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KLEIN-FOUR COVERS OF THE PROJECTIVE LINE IN CHARACTERISTIC TWO

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ABSTRACT. In this paper we examine curves defined over a field of characteristic 2 which are $(\mathbb{Z}/2\mathbb{Z})^2$-covers of the projective line. In particular, we determine which 2-ranks occur for such curves of a given genus and where possible we give explicit equations for such curves. As a corollary, we show that there exist hyperelliptic curves of genus g and 2-rank σ which contain an additional involution in their automorphism group if and only if $g \equiv \sigma \pmod{2}$.

1. INTRODUCTION

There are many ways to stratify the moduli space of curves. When working over an algebraically closed field of characteristic $p > 0$, one of the most natural stratifications comes from looking at the p-ranks of the curves. The p-rank of a curve X (or, more precisely, the p-rank of its Jacobian) can be defined as $\dim_{\mathbb{F}_p} \mathrm{Hom}(\mu_p, \mathrm{Jac}(X))$ where μ_p is the kernel of Frobenius on $\mathbb{G}_m$. In particular, curves of p-rank σ will have precisely p^σ distinct p-torsion points on their Jacobian defined over k.

It follows from [3] in characteristic $p > 2$ and [10] in characteristic 2 that there exist curves of each possible 2-rank for every genus. In this note, we investigate what one can say about the 2-ranks of curves which have multiple copies of $\mathbb{Z}/2\mathbb{Z}$ in their automorphism group. More precisely, we consider curves defined over an algebraically closed field of characteristic $p = 2$ which admit an action of $(\mathbb{Z}/2\mathbb{Z})^2$ and such that their quotient by this action is $\mathbb{P}^1$.

In Section 2 of this paper, we introduce notation and recall some results from [3] and [4] about Klein-four covers of the projective line. We also recall some results from the theory of Artin-Schreier covers that will be used to compute the genera and 2-ranks of the relevant curves. Section 3 is concerned with some nonexistence results, and we prove a number of results about the necessary conditions for a given 2-rank to occur. In the fourth section, we prove that the necessary conditions proven in Section 3 are in fact sufficient, and in particular we prove (a stronger version of) the following theorem.

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Theorem 1.1. *Let $g \geq 0$ and $0 \leq \sigma \leq g$. Then there exists a curve X with $G \cong (\mathbb{Z}/2\mathbb{Z})^2 \subseteq \mathrm{Aut}(X)$ and $X/G \cong \mathbb{P}^1$ such that X has genus g and 2-rank σ unless $\sigma = g-1$ or unless g is even and $\sigma = 1$.*

It will follow from the constructions of these curves that they are all defined over the finite field $\mathbb{F}_4$ and in most cases they can be chosen to be defined over $\mathbb{F}_2$. However, it will not always be the case that the 2-torsion points are themselves defined over $\mathbb{F}_4$.

We also relate our results to a result of Zhu in [10] which shows that there exist hyperelliptic curves of every possible 2-rank with no extra automorphisms. The following theorem shows precisely when a hyperelliptic curve can have extra involutions.

Theorem 1.2. *There are hyperelliptic curves of genus g and 2-rank σ which contain an additional involution in their automorphism group if and only if $g \equiv \sigma \pmod 2$.*

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2. Notation

In this article, we work over an algebraically closed field k of characteristic $p = 2$. We wish to examine curves that are $(\mathbb{Z}/2\mathbb{Z})^2$-covers of the projective line $\mathbb{P}^1_k$. In [3], we examined such curves defined over algebraically closed fields of characteristic $p > 2$ and in particular we used such curves to construct hyperelliptic curves with particular group schemes arising as the p-torsion of their Jacobians. When the characteristic of k is not equal to two, this Hurwitz space of such covers is well-defined (for details, see the results of Wewer in [9]) and in [3] we denoted the moduli space of genus g curves which are $(\mathbb{Z}/2\mathbb{Z})^2$-covers of $\mathbb{P}^1$ by $\mathcal{H}_{g,2}$. However, when the characteristic of k is equal to two we are in the situation of wild ramification, and Wewer's results do not hold. In particular, it is not clear whether $\mathcal{H}_{g,2}$ will be well-defined as a smooth moduli space due to the wild ramification.

