ALBANIAN JOURNAL
OF MATHEMATICS
Volume 1, Number 3, Pages 133–143
ISSN 1930-1235: (electronic version)

# EXTENDING TOPOLOGICAL GROUP ACTIONS TO CONFORMAL GROUP ACTIONS

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Abstract. A consequence of the resolution of the Nielsen Realization Problem is that if $G$ is any finite topological group of automorphisms of a compact oriented surface $S$, then there exists a complex structure on $S$ so that the action of $G$ extends to a conformal action on $S$. We show that this complex structure is unique if and only if $G$ is a triangle group, and for all other groups, there are infinitely many such actions.

## 1. Introduction

Suppose that $G$ is a group of homeomorphisms of a compact, connected, oriented surface $S$ of genus $g \geqslant 2$. A consequence of the resolution of the Nielsen Realization Problem (NRP), see [8], is that there exists a group of homeomorphisms of $S$ which is topologically equivalent to $G$ (which by abuse of notation we call $G$) and some complex structure on $S$ so that the action of $G$ extends to a conformal action on $S$. It is natural to ask for a given finite group of homeomorphisms $G$ of $S$, how many such complex stuctures exist which extend the action of $G$ to a conformal action, and are there any groups for which the struture is unique? In most circumstances, the complex structure is not unique, and in fact there are infinitely many different complex structures which can be imposed on $S$ so that $G$ acts conformally (see Lemma 3.3). However, in the special case that $G$ is a triangle group, there are only finitely many conformal structures which can be imposed on $S$ so that $G$ acts conformally (see Lemma 3.2). We shall strengthen this result and show that if $G$ is a triangle group, then there is a unique structure which can be imposed on $S$ so that $G$ extends to a conformal action (see Theorem 3.6). A consequence of our results is an enumeration method for the number of different topological group actions of a triangle group $G$ on a surface $S$ in terms of surface kernel epimorphisms (for which we may utilize techniques developed such as those in [6] or [13]).

Following the NRP, there has been tremendous progress in the study of topological equivalence classes of group actions on surfaces due to the interactions with conformal group actions, see for example [3], [11], [12]. For details on this, see Section 2, or [3] for a more thorough exposition. One of the motivational reasons for this is that there is a one-one correspondence between finite subgroups of the mapping class group $\mathcal{M}_\sigma$ of a surface of genus $\sigma$ and the topological equivalence classes of finite groups of homeomorphisms which can act on a surface of genus $\sigma$. In general, the problem of enumerating conjugacy classes of subgroups of $\mathcal{M}_\sigma$ for arbitrary $\sigma$ is highly computational and depends very much upon how a group

Received by the editors June 30, 2007 and in revised form, August 30, 2007.
2000 *Mathematics Subject Classification.* Primary: 14H45, 14H37, 14H30, 20F34.
*Key words and phrases.* Mapping Class Group, Quasiplatonic Surface, Triangle Group.

$G$ acts on a surface $S$ as well as the general structure of $G$. For triangle groups however, we have much more control over how $G$ may act on $S$, and so a general enumeration formula is much more realistic (see Proposition 4.4). Another important fact about triangle groups is that many such groups will be maximal as finite subgroups of $\mathcal{M}_\sigma$, and for those which are not, there are computational methods to determine precisely which ones are not maximal, see for example [5] or [9]. Thus an enumeration method for the number of different topological group actions which are triangle groups can be used to provide a lower bound on the number of maximal finite subgroups of $\mathcal{M}_\sigma$, thus providing insight into the general structure of $\mathcal{M}_\sigma$.

Our paper is structured as follows. In Secion 2 we develop the necessary preliminary results regarding topological group actions summarizing the results from [3]. We shall also outline the preliminaries for counting the conformal equivalence classes of complex structures which can be imposed on a surface of genus $g$. Following this, we shall prove the main result in Section 3. Though on initial consideration, it may seem to be a highly computational result, the proof is suprisingly straight forward with the combination of some classical results and more recent techniques. In Section 4, we use some more recent results to determine an enumeration method to count the number of such groups. We finish by presenting some explicit examples. We note that throughout the paper, by surface we mean a compact, oriented, 2-manifold.

## 2. Preliminaries on Topological and Conformal Group Actions

Let $G$ be a finite group. The group $G$ is said to act topologically (in an orientation preserving manner) on surface a $S$ of genus $\sigma \geqslant 2$ if there is an injection

$$\varepsilon : G \hookrightarrow \mathrm{Homeo}^+(S)$$

into the group of orientation preserving homeomorphisms (we shall identify $G$ with its image under $\varepsilon$). Two actions $\varepsilon_1, \varepsilon_2$ are said to be *topologically equivalent* if there is a homeomorphism $h$ of $S$ and an automorphism $\omega$ of $G$ such that

$$\varepsilon_2(\omega(g)) = h \circ \varepsilon_1(g) \circ h^{-1}.$$

This is equivalent to saying that the images $\varepsilon_1(G)$ and $\varepsilon_2(G)$ are conjugate in $\mathrm{Homeo}^+(S)$.

