Costa and R. M. Mir-Roig, Quantum cohomology of projective bundles over \( \mathbb{P}^{n+1} \times \cdots \times \mathbb{P}^{n+s} \), Internat. J. Math. 11 (2000), 6, 761-797.

A. Elezi
Department of Mathematics and Statistics,
American University,
4400 Massachusetts Ave NW,
Washington DC 20016, USA
Email: aelezi@american.edu