From now on, X will be a k-curve of genus g and 2-rank σ which is a $(\mathbb{Z}/2\mathbb{Z})^2$-cover of $\mathbb{P}^1$. Let H_1, H_2, and H_3 be the three subgroups of $(\mathbb{Z}/2\mathbb{Z})^2$ with respect to a fixed basis. Furthermore, let C_1, C_2, and C_3 be the three quotient curves of X by these subgroups. Finally, we define g_i to be the genus of C_i and σ_i to be the 2-rank of C_i. By results of Kani and Rosen in [5], $\mathrm{Jac}(X) \sim \prod \mathrm{Jac}(C_i)$ and therefore it follows that $g_X = g_1 + g_2 + g_3$ and $\sigma_X = \sigma_1 + \sigma_2 + \sigma_3$. We note that X can be viewed as the normalization of the fibre product of any pair of the C_i, and if the C_i are defined over $\mathbb{F}_2$ then X will be defined over $\mathbb{F}_4$. Throughout this paper, we will use α to denote one of the elements of $\mathbb{F}_4$ other than one or zero.

The fact that we have wild ramification restricts some of the information we can learn from this situation, but there is more that we can say. In particular, we know that C_1, C_2, and C_3 must be Artin-Schreier covers, and therefore can be put into the form $C_i : y^2 + y = f_i(x)$ where f_i is a rational function in $xk(x^2)$. In this case, it follows from results of van der Geer and van der Vlugt in [8] that the third quotient is of the form $y^2 + y = f_3(x)$ where $f_3(x) = f_1(x) + f_2(x)$.

Given a cover of curves $X \to Y$, their genera are related by the Riemann-Hurwitz formula (see [7] for details) and if the Galois group is a p-group then their p-ranks are related by the Deuring-Shafarevich formula (see [1] for details). In particular, if the characteristic of k is two and we have a $\mathbb{Z}/2\mathbb{Z}$-cover $X \to Y$ branched at j

points then the Riemann-Hurwitz formula says that the genera of X and Y are related by the formula $g_X = 2g_Y - 1 + \frac{r}{2}$ where r is the degree of the ramification divisor. It follows immediately that $g_Y \leq \frac{g_X+1}{2}$. The Deuring-Shafarevich formula further says that $\sigma_Y = 2\sigma_X - 1 + j$. The following results about the genus and 2-rank of Artin-Schreier curves in characteristic two follow immediately and will be used throughout this note without reference.

Theorem 2.1. *Let $y^2 + y = f(x)$ define a hyperelliptic curve C in characteristic two. Let $f(x)$ have j poles given by $x_1, \ldots, x_j$ and let n_i be the order of the pole at x_i. Without loss of generality we can assume that all of the n_i are odd. Then the genus of C is given by the formula $-1 + \frac{1}{2}\sum(n_i + 1)$ and the 2-rank of C is given by $j - 1$.*

To conclude this introduction we define the $\mathcal{K}$-type of a Klein-four cover $X \to \mathbb{P}^1$ to be the unordered triple $\mathfrak{p} = \{g_1, g_2, g_3\}$ consisting of the genera of the three $\mathbb{Z}/2\mathbb{Z}$ quotients of X. In particular, it follows that the g_i are integers such that $0 \leq g_i \leq \frac{g+1}{2}$ and $g_1 + g_2 + g_3 = g$, so that $\mathfrak{p}$ is a partition of g. We define a partition $\mathfrak{p}$ – and by extension the $\mathcal{K}$-type of a curve – to be unbalanced if it contains an element which is at least $\frac{g}{2}$. In particular, unbalanced partitions are of the form $\{\frac{g}{2}, g_1, g_2\}$ or $\{\frac{g+1}{2}, g_1, g_2\}$ depending on the parity of g. Note that if $0 \in \mathfrak{p}$ it follows immediately that $\mathfrak{p}$ is unbalanced. On the other extreme, a totally balanced partition is when in which all three elements are the same, and therefore $\mathfrak{p} = \{\frac{g}{3}, \frac{g}{3}, \frac{g}{3}\}$.

We note that the $\mathcal{K}$-type of X is technically the type of the cover $X \to \mathbb{P}^1$, and in a small number of cases a curve X can be considered a $(\mathbb{Z}/2\mathbb{Z})^2$-cover of $\mathbb{P}^1$ in more than one way leading to different types. However, we show in [4] that this is rare in characteristic $p \neq 2$ (and happens exactly in the case where $1 \in \mathfrak{p}$). While not stated in that paper, the proof also works in characteristic 2.