For $\sigma \geqslant 2$, due to the NRP, Fuchsian groups provide us with a way to examine topological group actions. Specifically, a surface $S$ of genus $\sigma \geqslant 2$ is topologically equivalent to a quotient of the upper half plane $\mathbb{H}/\Lambda$ where $\Lambda$ is any torsion free Fuchsian group isomorphic to the fundamental group of $S$ called a surface group for $S$. A finite group $G$ acts on $S$ if and only if $G$ is topologically equivalent to $\Gamma/\Lambda$ for some Fuchsian group $\Gamma$ containing such a $\Lambda$ as a normal subgroup of index $|G|$. The structure of $\Gamma$ is completely determined by the ramification data of the quotient map $\pi_G\colon S \to S/G$ which must satisfy the Riemann-Hurwitz formula. Specifically, if the quotient map $\pi_G$ branches over $r$ points with ramification indices $m_i$ for $1 \leqslant i \leqslant r$ and the quotient space $S/G$ has genus $g$, then a presentation for $\Gamma$ is:

$$\Gamma = \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_r | c_1^{m_1}, \dots, c_r^{m_r}, \prod_{i=1}^{r} c_i \prod_{j=1}^{g} [a_j, b_j] \rangle \tag{1}$$

where

$$\sigma = 1 + |G|(g-1) + \frac{|G|}{2}\sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right).$$

Such a group is described by the tuple $(g; m_1, \ldots, m_r)$ called the **signature** of $\Gamma$ (we also say that $G$ has signature $(g; m_1, \ldots, m_r)$). In the special case that $g = 0$ and $r = 3$, we call $G$ and $\Gamma$ **triangle groups**. Such group actions are usually described through the use of surface kernel epimorphisms, so we interpret our observations accordingly.

**Theorem 2.1.** *A finite group $G$ acts on a surface $S$ of genus $\sigma \geqslant 2$ with signature $(g; m_1, \ldots, m_r)$ if and only if there exists a Fuchsian group with signature $(g; m_1, \ldots, m_r)$ and an epimorphism $\varrho\colon \Gamma \to G$ preserving the orders of the elements of finite order (called a surface kernel epimorphism) such that*

$$\sigma = 1 + |G|(g-1) + \frac{|G|}{2}\sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right).$$

A useful way to describe surface kernel epimorphisms is through the use of generating vectors defined as follows (see [3]).

**Definition 2.2.** A vector of group elements $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \eta_1, \ldots, \eta_r)$ in a finite group $G$ is called a $(g; m_1, \ldots, m_r)$-generating vector for $G$ if all of the following hold:

(*i*) $G = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \eta_1, \ldots, \eta_r \rangle$

(*ii*) $\Pi_{i=1}^{g}[\alpha_i, \beta_i] \cdot \Pi_{j=1}^{r}\eta_j = 1$ (where $[,]$ denotes the commutator).

(*iii*) $O(c_i) = m_i$ (where $O(.)$ denotes group order).

Clearly any $(g; m_1, \ldots, m_r)$-generating vector $\mathcal{V}$ for $G$ defines a unique surface kernel epimorphism from a fixed $\Gamma$ with signature $(g; m_1, \ldots, m_r)$ onto $G$, called the surface kernel epimorphism of $\mathcal{V}$. Conversely, any surface kernel epimorphism $\varrho\colon \Gamma \to G$ uniquely defines a generating vector called the generating vector of $\varrho$. Thus topological group actions can be described through the utilization of generating vectors of finite groups. The exact correspondence is given in the following.

**Theorem 2.3.** *Two equivalence classes of $(g; m_1, \ldots, m_r)$-generating vectors of the finite group $G$ define the same topological equivalence class of $G$-actions if and only if the generating vectors lie in the same* $\mathrm{Aut}\,(G) \times \mathrm{Aut}\,(\Gamma)$*-class where the action of* $\mathrm{Aut}\,(G) \times \mathrm{Aut}\,(\Gamma)$ *on a generating vector $\mathcal{V}$ is defined by the action on a surface kernel epimorphism $\varrho$ with generating vector $\mathcal{V}$ given by $(\alpha, \gamma) \cdot \varrho = \alpha \circ \varrho \circ \gamma^{-1}$ for* $\alpha \in \mathrm{Aut}\,(G)$*,* $\gamma \in \mathrm{Aut}\,(\Gamma)$*.*

*Proof.* See Proposition 2.2 of [3]. □

Thus given a generating vector $\mathcal{V}$ for a group $G$, it defines a topological action, namely the action of $\Gamma/\Lambda$ on $S = \mathbb{H}/\Lambda$ where $\Lambda$ is the kernel of the surface kernel epimorphism of $\mathcal{V}$ where $\Gamma$ is a Fuchsian group with signature $(g; m_1, \ldots, m_r)$ (note that $\Gamma$ can be any subgroup of $\mathrm{PSL}\,(2, \mathbb{R})$ with signature $(g; m_1, \ldots, m_r)$). Conversely, given a group acting topologically on $S$ with signature $(g; m_1, \ldots, m_r)$, it defines a $(g; m_1, \ldots, m_r)$-generating vector up to $\mathrm{Aut}\,(G) \times \mathrm{Aut}\,(\Gamma)$ equivalence. Specifically, it defines the class containing the generating vector of $\varrho\colon \Gamma \to G$ where $\varrho$ is the surface kernel epimorphism from $\Gamma$ with signature $(g; m_1, \ldots, m_r)$ and kernel $\Lambda$ such that $S$ is topologically equivalent to $\mathbb{H}/\Lambda$ and $G$ is topologically equivalent to $\Gamma/\Lambda$. We call any generating vector from this $\mathrm{Aut}\,(G) \times \mathrm{Aut}\,(\Gamma)$ class **a generating vector of** $G$.