3. Nonexistence Results

Throughout this section, X will be $(\mathbb{Z}/2\mathbb{Z})^2$-cover of $\mathbb{P}^1$ with 2-rank equal to σ and with $\mathcal{K}$-type $\mathfrak{p}$. We will give necessary conditions on σ and $\mathfrak{p}$ in order for such a curve X to exist. Recall that a curve is said to be almost-ordinary if it has 2-rank equal to $g - 1$.

Lemma 3.1. *The 2-rank of X cannot equal $g - 1$.*

Proof. Assume X is almost-ordinary. It follows that one of its $\mathbb{Z}/2\mathbb{Z}$ quotients must be almost-ordinary and the other two must be ordinary. Let C_1 and C_2 be the two quotients which are ordinary so that C_1 (resp. C_2) is defined by the equation $y^2 + y = f_1(x)$ (resp. $f_2(x)$) where f_1 (resp. f_2) only has simple poles. Then $f_1 + f_2$ must also have only simple poles and therefore the curve C_3, which is defined by $y^2 + y = f_1(x) + f_2(x)$, must also be ordinary. This gives a contradiction. □

In some cases it happens that a given 2-rank can occur for curves of some $\mathcal{K}$-types but not for curves of other $\mathcal{K}$-types, as the following results indicate.

Lemma 3.2. *If $\sigma = 0$ then $\mathfrak{p} = \{g_1, g_1, g_3\}$ where $g_1 \geq g_3$.*

Proof. Assume X is a curve with 2-rank equal to zero. It follows that all three of the hyperelliptic quotients have 2-rank zero and therefore they can each be defined by $y^2 + y = f_i(x)$ where each f_i has a single pole at the same point. It follows that (at least) two of these three functions must have a pole of the same order and that the order of the third pole is no larger than these two, and therefore the same statement can be made about the genera.

□

Lemma 3.3. *If $\sigma = 1$ then g is odd and $\mathfrak{p} = \{\frac{g+1}{2}, g_2, g_3\}$ is unbalanced.*

Proof. Assume that σ equals 1. Then two of the hyperelliptic quotients must have 2-rank zero while the third has 2-rank one. It follows without loss of generality that f_1 has a pole of order a at one point and f_2 has a pole of order b at another point where a and b are both odd. In that case we can compute that the curve X is of type $\{\frac{a-1}{2}, \frac{b-1}{2}, \frac{a+b}{2}\}$ which in turn implies that the genus of the curve X is $a + b - 1$ (and is thus odd) while the genus of the curve C_3 is $\frac{a+b}{2} = \frac{g+1}{2}$.

□

A quite different result holds if we look at curves with 2-rank equal to 2.

Lemma 3.4. *If $\sigma = 2$ then $\mathfrak{p} \neq \{g_1, g_1, g_1\}$.*

Proof. Assume X is a curve whose 2-rank is equal to 2. Let C_1, C_2, and C_3 be the three quotient curves and let σ_i be the 2-rank of C_i. Then it follows without loss of generality that either $\sigma_1 = 2$ and $\sigma_2 = \sigma_3 = 0$ or $\sigma_1 = \sigma_2 = 1$ and $\sigma_3 = 0$. However, the first case cannot happen, because it would imply that f_1 would have 3 poles while each of f_2 and f_3 would have a unique pole.

Therefore we must be in the second case, in which f_1 and f_2 each have two poles and f_3 has one pole. We can assume that f_1 and f_2 each have poles at zero which cancel each other out and poles at infinity and that f_3 has a pole only at infinity. Without loss of generality, we may assume that $ord_\infty(f_1) \geq ord_\infty(f_3)$ which will in turn imply that $g_1 > g_3$. Therefore, $\mathfrak{p}$ cannot be a totally balanced partition.

□

Lemma 3.5. *If $\mathfrak{p}$ is unbalanced then $g \equiv \sigma \pmod 2$.*

Proof. If g is odd and $\frac{g+1}{2} \in \mathfrak{p}$ then there exists an involution $\tau \in \mathrm{Aut}(X)$ such that the genus of $C_1 = X/ < \tau >$ is equal to $\frac{g+1}{2}$. It follows from the Riemann-Hurwitz formula that the cover $X \to C_1$ must be étale. Therefore, if we apply the Deuring-Shafarevich formula to $X \to C_1$ we see that $\sigma_X = 2\sigma_{C_1} - 1$ is odd.

Similarly, if g is even and $\frac{g}{2} \in \mathfrak{p}$ then it follows from the Riemann-Hurwitz formula that the cover $X \to C_1$ must be ramified at a single point. Again, it will follow from the Deuring-Shafarevich formula that $\sigma_X = 2\sigma_{C_1}$ must be even.

Therefore, in both cases where we look at curves whose $\mathcal{K}$-types are unbalanced we see that $\sigma_X \equiv g_X \pmod 2$.

□

4. Existence Results

The main result in this section is that the necessary conditions on σ and $\mathfrak{p}$ which were shown in the previous section are also sufficient. In particular, we will prove the following theorem.

Theorem 4.1. *There exist curves of genus g, 2-rank σ, and $\mathcal{K}$-type $\mathfrak{p}$ under the following conditions:*

(1) $\sigma \neq g-1$.
(2) *If* $\sigma = 0$ *then* $\mathfrak{p} = \{g_1, g_1, g_3\}$ *with* $g_3 \leq g_1$.
(3) *If* $\sigma = 1$ *then* $\frac{g+1}{2} \in \mathfrak{p}$.
(4) *If* $\sigma = 2$ *then* $\mathfrak{p} \neq \{g_1, g_1, g_1\}$.
(5) *If* $\mathfrak{p}$ *is unbalanced, then* $g \equiv \sigma$ *(mod 2).*

We will prove this theorem by induction on σ after looking at some base cases. In particular, we will use the following inductive lemma which says that if Theorem 4.1 holds for σ then it is *almost* immediate that it will hold for $\sigma + 3$.

Lemma 4.2. *If there exists a curve X of genus g, 2-rank σ and $\mathcal{K}$-type $\mathfrak{p} = \{g_1, g_2, g_3\}$ then there exists a curve $\tilde{X}$ of genus $g+3$ and 2-rank $\sigma+3$ which has $\mathcal{K}$-type $\hat{\mathfrak{p}} = \{g_1+1, g_2+1, g_3+1\}$.*

Proof. Assume that the three hyperelliptic quotients of X are defined by the equations $y^2 + y = f_i(x)$, where without loss of generality we may assume that none of the f_i have poles at infinity. Then we define $\tilde{f}_1 = f_1 + x$, $\tilde{f}_2 = f_2 + \alpha x$ and $\tilde{f}_3 = f_3 + (\alpha+1)x$ where α is one of the elements of $\mathbb{F}_4$ other than one or zero. It is clear that $\tilde{f}_3 = \tilde{f}_1 + \tilde{f}_2$ and that the curve $\tilde{X}$ defined by the fibre product of $y^2 + y = \tilde{f}_1(x)$ and $y^2 + y = \tilde{f}_2(x)$ will have the desired properties. □

We begin proving the necessary base cases by showing that Theorem 4.1 is true for small values of σ.

Lemma 4.3. *Let $\mathfrak{p} = \{g_1, g_1, g_3\}$ with $g_3 \leq g_1$. Then there exist curves of $\mathcal{K}$-type $\mathfrak{p}$ and 2-rank $\sigma = 0$.*

Proof. Let $a = 2g_1 + 1$ and $b = 2g_3 + 1$ and define $f_1 = x^a$ and $f_3 = x^b$ so that $f_2 = f_1 + f_3 = x^a + x^b$. Then the curves defined by $y^2 + y = f_i(x)$ all have 2-rank equal to zero, and the genera of the curves $y^2 + y = f_1(x)$ and $y^2 + y = f_2(x)$ will each be g_1 while the genus of the curve defined by $y^2 + y = f_3(x)$ will be g_3. Our construction now shows that the relevant fibre product will have the desired properties. □

Lemma 4.4. *Let g be odd and let $\mathfrak{p}$ be an unbalanced partition (ie $\frac{g+1}{2} \in \mathfrak{p}$). Then there are curves X of genus g and $\mathcal{K}$-type $\mathfrak{p}$ with 2-rank equal to one.*

Proof. Let $\mathfrak{p} = \{\frac{g+1}{2}, g_2, g_3\}$ and define $a = 2g_2 + 1$ and $b = 2g_3 + 1$. The curve C_2 defined by $y^2 + y = x^a$ will have genus g_2 and 2-rank equal to 0 and the curve C_3 defined by $y^2 + y = \frac{1}{x^b}$ will have genus g_3 and 2-rank equal 0. If we look at the fibre product of C_2 and C_3, it will be a $(\mathbb{Z}/2\mathbb{Z})^2$-cover of $\mathbb{P}^1$ and the third hyperelliptic quotient C_1 will be defined by the equation $y^2 + y = x^a + \frac{1}{x^b}$. In particular, the genus of C_1 will be $g_1 = \frac{a+b}{2} = g_2 + g_3 + 1 = \frac{g+1}{2}$ where $g = g_1 + g_2 + g_3$ is the genus of the fibre product C. Similarly, we see that the 2-rank of C is equal to one as desired. □