Now observe that given a surface kernel epimorphism $\varrho \colon \Gamma \to G$ with kernel $\Lambda$, there exists a complex structure which can be imposed on $S = \mathbb{H}/\Lambda$ so that the group $G$ acts conformally on $S$ - namely the natural structure inherited from the complex structure on $\mathbb{H}$. This motivates the following definition.

**Definition 2.4.** Suppose $G$ is a finite group of homeomorphisms of a surface $S$ and $G$ is topologically equivalent to $\Gamma/\Lambda$ where $\Gamma$ and $\Lambda$ are Fuchsian groups with $\Lambda$ isomorphic to the fundamental group of $S$ so $S$ is topologicially equivalent to $\mathbb{H}/\Lambda$. Then we say the complex structure which can be imposed on $S$ inherited from the complex structure on $\mathbb{H}$ **extends the action of** $G$ **to a conformal action on** $S$.

Unlike toplogical actions, this structure depends upon a choice for $\Gamma$. Specifically, we have the following.

**Theorem 2.5.** *Two surfaces $S_1 = \mathbb{H}/\Lambda_1$ and $S_2 = \mathbb{H}/\Lambda_2$ are conformally equivalent if and only if $\Lambda_1$ and $\Lambda_2$ are conjugate in* $\mathrm{PSL}\,(2, \mathbb{R})$.

In particular, given a topological group action $G$ on $S$, there may exist multiple inequivalent structures on $S$ so that the action of $G$ extends to a conformal action on $S$. Indeed, our observations imply the following.

**Theorem 2.6.** *Suppose $G$ is a group acting on a surface $S$ of genus $\sigma \geqslant 2$ with generating vector $\mathcal{V}$. Then the number of conformal structures which can be imposed on $S$ so that the action of $G$ extends to a conformal action is equal to the number of* $\mathrm{PSL}\,(2, \mathbb{R})$*-conjugacy classes of kernels of surface kernel epimorphism whose generating vector is $\mathcal{V}$.*

Thus in order to determine the number of structures which can be imposed on $S$ so that $G$ with generating vector $\mathcal{V}$ extends to a conformal action on $S$, we need to count $\mathrm{PSL}\,(2, \mathbb{R})$-classes of surface subgroups which are normal in all Fuchsian groups with signature given by the generating vector $\mathcal{V}$. It is a well known fact that the set of conjugacy classes of groups with signature $(g; m_1, \ldots, m_r)$ is homeomorphic to $\mathbb{R}^{6g-6+2r}$ (see [1]). This fact coupled with Theorem 2.6 suggests that for a given a group $G$ with generating vector $\mathcal{V}$ acting on $S$, unless $G$ is a triangle group, there will be an infinite number of structures which can be imposed on $S$ so that $G$ extends to a conformal action, and in the special case that $G$ is a triangle group, there are finitely many such structures. This leads to the following questions which are the central focus of our work .

**Question 2.7.** Suppose $G$ is a group acting on a surface $S$ of genus $\sigma \geqslant 2$ with generating vector $\mathcal{V}$. How many conformal structures do there exist which can be imposed on $S$ so that $G$ extends to a conformal action on $S$? In addition, are there any groups actions for which there exists a unique conformal structure extending the action of $G$ to a conformal action?

## 3. The Main Result

We shall prove our main result through a series of Lemmas. Henceforth, assume that $G$ is a group acting on $S$ with generating vector $\mathcal{V}$ and, where relevant, that $G$ is topologically $\Gamma/\Lambda$ for Fuchsian groups $\Gamma$ and $\Lambda$ where $\Lambda$ is isomorphic to the fundamental group of $S$. We need the following important result regarding

PSL $(2,\mathbb{R})$-conjugacy classes of Fuchsian groups and the existence of overgroups (see [1] and [10]).

**Theorem 3.1.** *Suppose that $\Gamma$ is a Fuchsian group with signature $\mathcal{S}$.*

(*i*) *If $G$ is a triangle group, there is a unique* PSL $(2,\mathbb{R})$*-conjugacy class of Fuchsian groups with signature $\mathcal{S}$.*

(*ii*) *If $G$ is not a triangle group, there exist infinitely many* PSL $(2,\mathbb{R})$*-conjugacy classes of Fuchsian groups with signature $\mathcal{S}$. Moreover, we have the following additional information about the existence of overgroups:*

(a) *If the signature of $\Gamma$ does not appear in Singermans list,* [10]*, there are infinitely many* PSL $(2,\mathbb{R})$*-conjugacy classes of finitely maximal Fuchsian groups with signature $\mathcal{S}$ ($\Gamma$ is finitely maximal if there is no Fuchsian group $\Delta$ with $\Gamma \leqslant \Delta$ and $[\Delta : \Gamma] < \infty$).*