Lemma 4.5. *Let $\mathfrak{p}$ be a partition which is neither completely balanced or, if g is odd, unbalanced. Then there exist curves of $\mathcal{K}$-type $\mathfrak{p}$ and 2-rank equal to two.*

Proof. Let $\mathfrak{p} = \{g_1, g_2, g_3\}$ with $g_1 \geq g_2 \geq g_3$. Let $a = 2g_3 + 1$, $b = 2(g_1 - g_3) - 1$ and $c = 2(g_2 + g_3 - g_1) + 1$. It is clear that a, b, and c are all odd, and that

$a \geq c$. Furthermore, $b \geq 1$ because $\mathfrak{p}$ is not completely balanced and $c \geq 1$ because $g_1 \leq g/2$. If we now let $f_1 = x^a + \frac{1}{x^b}$ and $f_2 = \alpha x^c + \frac{1}{x^b}$ we see that $f_3 = f_1 + f_2 = x^a + \alpha x^c$ and a simple computation shows that the fibre product X will have 2-rank equal to 2 and $\mathcal{K}$-type $\mathfrak{p}$.

□

Note that in all of the above situations, the case of $\sigma = g-1$ is eliminated. Next, we show that the necessary condition on the 2-ranks of curves with unbalanced $\mathcal{K}$-types from Theorem 3.5 is actually sufficient.

Lemma 4.6. *For any unbalanced partition $\mathfrak{p}$, there will be curves of $\mathcal{K}$-type $\mathfrak{p}$ and 2-rank σ as long as $g \equiv \sigma$ (mod 2).*

Proof. In order to prove this lemma we must show that there are curves of $\mathcal{K}$-type $\mathfrak{p} = \{\frac{g}{2}, g_1, g_2\}$ for all even 2-ranks and curves of $\mathcal{K}$-type $\mathfrak{p} = \{\frac{g+1}{2}, g_1, g_2\}$ for all odd σ.

We first consider the case when g is odd and $\mathfrak{p}$ is unbalanced, so that $\mathfrak{p} = \{\frac{g+1}{2}, g_1, g_2\}$ with $g_1 \geq g_2$. We note that we can construct hyperelliptic covers $C_1 \to \mathbb{P}^1$ and $C_2 \to \mathbb{P}^1$ so that the genus of C_i is g_i and the 2-rank of C_i is k_i for all $0 \leq k_i \leq g_i$. Furthermore, after modifying $C \to \mathbb{P}^1$ by a projective linear transformation of $\mathbb{P}^1$, one can assume that the branch loci of the two covers are distinct. If we let X be the fibre product of C_1 and C_2 and consider the third hyperelliptic quotient of X we see that it will have genus $g_1 + g_2 + 1$ and 2-rank $k_1 + k_2 + 1$. If we choose k_1 and k_2 so that $k_1 + k_2 = k$ then X will have 2-rank equal to σ and $\mathcal{K}$-type $\mathfrak{p}$.

Next, we will construct a curve with 2-rank equal to $2m$ and $\mathcal{K}$-type $\{\frac{g}{2}, \frac{g}{2}, 0\}$. We first note that we can find a hyperelliptic curve of genus $\frac{g}{2}$ with 2-rank equal to m for $0 \leq m \leq \frac{g}{2}$. Let us assume that this curve C_1 is defined by the equation $y^2 + y = f_1(x)$ where f_1 has a pole at infinity. Let f_2 be some constant multiple of x so that $f_3 = f_1 + f_2$ will have the same poles (with the same orders) as f_1. Note that if the order of the pole of f_1 at ∞ is greater than one then we can choose this constant multiple to simply be x. If $ord_\infty(f_1) = 1$ then we need to choose a multiple so that f_3 still has a pole at infinity, but we are guaranteed a choice of this multiple defined over $\mathbb{F}_4$. It follows from our construction that the curve X will have 2-rank $2k$ and the desired $\mathcal{K}$-type.

Finally, we consider the case where $\mathfrak{p} = \{\frac{g}{2}, g_1, g_2\}$ with g_1 and g_2 both positive and we wish to show that there will be curves of all even 2-ranks. We note that $\hat{\mathfrak{p}} = \{\frac{g}{2} - 1, g_1 - 1, g_2 - 1\}$ gives an unbalanced partition of $g - 3$. We may assume that $\sigma \geq 4$ (the case $\sigma = 0$ was handled in Lemma 4.3 and the case of $\sigma = 2$ was taken care of by Lemma 4.5), so $\sigma - 3$ will be a positive odd number. In particular, the above argument shows that there exists a curve $\hat{X}$ of $\mathcal{K}$-type $\hat{\mathfrak{p}}$ and 2-rank $\sigma - 3$. The conclusion now follows from Lemma 4.2.

□

For all $\sigma \geq 3$, Theorem 4.1 says that there are no restrictions other than this parity condition for unbalanced $\mathcal{K}$-types and the case where $\sigma = g - 1$. We now show this concretely for $\sigma = 3, 4$, and 5.

Lemma 4.7. *If $\sigma = 3$ then Theorem 4.1 holds. In particular, there are curves of 2-rank equal to three of all $\mathcal{K}$-types except the case where g is even and $\frac{g}{2} \in \mathfrak{p}$.*

Proof. If g is odd and $\mathfrak{p}$ is unbalanced then the result follows from Lemma 4.6. Therefore, it suffices to consider the case where $\mathfrak{p} = \{g_1, g_2, g_3\}$ with $1 \leq g_3 \leq g_2 \leq g_1 \leq \frac{g-1}{2}$. Set $a = 2(g_1 - g_3) + 1$, $b = 2g_3 - 1$ and $d = 2(g_2 + g_3 - g_1) - 1$. We note that our hypotheses imply that a, b, and d are all odd positive numbers with $b \geq d$. Now, let $f_1 = x^a + \frac{\alpha}{x^b}$, $f_2 = x^a + \frac{1}{x^d}$ and $f_3 = f_1 + f_2$. Then the curve defined by $y^2 + y = f_i$ will have genus g_i and 2-rank equal to one, and therefore X will be a curve of 2-rank equal to three.

□

Lemma 4.8. *If $\sigma = 4$ then Theorem 4.1 holds. In particular, there are curves of 2-rank equal to four of all $\mathcal{K}$-types except the case where $g = 5$ or when g is odd and $\frac{g+1}{2} \in \mathfrak{p}$.*

Proof. Assume that $\mathfrak{p} = \{g_1, g_2, g_3\}$ where $g_1 > g_2 \geq g_3$. Let $a = 2g_2 - 1$, $b = 2(g_1 - g_2) - 1$, and $c = 2(g_2 + g_3 - g_1) + 1$. One can easily check that a, b, and c are all positive odd numbers as the fact that $\frac{g+1}{2} \notin \mathfrak{p}$ implies that $g_1 \leq g_2 + g_3$. Furthermore, we see that $a \geq c$. Let $f_1 = x^a + \frac{1}{x^b} + \frac{1}{x+1}$, $f_3 = x^c + \frac{1}{x^b}$, and $f_2 = f_1 + f_3$. Then the curve defined by the equation $y^2 + y = f_i(x)$ has genus g_i and the fibre product X will have genus g and 2-rank $\sigma = 4$ as desired.

On the other hand, assume that $g_1 = g_2 \geq g_3 \geq 2$. In this case, let $a = 2g_1 - 1$ and $b = 2g_3 - 3$. Then it is clear that a and b are positive odd integers with $a > b$. If we define $f_1 = x^a + \frac{1}{x}$ and $f_3 = x^b + \frac{1}{x} + \frac{1}{x+1}$ we can see that the curves will have the desired properties.

For the partition $\mathfrak{p} = \{\frac{g}{2}, \frac{g}{2}, 0\}$ the lemma follows from Lemma 4.6, so it suffices to consider the case where g is odd and $\mathfrak{p} = \{\frac{g-1}{2}, \frac{g-1}{2}, 1\}$. We note that $g \neq 5$, so we may assume that $g \geq 7$. Let $f_1 = x^{g-4} + \frac{1}{x} + \frac{1}{x+1}$ and $f_2 = \alpha x$. These equations define curves with the desired genera and 2-ranks.