(b) *If the signature of $\Gamma$ does appear in Singermans list,* [10]*, there is a list of signatures $\mathcal{L}$ such that given any group $\Gamma$ with signature $\mathcal{S}$ and any signature $\mathcal{S}_\Delta \in \mathcal{L}$, there is a Fuchsian group $\Delta$ with signature $\mathcal{S}_\Delta$ with $\Gamma \leqslant \Delta$ and $[\Delta : \Gamma] < \infty$. Moreover, there exist infinitely many* PSL $(2,\mathbb{R})$*-conjugacy classes of subgroups with signature $\mathcal{S}$ such that if $\Gamma$ is any such given group with signature $\mathcal{S}$, the only possible signatures for a group $\Delta$ with $\Gamma \leqslant \Delta$ and $[\Delta : \Gamma] < \infty$ are those from $\mathcal{L}$.*

We can use this result to show that the only possible candidates are triangle groups.

**Lemma 3.2.** *If $G$ is a triangle group with generating vector $\mathcal{V}$ acting on $S$, then there are finitely many structures which can be imposed on $S$ extending the action of $G$ to a conformal action.*

*Proof.* Suppose that $G$ is a triangle group. We shall show that there exists only finitely many structures extending the action of $G$ to a conformal action independent of the generating vector for $G$ and hence there can only be finitely many once a generating vector has been specified.

If $G$ has signature $(0; m_1, m_2, m_3)$, let $\Gamma$ be a fixed Fuchsian group with signature $(0; m_1, m_2, m_3)$. We first observe that since all triangle groups are conjugate in PSL $(2,\mathbb{R})$, given any surface kernel $\Lambda_1$ such that $\Gamma_1/\Lambda \cong G$ where $\Gamma_1$ also has signature $(0; m_1, m_2, m_3)$, there exists a surface kernel $\Lambda \leqslant \Gamma$ with $\Gamma/\Lambda \cong G$ which is PSL $(2,\mathbb{R})$-conjugate to $\Lambda_1$. It follows that the number of PSL $(2,\mathbb{R})$-conjugacy classes of surface subgroups of Fuchsian groups with signature $(0; m_1, m_2, m_3)$ and quotient group $G$ is bounded above by the number of surface kernels of the fixed Fuchsian group $\Gamma$ with quotient group $G$. Since $G$ is finite, there are only finitely many surface kernel epimorphisms from $\Gamma$ to $G$, so only finitely many surface kernels with quotient group $G$ and hence only finitely many conformal structures which can be imposed on $S$ so that $G$ acts conformally.

□

**Lemma 3.3.** *If a group $G$ with generating vector $\mathcal{V}$ acting on $S$ is not a triangle group, then there exist infinitely many different structures which can be imposed on $S$ so that the action of $G$ extends to a conformal action on $S$.*

*Proof.* Suppose that $G$ with signature $(g; m_1, \ldots, m_r)$ and generating vector $\mathcal{V}$ acting on $S$ is not a triangle group. If the signature of $G$ does not appear in Singermans list, Theorem 3.1 implies there exist infinitely many $\mathrm{PSL}(2,\mathbb{R})$-conjugacy classes of finitely maximal subgroups of $\mathrm{PSL}(2,\mathbb{R})$ with signature $(g;m_1,\ldots,m_r)$. For such a $\Gamma$, let $\varrho\colon \Gamma \to G$ denote a surface kernel epimorphism with generating vector $\mathcal{V}$. Since automorphism groups of compact Riemann surfaces of genus $g \geqslant 2$ are finite, it follows that $\Gamma$ is the normalizer of $\mathrm{Ker}(\varrho)$ in $\mathrm{PSL}(2,\mathbb{R})$. It follows that if two finitely maximal groups $\Gamma_1$ and $\Gamma_2$ with signature $(g;m_1,\ldots,m_r)$ are not $\mathrm{PSL}(2,\mathbb{R})$-conjugate, and $\varrho_1\colon \Gamma_1 \to G$ and $\varrho_2\colon \Gamma_2 \to G$ are surface kernel epimorphisms with generating vector $\mathcal{V}$, then the kernels $\mathrm{Ker}(\varrho_1)$ and $\mathrm{Ker}(\varrho_2)$ are not $\mathrm{PSL}(2,\mathbb{R})$-conjugate (since normalizers of conjugate subgroups of a group are conjugate). Since there are infinitely many different $\mathrm{PSL}(2,\mathbb{R})$-conjugacy classes of finitely maximal subgroups of $\mathrm{PSL}(2,\mathbb{R})$ with signature $(g;m_1,\ldots,m_r)$, Theorem 2.6 implies there are infinitely many different structures which can be imposed on $G$ such that the action of $G$ extends to a conformal action.

Now suppose that the signature of $G$ appears in Singermans list and let $\mathcal{L}$ be the list of signatures such that given any $\Gamma$ with signature $(g;m_1,\ldots,m_r)$, for each signature $\mathcal{S} \in \mathcal{L}$, there always exists an overgroup $\Delta$ of $\Gamma$ with signature $\mathcal{S}$ and $[\Delta : \Gamma] < \infty$. Now for a fixed epimorphism $\varrho\colon \Gamma \to G$, Theorem 5.1 of [5] gives complete conditions for $\mathrm{Ker}(\varrho)$ to be normal in $\Delta \geqslant \Gamma$ with signature from $\mathcal{L}$. In particular, these conditions are dependent only on $\varrho$ and either hold true for all possible $\Gamma$ with signature $(g;m_1,\ldots,m_r)$ or none. Let $\mathcal{S}_\Delta$ denote the signature of the largest Fuchsian group with signature from $\mathcal{L}$ in which any $\mathrm{Ker}(\varrho)$ is normal (this is well defined by our remarks). Observe that since signature determines a group up to isomorphism, if $\Delta$ is a group with signature $\mathcal{S}_\Delta$, there always exists a subgroup $\Gamma$ with signature $(g;m_1,\ldots,m_r)$ and a normal surface subgroup $\Lambda$ with $\Lambda \lhd \Gamma$ such that $\eta\colon \Gamma \to \Gamma/\Lambda \cong G$ is $\mathrm{Aut}(G)$ (and hence $\mathrm{Aut}(G) \times \mathrm{Aut}(\Gamma)$) equivalent to $\varrho$ as specified above. To finish, we note that since any group with signature from $\mathcal{L}$ is not a triangle group (since $\Gamma$ is not a triangle group), Theorem 3.1 implies there exist infinitely many $\mathrm{PSL}(2,\mathbb{R})$-classes of subgroups with signature $\mathcal{S}_\Delta$ which can only be subgroups of Fuchsian groups with signature from $\mathcal{L}$. In particular, if $\Delta$ is a group with signature $\mathcal{S}_\Delta$ from one of these classes and $\Gamma$ and $\Lambda$ are as specified above, then $\Delta = N(\Lambda)$, so we can apply an identical argument to the previous case and the result follows.

□

This result means that the only possible candidates for which there exists a unique structure extending the action of $G$ are triangle groups. Since we are looking for $\mathrm{Aut}(G) \times \mathrm{Aut}(\Gamma)$-classes of generating vectors, we need to examine the group $\mathrm{Aut}(\Gamma)$ in more detail when $\Gamma$ is a triangle group.

**Theorem 3.4.** *Suppose that* $\Gamma$ *has signature* $(0;m_1,m_2,m_3)$.

(*i*) *If all the* $m_i$ *are distinct, then* $\mathrm{Aut}(\Gamma) = \mathrm{Inn}(\Gamma)$.
(*ii*) *If precisely two of the* $m_i$ *are equal, then* $[\mathrm{Aut}(\Gamma) : \mathrm{Inn}(\Gamma)] = 2$.
(*iii*) *If all of the* $m_i$ *are equal, then* $[\mathrm{Aut}(\Gamma) : \mathrm{Inn}(\Gamma)] = 6$.

*Proof.* A consequence of [4] is that if $F$ is the free group on two generators, then $[\mathrm{Aut}(F) : \mathrm{Inn}(F)] = 6$. Since the automorphisms of $\Gamma$ will be the same as the automorphisms of $F$ which preserve orders of elements, the result follows. Alternatively, we could construct the $\mathrm{Aut}(\Gamma)$ explicitly using the generators of general mapping class groups given in [2]. □

**Lemma 3.5.** *Suppose $\Gamma$ is a Fuchsian triangle group. Then there exists another Fuchsian group $\Delta$ containing $\Gamma$ as a normal subgroup and an isomorphism $\Phi\colon \Delta \to \mathrm{Aut}(\Gamma)$ induced by the action of conjugation of $\Delta$ on $\Gamma$.*

*Proof.* Suppose that $\Gamma$ has signature $(0; m_1, m_2, m_3)$ and let $\Delta$ be a Fuchsian group with $\Gamma \lhd \Delta$. We shall first show that the map $\Phi\colon \Delta \to \mathrm{Aut}(\Gamma)$ induced by the action of conjugation of the elements of $\Delta$ on $\Gamma$ is injective. To see this, suppose that conjugation by $\gamma \in \Delta$ and $\delta \in \Delta$ induce the same automorphism. Then it follows that for all $c \in \Gamma$, $\gamma c \gamma^{-1} = \delta c \delta^{-1}$, or equivalently $\delta^{-1}\gamma$ commutes with every element in $\Gamma$. However, two non-identity elements in a Fuchsian group $\Gamma$ commute if and only if they have the same fixed point set (see for example Theorem 5.2.4 of [7]). If $\delta^{-1}\gamma$ is non-trivial, it follows that all the elements of $\Gamma$ have the same fixed point set and hence $\Gamma$ is commutative which is not true. Hence $\delta^{-1}\gamma$ is trivial so $\delta = \gamma$, so the map $\Phi$ is injective.