□

Lemma 4.9. *If $\sigma = 5$ then Theorem 4.1 holds. In particular, if $g \geq 7$ there are curves of 2-rank equal to five of all $\mathcal{K}$-types except the case where $g = 6$ or the case where g is even and $\frac{g}{2} \in \mathfrak{p}$.*

Proof. Let $\mathfrak{p} = \{g_1, g_2, g_3\}$ be a partition of g with $0 \leq g_3 \leq g_2 \leq g_1 \leq \frac{g+1}{2}$. We wish to show that there are curves of $\mathcal{K}$-type $\mathfrak{p}$ and 2-rank equal to five unless $g_1 = \frac{g}{2}$ (in which case g will be even). If $g_1 = \frac{g+1}{2}$ then the result follows from Lemma 4.6.

If $g_1 \leq \frac{g-1}{2}$ then it follows that $g_3 > 0$ and thus $\hat{\mathfrak{p}} = \{g_1 - 1, g_2 - 1, g_3 - 1\}$ gives a partition of $g - 3$ all of whose entries are at most $\frac{g-3}{2}$. Thus, by Lemma 4.5 there are curves of $\mathcal{K}$-type $\hat{p}$ of 2-rank equal to 2 unless $\hat{p}$ (and therefore $\mathfrak{p}$) is completely balanced. By the induction argument in Lemma 4.2 we therefore have curves whose 2-rank is equal to five in of $\mathcal{K}$-type $\mathfrak{p}$.

It remains to consider the case where $\mathfrak{p}$ is totally balanced: that is, where $g_1 = g_2 = g_3 = a > 2$. To deal with this case, let $f_1 = x^a + \frac{1}{x^a}$ and $f_2 = x^a + \frac{1}{(x-1)^{a-2}} + \frac{1}{x-\alpha}$ and $f_3 = f_1 + f_2$. One can easily compute that these choices will lead to a curve X of $\mathcal{K}$-type $\{a, a, a\}$ whose 2-rank is equal to 5.

□

Before proving the main theorem, there is one more base case that we need to consider.

Lemma 4.10. *Let g be odd and $\frac{g-1}{2} \in \mathfrak{p}$ but $\frac{g+1}{2} \notin \mathfrak{p}$. Then there are curves in of $\mathcal{K}$-type $\mathfrak{p}$ with 2-rank equal to $2m$ for all $0 \leq m \leq \frac{g-3}{2}$.*

Proof. Let $\mathfrak{p} = \{\frac{g-1}{2}, g_1, g_2\}$ with $g_1 \geq g_2 > 0$ and let $\sigma = 2m$ be as above. Because $\sigma \leq g-3$ we have that $m \leq g_1 + g_2 - 2$ and therefore we can choose m_1 and m_2 so that $m_1 + m_2 = m$ but $m_i < g_i$. In particular, we can define a function $h_1(x)$ which has m_1 poles (none of which are at infinity) so that the curve C_1 defined by $y^2 + y = x^3 + h_1(x)$ will have genus g_1 and 2-rank m_1. Similarly, we can choose h_2 with poles distinct from those of h_1 so that the curve C_2 defined by $y^2 + y = \alpha x^3 + h_2(x)$ will have genus g_2 and 2-rank m_2.

If we look at the normalization of the fibre product of C_1 and C_2 we see that the third quotient will be defined by the equation $y^2 + y = (\alpha + 1)x^3 + h_1(x) + h_2(x)$ and therefore will have genus $g_1 + g_2 - 1$ and 2-rank $m_1 + m_2 = m$. Thus, the curve X has $\mathcal{K}$-type $\{\frac{g-1}{2}, g_1, g_2\}$ and has 2-rank equal to $2m$, as desired. □

We are finally ready to prove Theorem 4.1.

Proof. Given the results of the above lemmata, it suffices to consider the case where $\sigma \geq 6$. In this case, we only need to prove that there are curves of 2-rank equal to σ in every partition if $g \equiv \sigma \pmod 2$ and that there are curves of 2 rank equal to σ in every partition whose entries are all at most $\frac{g-1}{2}$ if $g \not\equiv \sigma \pmod 2$.

If $0 \in \mathfrak{p}$ then $\mathfrak{p}$ must be unbalanced, and therefore we only need to consider the case where $g \equiv \sigma \pmod 2$. The result then follows from Lemma 4.6. Similarly, if $\frac{g+1}{2} \in \mathfrak{p}$ the result follows from Lemma 4.6.