To finish the proof, we observe that under the map $\Phi$ induced by conjugation, $\Phi(\Gamma) = \mathrm{Inn}(\Gamma)$ (the inner automorphism group of $\Gamma$). The result then follows through observation of the different possible overgroups of $\Gamma$ given in Singermans list, [10]. Specifically if all the $m_i$ are distinct, then $\mathrm{Aut}(\Gamma) = \mathrm{Inn}(\Gamma)$ so the result trivially holds. If precisely two of the $m_i$ equal, there exists a Fuchsian group $\Delta$ with $\Gamma \lhd \Delta$ and $[\Delta : \Gamma] = 2$, and if all the $m_i$'s are equal, then there exists a Fuchsian group $\Delta$ with $\Gamma \lhd \Delta$ and $[\Delta : \Gamma] = 6$.

□

We are now ready to prove our main result.

**Theorem 3.6.** *There exists a unique conformal structure extending the action of a topological group of automorphisms $G$ to a conformal action if and only if $G$ is a triangle group. For all other groups, there exist infinitely many different structurse extending the action of $G$ to a conformal action.*

*Proof.* If $G$ is not a triangle group, Lemma 3.3 proves the result. Therefore, suppose $G$ is a triangle group with signature $\mathcal{S}$ acting on a surface $S$ of genus $\sigma \geqslant 2$ with generating vector $\mathcal{V}$. By Theorem 2.6, we need to show that there is just one $\mathrm{PSL}(2,\mathbb{R})$-conjugacy class of kernels of surface kernel epimorphism whose generating vector $\mathrm{Aut}(G) \times \mathrm{Aut}(\Gamma)$-equivalent to $\mathcal{V}$. To prove this, we shall first show that for a fixed triangle group $\Gamma$ with signature $\mathcal{S}$, all kernels of surface kernel epimorphisms with generating vector $\mathrm{Aut}(G) \times \mathrm{Aut}(\Gamma)$-equivalent to $\mathcal{V}$ are $\mathrm{PSL}(2,\mathbb{R})$-conjugate. We shall then show that if $\Gamma_1$ is any other triangle group with signature $\mathcal{S}$, any kernel of a surface kernel epimorphism from $\Gamma_1$ to $G$ with generating vector $\mathrm{Aut}(G) \times \mathrm{Aut}(\Gamma)$-equivalent to $\mathcal{V}$ is $\mathrm{PSL}(2,\mathbb{R})$-conjugate to one of the surface kernels in $\Gamma$.

Suppose that $\Gamma$ is some fixed triangle group with signature $\mathcal{S}$ and let $\mathcal{K}$ denote the set of all kernels of surface kernel epimorphisms from $\Gamma$ to $G$ with generating vector $\mathrm{Aut}(G) \times \mathrm{Aut}(\Gamma)$-equivalent to $\mathcal{V}$. If $\Lambda_1, \Lambda_2 \in \mathcal{K}$, let $\varrho_1, \varrho_2\colon \Gamma \to G$ denote corresponding surface kernel epimorphisms. Since the generating vectors of $\varrho_1, \varrho_2$ are $\mathrm{Aut}(G) \times \mathrm{Aut}(\Gamma)$-equivalent to $\mathcal{V}$, they are $\mathrm{Aut}(G) \times \mathrm{Aut}(\Gamma)$-equivalent to each other and so $\varrho_1 = \alpha \circ \varrho_2 \circ \gamma^{-1}$ for some $\alpha \in \mathrm{Aut}(G)$ and $\gamma \in \mathrm{Aut}(\Gamma)$. It follows that $\mathrm{Ker}(\varrho_1) = \mathrm{Ker}(\varrho_2 \circ \gamma^{-1})$, or equivalently $\gamma(\Lambda_2) = \mathrm{Ker}(\Lambda_1)$. However, by Lemma 3.5, every element of $\mathrm{Aut}(\Gamma)$ is induced by the action of conjugation by some overgroup $\Delta$ of $\Gamma$ in $\mathrm{PSL}(2,\mathbb{R})$, and in particular, it follows that $\Lambda_1$ and $\Lambda_2$ are $\mathrm{PSL}(2,\mathbb{R})$-conjugate. Thus for a fixed triangle group $\Gamma$, any two surface kernel epimorphisms with $\mathrm{Aut}\,(G) \times \mathrm{Aut}\,(\Gamma)$-equivalent generating vectors have $\mathrm{PSL}\,(2,\mathbb{R})$-conjugate kernels.

Now suppose $\Gamma_1 \neq \Gamma$ is a triangle group with signature $\mathcal{S}$ and suppose that $\eta\colon \Gamma_1 \to G$ is a surface kernel epimorphism with generating vector $\mathrm{Aut}\,(G) \times \mathrm{Aut}\,(\Gamma)$-equivalent to $\mathcal{V}$. To finish the proof, we need to show that $\mathrm{Ker}\,(\eta)$ is $\mathrm{PSL}\,(2,\mathbb{R})$-conjugate to a group in the set $\mathcal{K}$. Since $\Gamma_1$ and $\Gamma$ are triangle groups, there exists $T \in \mathrm{PSL}\,(2,\mathbb{R})$ such that $T\Gamma_1 T^{-1} = \Gamma$. Now $T$ induces a conformal map between the quotient spaces (see for example Theorem 5.9.3 of [7])