If $\mathfrak{p} = \{\frac{g-1}{2}, g_1, g_2\}$ then it follows from Lemma 4.10 that there are curves of every even 2-rank strictly less than $g-1$ of $\mathcal{K}$-type $\mathfrak{p}$. To construct the curves of odd 2-rank σ, we note that g_1 and g_2 must be positive, and therefore $\hat{\mathfrak{p}} = \{\frac{g-3}{2}, g_1 - 1, g_2 - 1\}$ gives an unbalanced partition of $g-3$. Furthermore, $g - 3 \equiv \sigma - 3 \pmod 2$ and therefore there are curves of $\mathcal{K}$-type $\hat{\mathfrak{p}}$ of 2-rank equal to $\sigma - 3$ by Lemma 4.6. The result then follows from the inductive process described in Lemma 4.2.

If all entries of $\mathfrak{p}$ are at least 1 and at most $\frac{g-2}{2}$, we note $\hat{\mathfrak{p}} = \{g_1 - 1, g_2 - 1, g_3 - 1\}$ gives a partition of $\hat{g} = g - 3$ such that each $\hat{g}_i = g_i - 1$ is at most $\frac{\hat{g}-1}{2}$ and therefore there exist curves of 2-rank $\sigma - 3$ and $\mathcal{K}$-type $\hat{\mathfrak{p}}$. By the inductive procedure described in Lemma 4.2 we can construct a curve of $\mathcal{K}$-type $\mathfrak{p}$ with 2-rank equal to σ, proving the theorem. □

In [10], Zhu proves that there exist hyperelliptic curves with no extra automorphisms of every possible 2-rank. The following result shows that, depending on the 2-rank, there may or may not be hyperelliptic curves that *do* admit an extra involution.

Corollary 4.11. *There are hyperelliptic curves of genus g and 2-rank σ which contain an additional involution in their automorphism group if and only if $g \equiv \sigma$ (mod 2).*

Proof. It is well known that the hyperelliptic involution is contained in the center of the automorphism group of a curve (see [6] for one proof in characteristic two). Therefore, if there is another involution in the automorphism group then we must have a Klein-four action on the curve and therefore we will be in the setup above.

Furthermore, it follows that the partition $\mathfrak{p}$ corresponding to this curve contains a zero and is therefore either $\mathfrak{p} = \{\frac{g+1}{2}, \frac{g-1}{2}, 0\}$ or $\mathfrak{p} = \{\frac{g}{2}, \frac{g}{2}, 0\}$. In either case, the partition is unbalanced and therefore $g \equiv \sigma \pmod 2$ by Theorem 3.5.

Conversely, it follows from Theorem 4.1 that if $g \equiv \sigma \pmod 2$ then there will exist curves in this partition, which will therefore be both hyperelliptic and contain an extra involution.

□

We note that this does not answer the question of the automorphism groups fully, as the curves may have automorphisms of degree greater than two. We examine the question of the possible 2-ranks of hyperelliptic curves with extra automorphisms in depth in [2].

References

[1] R. M. Crew. Etale p-covers in characteristic p. *Compositio Math.*, 52(1):31–45, 1984.

[2] D. Glass. The 2-ranks of hyperelliptic curves with extra automorphisms. *preprint*, 2006.

[3] D. Glass and R. Pries. Hyperelliptic curves with prescribed p-torsion. *Manuscripta Math.*, 117(3):299–317, 2005.

[4] D. Glass and R. Pries. On the moduli space of Klein four covers of the projective line. In *Computational aspects of algebraic curves*, volume 13 of *Lecture Notes Ser. Comput.*, pages 58–70. World Sci. Publ., Hackensack, NJ, 2005.

[5] E. Kani and M. Rosen. Idempotent relations and factors of Jacobians. *Math. Ann.*, 284(2):307–327, 1989.

[6] C. Lehr and M. Matignon. Automorphism groups for p-cyclic covers of the affine line. *Compos. Math.*, 141(5):1213–1237, 2005.

[7] J.-P. Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg.

[8] G. van der Geer and M. van der Vlugt. Fibre products of artin-schrier curves and generalized hamming weights of codes. *Journal of Combinatorial Theory, Series A*, 1995.

[9] S. Wewers. Construction of Hurwitz spaces. Thesis.

[10] H. J. Zhu. Hyperelliptic curves over $\mathbb{F}_2$ of every 2-rank without extra automorphisms. *Proc. Amer. Math. Soc.*, 134(2):323–331, 2006.

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