$$\bar{T}\colon \mathbb{H}/\,\mathrm{Ker}\,(\eta) \to \mathbb{H}/(T\,\mathrm{Ker}\,(\eta)T^{-1})$$

and with this map, we have $\bar{T}(\Gamma_1/\,\mathrm{Ker}\,(\eta))\bar{T}^{-1} = \Gamma/(T\,\mathrm{Ker}\,(\eta)T^{-1})$. Since $\bar{T}$ is conformal, it is a homeomorphism, and so it follows that the groups $\Gamma_1/\,\mathrm{Ker}\,(\eta)$ and $\Gamma/(T\,\mathrm{Ker}\,(\eta)T^{-1})$ are topologically equivalent. In particular, the generating vector of the map $\varrho\colon \Gamma \to G$ with kernel $T\,\mathrm{Ker}\,(\eta)T^{-1}$ will be $\mathrm{Aut}\,(G) \times \mathrm{Aut}\,(\Gamma)$-equivalent to $\mathcal{V}$ and so $T\,\mathrm{Ker}\,(\eta)T^{-1} \in \mathcal{K}$, hence the result.

□

## 4. Enumeration of Topological Equivalence Classes of Triangle Group Actions

Our results in the previous section imply the following result relating the number of topological equivalence classes of group actions of triangle groups and the $\mathrm{PSL}\,(2,\mathbb{R})$-conjugacy classes of subgroups of triangle groups.

**Proposition 4.1.** *The number of topological equivalence classes of group actions of a group $G$ with signature $(0; m_1, m_2, m_3)$ on $S$ of genus $\sigma \geqslant 2$ is equal to the number of $N(\Gamma)$-conjugacy classes of surface subgroups of a fixed Fuchsian triangle group $\Gamma$ with quotient group $G$.*

Thus in order to determine the number of topological equivalence classes of group actions of a triangle group $G$ with signature $(0; m_1, m_2, m_3)$, we just need to count the number of surface kernels in a Fuchsian group $\Gamma$ with signature $(0; m_1, m_2, m_3)$ up to conjugation in the normalizer of $\Gamma$ in $\mathrm{PSL}\,(2,\mathbb{R})$. The next result provides a way to determine $N(\Lambda) \cap N(\Gamma)$ dependent on the maps defined below.

**Definition 4.2.** Suppose $(x, y, z)$ is a generating vector for a triangle group $G$. Then we define the following identifications:

- $i_1 : x \mapsto y, y \mapsto x, z \mapsto yzy^{-1}$
- $i_2 : x \mapsto y^{-1}xy, y \mapsto z, z \mapsto y$
- $i_3 : x \mapsto z, y \mapsto xyx^{-1}, z \mapsto x$
- $j : x \mapsto y, y \mapsto z, z \mapsto x$

**Theorem 4.3.** *Suppose that $\Lambda$ is a surface kernel subgroup of a triangle group $\Gamma$ and $\Gamma \lhd \Delta$. Let $\varrho\colon \Gamma \to G = \Gamma/\Lambda$ be the corresponding surface kernel epimorphism and suppose $(x, y, z)$ is the generating vector of $\varrho$.*

(*i*) *If $\Gamma$ has signature $(0; m_1, m_1, m_2)$, then $\Lambda$ is normal in $\Delta$ with signature $(0; 2, m_1, 2m_2)$ if and only if the identification $i_1$ extends to an automorphism of $G$.*

(*ii*) *If $\Gamma$ has signature $(0; m, m, m)$, then $\Lambda$ is normal in a group $\Delta$ with signature $(0; 2, m, 2m)$ and is normal in no larger group if and only if only precisely one of the identifications $i_1$, $i_2$, or $i_3$ extends to an automorphism of $G$*

($iii$) *If* $\Gamma$ *has signature* $(0; m, m, m)$, *then* $\Lambda$ *is normal in a group* $\Delta$ *with signature* $(0; 3, 3, m)$ *and is normal in no larger group if and only if the identification* $j$ *extends to an automorphism of* $G$ *but the identification* $i_1$ *does not.*

($iv$) *If* $\Gamma$ *has signature* $(0; m, m, m)$, *then* $\Lambda$ *is normal in a group* $\Delta$ *with signature* $(0; 2, 3, 2m)$ *if and only if the identifications* $i_1$ *and* $j$ *extend to automorphisms of* $G$.

*Proof.* This is a consequence of the results developed in [5].

□

Putting our results together, we get the following.

**Proposition 4.4.** *The number of topologically inequivalent topological* $G$*-actions* $\mathcal{T}$ *with signature* $(0; m_1, m_2, m_3)$ *on a surface* $S$ *can be calculated as follows.*

($i$) *If all the* $m_i$ *are distinct,*

$$\mathcal{T} = \frac{|\mathcal{V}_G|}{|\operatorname{Aut}(G)|}$$

*where* $\mathcal{V}_G$ *denotes the set of all generating vectors for surface kernel epimorphisms from* $\Gamma$ *to* $G$.

($ii$) *If* $\Gamma$ *has signature* $(0; m_1, m_1, m_2)$ *with* $m_2 \neq m_1$, *then*

$$\mathcal{T} = \frac{|\mathcal{V}_G|}{2|\operatorname{Aut}(G)} + \frac{|\mathcal{V}_{G,i_1}|}{|\operatorname{Aut}(G)|}$$

*where* $\mathcal{V}_G$ *denotes the set of generating vectors of surface kernel epimorphisms from* $\Gamma$ *to* $G$ *for which the identification* $i_1$ *does not extend to an automorphism of* $G$ *and* $\mathcal{V}_{G,i_1}$ *denotes the set of generating vectors for which the map* $i_1$ *does extend to an automorphism of* $G$.

($iii$) *If* $\Gamma$ *has signature* $(0; m, m, m)$, *then the number of topological equivalence classes of group actions of* $G$ *on a surface* $S$ *with signature* $(0; m, m, m)$ *is equal to*

$$\mathcal{T} = \frac{|\mathcal{V}_G|}{6|\operatorname{Aut}(G)|} + \frac{|\mathcal{V}_{G,i}|}{3|\operatorname{Aut}(G)|} + \frac{|\mathcal{V}_{G,j}|}{2|\operatorname{Aut}(G)|} + \frac{|\mathcal{V}_{G,i,j}|}{|\operatorname{Aut}(G)|}$$

*where* $\mathcal{V}_G$ *denotes the set of generating vectors of surface kernel epimorphisms from* $\Gamma$ *to* $G$ *for which none of the transformations* $i_1, i_2, i_3$ *or* $j$ *extend to automorphisms of* $G$, $\mathcal{V}_{G,i}$ *denotes the set of generating vectors for which just one of* $i_1, i_2$ *or* $i_3$ *extends to an automorphism of* $G$, $\mathcal{V}_{G,j}$ *denotes the set of generating vectors for which* $j$ *extends to an automorphism, but* $i_1$ *does not, and* $\mathcal{V}_{G,i,j}$ *denotes the set of generating vectors for which* $i_1$ *and* $j$ *extend to automorphisms of* $G$.

*Proof.* For a given $\Gamma$, the proof is a simple enumeration of the size of the orbits of surface kernels under the action of conjugation by subgroups of $N(\Gamma)$.

□

We finish with some examples.

**Example 4.5.** Suppose $G$ is cyclic of order 7 generated by $x$ and $\Gamma$ has signature $(0; 7, 7, 7)$. Then any surface kernel epimorphism from $\Gamma$ with signature $(0; 7, 7, 7)$ has corresponding generating vector of the form $(x^a, x^b, x^{7-a-b})$, so there are 30 epimorphisms in total. For each of these generating vectors, we need to determine which identifications given in Definition 4.2 extend to automorphisms of $G$. Observe that if $(x, y, z)$ is any generating vector with a repeated entry, then switching the two repeated entries induces an automorphism of $G$ (namely the identity), so either $i_1$, $i_2$ or $i_3$ extends to an automorphism of $G$. Now note that since there can be at most repeated entry, if $(x, y, z)$ is a generating vector with a repeated entry, then the identification $j$ does not extend to an automorphism of $G$. Since there are 18 such generating vectors, we have $|\mathcal{V}_{G,i}| = 18$.

If a generating vector $(x, y, z)$ has three distinct entries, we can use a similar argument to show that the identification $j$ always extends to an automorphism of $G$ but the identification $i_1$ does not. Since there are 12 such generating vectors, we have $|\mathcal{V}, |_G| = 18$. There are no other generating vectors, so applying Proposition 4.4, we get

$$\mathcal{T} = \frac{|\mathcal{V}_{G,i_1}|}{3|\operatorname{Aut}(G)|} + \frac{|\mathcal{V}_{G,j}|}{2|\operatorname{Aut}(G)|} = \frac{18}{18} + \frac{12}{12} = 2.$$

Therefore there are two distinct topological group actions of the cyclic group of order 7 with signature $(0; 7, 7, 7)$ on a surface of genus 3.

As we briefly mentioned in the introduction, there are numerous computational methods developed to help calculate the size of the sets of generating vectors (see for example [6] or [13]). We illustrate with an example.

**Example 4.6.** Suppose that $G = C_{13} \rtimes C_3$ (the non-trivial semidirect product). Then $G$ acts on a surface $S$ of genus 6 with signature $(0; 3, 3, 13)$. Applying Theorem 3 of [6], we get at most 156 different generating vectors for $G$, but since the smallest group containing an element of order 3 and order 13 is 39, it follows that all these generating vectors are generating vectors for $G$. Since $|\operatorname{Aut}(G)| = 78$, it follows that there are 2 different surface kernels in $\Gamma$ with signature $(0; 3, 3, 13)$ with quotient $G$. Since there are no groups which act on a surface $S$ of genus 6 with signature $(0; 2, 3, 26)$, it follows that these two surface kernels are conjugate by an element in the Fuchsian $\Delta \geqslant \Gamma$ with signature $(0; 2, 3, 26)$. Hence there is just one $\mathrm{PSL}(2, \mathbb{R})$-class of surface kernels and thus a unique topological action of $G$ on $S$ with signature $(0; 3, 3, 13)$.

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